Gains in evolutionary dynamics
unifying rational framework for dynamic stability of ESS

Dai ZUSEI

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Summary

Goal
We give a general and intuitive proof of dynamic stability of (regular) ESS.
★ Nash stability in contractive (negative definite) games is proven as a preliminary step.
★ With a general proof in hand, the stability can be easily extended to the dynamic of multiple populations with heterogeneity in payoff functions and/or revision protocols, regardless of observability or unobservability of heterogeneous types.
★ We prove stability by finding a general formula of a Lyapunov function.

Key idea
Gain from strategy revision should diminish over time, as long as each individual wholly exploits the opportunity to improve its payoff.
★ Nash eqm ⇔ “no profitable deviation”, i.e., no gain for anyone.
★ In a strategic situation, there is a second order effect: the payoffs may change and it may revive potential gains.
★ The negative definiteness guarantees that the second order effect only lessens the gain.

Challenge
Other than BRD, agents may still switch to a suboptimal strategy and may not exploit the whole possible gain.
★ Suppose that we simply define “gross gain” as the payoff difference.
  · Positive correlation: total gross gain $\dot{x} \cdot \pi \geq 0$ always, and $= 0$ only at a Nash eqm.
  · Yet, it may not decrease over time unless the game has a concave potential function.
★ We reformulate evolutionary dynamics to rationalize switches to suboptimal strategies by a (stochastic) switching cost and a restricted available strategy set.
★ Then, we define the net gain as the payoff difference minus the switching cost.
★ This works: the total net gain monotonically diminishes to zero in contractive games.
Population game: notation

We start from a single-population game, defined in a strategic form as follows.

- A continuum of agents; total mass=1
- Each agent chooses an action (strategy) from $\mathcal{A} = \{1, \ldots, A\}$.
  - $x \in \Delta^A$: the action distribution (the social state). $\Delta^A := \{x \in \mathbb{R}^A | \sum_{a \in \mathcal{A}} x_a = 1\}$.
- $F: \Delta^A \to \mathbb{R}^A$: payoff function. $F_a(x) \in \mathbb{R}$: payoff of action $a$ in state $x$.
  - We assume that $F$ is continuously differentiable.

<table>
<thead>
<tr>
<th>Definition</th>
<th>Contractive (negative definite) game</th>
<th>Hofbauer &amp; Sandholm (2009, JET)</th>
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<tr>
<td>A population game $F$ is a <strong>contractive game</strong> if</td>
<td>$$(y - x) \cdot (F(y) - F(x)) \leq 0$$ for all $x, y \in \Delta^A$.</td>
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<td>or (if $F$ is $C^1$) equivalent to</td>
<td>$z \cdot \frac{dF}{dx}(x)z \leq 0$ for any $z \in \mathbb{R}^A$.</td>
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- e.g. Congestion games with payoff heterogeneity; Two-player zero sum games; War of attrition
- regular ESS: local version. (We will see it later.)

Nash stability has been verified for major dynamics by finding a Lyapunov function specifically for each dynamic: see the references in HS, as well as Fox & Shamma (2013, Games).
Evolutionary dynamics and revision protocol

We construct an evolutionary dynamic from a revision protocol:

### Standard formulation of an evolutionary dynamic

| 1° Each agent occasionally receives an opportunity to revise the action |
| Revision opportunities follow a Poisson process; the arrival rate is fixed at 1. |
| 2° A revising agent uses a certain revision protocol to choose a new action, depending on the current payoff vector $\pi \in \mathbb{R}^A$ and the social state $x \in \Delta A$. |
| 3° Evolutionary dynamic $\dot{x} \in V(x)$ is obtained as the aggregation of such individual revision processes in large population. |

A variety of evolutionary dynamics comes from a variety of revision protocols.

- e.g. BRD: optimization revision protocol: pick an action that yields the greatest payoff.
- e.g. Smith (a canonical payoff comparison dynamic):
  1) randomly pick another action $b$ with equal probability
  2) and then switch to it with probability proportional to the payoff difference, as long as $b$ is better than $a$. 
Reconstruction of evolutionary dynamics from optimization

To rationalize deviation from exact optimization, I propose to reconstruct an evolutionary dynamic from an *optimization* protocol with a *switching cost* and a *restricted action set*. Then, I regain a variety of dynamics by allowing randomness in these two components and a variety of probability distributions.

General reconstruction by optimization-based revision protocol

- Upon receipt of a revision opportunity, a revising agent compares payoffs of current action $a$ and of other *available* actions.
- The *available action set* $A'_a$ is a nonempty subset of $\mathcal{A} \setminus \{a\}$.
- If the revising agent chooses to switch an action, he needs to pay *switching cost* $q$.
  
  The realized switching cost $q$ is commonly incurred to all the available actions.
- Given payoff vector $\pi \in \mathbb{R}^A$, the revising agent chooses an action to optimize his *net payoff* from available action set $A'_a$ or current action $a$: i.e., choose an action so as to...
  
  $\max\{\pi_a, \max\{(\pi_b - q) \mid b \in A'_a\}\}$

N.B. In other words, current action $a$ is *status quo* in the decision of new action and $q$ is the bias to prefer the status quo.

Allowing the randomness

- $A'_a$ is drawn from from probability distribution $\mathbb{P}_{A\setminus\{a\}}$ over the power set of $\mathcal{A} \setminus \{a\}$.
- $q$ is drawn from probability distribution $\mathbb{P}_Q$ over $\mathbb{R}$.
- Assume that these random draws are independent of each other, the agent’s current action and the social state and also their probability distributions are invariant over time.
Note that switching occurs with prob $Q(\pi_\pi[A'_a] - \pi_a)$, where $\pi_\pi[A'_a] = \max\{\pi_b \mid b \in A'_a\}$ is the payoff from an optimal available action. There may be multiple optimal actions in $A'_a$.

By formulating a dynamic as a differential inclusion (set-valued differential equation), we don’t have to reduce multiplicity. That is, we allow switching agent who face $A'_a$ to switch to any mixed strategy (distribution) $y_\pi[\pi; A'_a]$ of the optimal available actions among $A'_a$, i.e., any $y_\pi[\pi; A'_a] \in \Delta^A(A') := \{x \in \Delta^A \mid x_b > 0 \Rightarrow b \in A'\}$.

$\mathcal{V}$: the dynamic of the social state, induced from the constrained optimization protocol:

$$\dot{x} \in \mathcal{V}(x)[\pi] := \sum_{a \in A} x_a \mathcal{V}_a[\pi] \sum_{A'_a \subset A \setminus \{a\}} P_{A_a}[A'_a] Q(\pi_\pi[A'_a] - \pi_a)(\Delta^A(b_\pi[\pi; A'_a]) - e_a).$$

Plugging payoff function $F(x)$ into $\pi$ in the above eq’n, we can pin down a dynamic.

We denote $\mathcal{V}[F(x)]$ by $\mathcal{V}^F(x)$. 
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Assumptions

**Assumption** Assumptions on the revision protocol

(A-Q1) Nonnegativity of switching costs \( q \):

\[
Q(0) = \mathbb{P}_Q((-\infty, 0)) = 0, \quad \text{i.e., } q \geq 0 \text{ almost surely.}
\]

(A-Q2) Infinitesimally small positive \( q \):

\[
\forall q > 0, \quad Q(q) = \mathbb{P}_Q((-\infty, q)) > 0.
\]

(A-A1) Ex ante availability of all actions:

\[
\forall a \in \mathcal{A}, b \in \mathcal{A} \setminus \{a\}, \quad \mathbb{P}_{Aa}(\{A'_a | b \in A'_a\}) > 0
\]

\( b \) is switchable from \( a \)

(A-A2) Symmetry of available action set distribution:

\[
\forall a, b \in \mathcal{A}, A^0 \subset \mathcal{A} \setminus \{a, b\}, \quad \mathbb{P}_{Aa}(\{A'_a | A'_a \cap A^0 \neq \emptyset\}) = \mathbb{P}_{Ab}(\{A'_b | A'_b \cap A^0 \neq \emptyset\}).
\]

\( \exists \) an action in \( A^0 \) switchable from \( a \) \quad \( \exists \) an action in \( A^0 \) switchable from \( b \)

(A-A3) Invariance of \( A'_a \) to the social state:

\[\mathbb{P}_{Aa} \text{ is independent of } x.\]

Observational dynamics (e.g. imitative dynamics like the replicator) do not meet A3.

(cf. Mertikopoulus & Sandholm “Riemannian dynamics.” provides a general proof for such dynamics.)
Examples of rationalizable dynamics

If an evolutionary dynamic can be reconstructed in this way and satisfies Q1–2 & A1–3, we call it a rationalizable dynamic.

The class of rationalizable dynamics includes...

- **Standard BRD** (Gilboa & Matsui, Hofbauer): always switch to the optimal action
- **Tempered BRD** (Zusai, IJGT forthcoming): switch to the optimal action with prob \( Q(\pi^* - \pi_a) \)
- **Symmetric pairwise payoff comparison** (e.g. Smith): randomly pick another action \( b \) and switch to it with prob \( Q([\pi_b - \pi_a]_+) \) (Smith: \( Q(\bar{\pi}) = \bar{\pi} \)).

Some new dynamics can be though of, such as...

- **BRD with randomaly restricted action set.** An agent randomly observes only a subset of (non-incumbent) action sets in either of the two ways below, compares these observed actions with the current action in their payoffs and then switches to the best.
  - Each action is observed independently with the same probability.
  - First, the number of observed actions is determined, possibly randomly. Then, each subset with this number of actions is picked with the same probability.

With some twists in formulation, we can include...

- **Excess payoff target dynamics** (e.g. Brown-von Neumann-Nash dyn): Take mixed strategy \( x \) (random assignment based on the current action distribution) as a status quo and charge the cost on switching from this status quo.

Then, the gain function satisfies the properties that we will see. The proof differs from rationalizable dynamics but remains similar.
To prove stability, I use a Lyapunov function (a modified version for a differential inclusion, proved in the appendix of my tBRD paper.):

**Theorem** Lyapunov stability theorem  

\[ \text{Zusai (IJGT, forthcoming)} \]

Let \( A \) be a closed subset of a compact space \( \mathcal{X} \) and \( A' \) be a neighborhood of \( A \). Suppose that two continuous functions \( W : \mathcal{X} \to \mathbb{R} \) and \( \tilde{W} : \mathcal{X} \to \mathbb{R} \) satisfy

(i) \( W(x) \geq 0 \) and \( \tilde{W}(x) \leq 0 \) for all \( x \in \mathcal{X}' \),

(ii) \( W^{-1}(0) = \tilde{W}^{-1}(0) = A \).

In addition, assume that \( W \) is Lipschitz continuous in \( x \in \mathcal{X} \) with Lipschitz constant \( K \in (0, \infty) \). If any Carathéodory solution \( \{x_t\} \) starting from \( A' \) satisfies

\[ \dot{W}(x_t) \leq \tilde{W}(x_t) \text{ for almost all } t \in [0, \infty), \]

then \( A \) is asymptotically stable and \( A' \) is a basin of attraction to \( A \).

- I call \( W \) a (decreasing) **Lyapunov function** and \( \tilde{W} \) a **decaying rate function**.
- This is a general version for a differential inclusion (set-value dynamic to allow multiple transition vectors: e.g. best response dynamic).
- If it is a differential equation, it is sufficient to verify that

\[ [\dot{W}(x) \geq 0 \text{ for all } x \in A'] \quad \text{and} \quad [\dot{W}(x) = 0 \iff x \in A]. \]

No need to specify the decaying rate function \( \tilde{W} \).
Define net gains

Say, an agent has been taking action $a$ so far and receives a revision opportunity now, available action set $A'_a$ and switching cost $q$. Let the current payoff vector be $\pi$.

Then, the **net gain** from this revision opportunity is

$$
\begin{align*}
\pi^*_a[A'_a] - \pi_a - q & \quad \text{if switching to a different action, i.e., } \pi^*_a[A'_a] - \pi_a \geq q, \\
0 & \quad \text{if remaining at the current action, i.e., } \pi^*_a[A'_a] - \pi_a \leq q.
\end{align*}
$$

$$
= [\pi^*_a[A'_a] - \pi_a - q]^+,
$$

where $\pi^*_a[A'_a] := \max \{ \pi_b | b \in A'_a \}$. Note that $q$ follows $\mathbb{P}_Q$ and $A'_a$ is drawn from $\mathbb{P}_{Aa}$.

The total (first-order) net gain is

$$
G(x) [\pi] := \sum_{a \in A} x_a g^*_a [\pi],
$$

where $g^*_a [\pi]$ is the expected first-order net gain for an action-$a$ player (or namely, the average of net gains among such players):

$$
g^*_a [\pi] := \sum_{A'_a \subset A \setminus \{a\}} \mathbb{P}_{Aa}[A'_a] \mathbb{E}_Q[\pi^*_a[A'_a] - \pi_a - q]^+.
$$

Assumptions Q1–2 and A1–2 imply that $g^*_a [\pi] \geq 0$ and

$$
\pi_b \geq \pi_a \iff g^*_b [\pi] \leq g^*_a [\pi].
$$

$$
\therefore \text{ argmax}\{\pi_b | b \in A'_a\} = \text{ argmin}\{g^*_b [\pi] | b \in A'_a\}.
$$

We define the second-order gain $H$ by summing the change $\min_{b \in A'_a} g^*_b [\pi] - g^*_a [\pi]$ in first-order gain by revision and use it as a decaying function.
**Theorem**  Properties of aggregate gain functions

Consider a rationalizable dynamic. Then, $G$ and $H$ satisfy the followings.

(G) i) $G \geq 0$ and, ii) $G(x, \pi) = 0 \iff x \in \Delta^A(b_*[\pi]).$

(H) i) $H \leq 0$ and, ii) $H(x, \pi) = 0 \iff x \in \Delta^A(b_*[\pi]).$

(GH) $\forall x, \pi \quad \frac{\partial G}{\partial x} (x, \pi) \dot{x} = H(x, \pi), \quad \frac{\partial G}{\partial \pi} (x, \pi) \dot{\pi} = \dot{x} \cdot \dot{\pi}$ for any $\dot{x} \in V(x)[\pi], \dot{\pi} \in \mathbb{R}^A.$

These properties of $G, H$ imply $\delta$-passivity (Fox & Shamma):

$$\dot{G} = \frac{\partial G}{\partial x} (x, \pi) \dot{x} + \frac{\partial G}{\partial \pi} (x, \pi) \dot{\pi} = H(x, \pi) + \dot{x} \cdot \dot{\pi} \leq 0.$$

With $\dot{\pi} = (dF/dx) \dot{x},$

$$\dot{G} = \frac{\partial G}{\partial x} (x, \pi) \dot{x} + \frac{\partial G}{\partial \pi} (x, \pi) \dot{\pi} = H(x, \pi) + \dot{x} \cdot dF/dx \dot{x} \leq 0.$$

**Theorem**  Nash stability in contractive games

Consider a rationalizable dynamic in a contractive game. Then, the set of Nash equilibria is globally asymptotically stable.
**Definition** Regular ESS

Taylor & Jonker 1978

\( x^* \) is a **regular evolutionary stable state** if it satisfies both the following two conditions:

i) \( x^* \) is a quasi-strict equilibrium:

\[
F_s(x^*) = F^*_s(x^*) > F_u(x^*)
\]

for any \( s \in S \) and \( u \in U \).

Here \( S \) is the set of actions used in \( x^* \), i.e., \( S := \{ s \in A \mid x^*_s > 0 \} \) and \( U := A \setminus S \).

ii) \( DF(x^*) \) is **negative definite** with respect to \( \mathbb{R}^A_{S_0} := \{ z \in \mathbb{R}^A \mid z_s \neq 0 \text{ only if } s \in S \} \):

\[
z \cdot DF(x^*)z < 0 \quad \text{for all } z \in \mathbb{R}^A_{S_0} \setminus \{0\}.
\]

The payoff field vector would behave like a **contractive game** around \( x^* \),

if the state space was restricted to \( \Delta^A_S := \{ x \in \Delta^A \mid x_s > 0 \text{ only if } s \in S \} \).

Borrowing the idea in Sandholm (2010, TE), we can construct a Lyapunov function for a regular ESS from the one for contractive games:

\[
G^*(x) := G(x) + C \sum_{u \in U} x_u.
\]

**Theorem** Local stability of a regular ESS

Regular ESS \( x^* \) is asymptotically stable under a rationalizable dynamic.
We can utilize general findings in preceding literature:

- Dynamics with only finitely many agents.
  - Our continuous population dynamic is an “approximation” of finite population dynamics. (Roth & Sandholm, 2013, SIAM J. Control Opt.)
  - The Lyapunov function guarantees “fast convergence” in finite population dynamics. (Ellison, Fudenberg, Imhof, 2016, JET)

- Modification to agents’ recognition of payoffs.
  - Contractiveness is preserved and thus Nash stability is retained in the following situations. (Fox & Shamma, 2013, Games)
  - Smoothed payoff modification: agents react to weighted moving average of past payoffs for each action.
  - Anticipatory payoff modification: agents have a predictor $\pi^e$ of near-future payoffs such as $\dot{\pi}^e = \lambda (\pi - \pi^e)$ and react to a weighted average of the current payoff vector and the predictor.

Further, extension of stability results to multi-population dynamics is easily done.

- The whole society’s $G$ and $H$ functions can be defined just by summing those functions of each population. Properties $G$, $H$ and $GH$ for the society’s $G$ and $H$ are immediately derived from those for each population’s $G$ and $H$.

- Contractiveness is preserved, if constant but heterogeneous payoffs are added to contractive payoff function: $F^p_a(x) := F_a(x) + \theta^p_a$.

- It implies that “evolutionary implementation” of social optimum by Pigouvian pricing (Sandholm, 2003 RES) is robust to perturbation in payoffs.
Application of the idea

- I believe that the idea in this paper (constructing a Lyapunov function from gain) is generally applicable to a various (possibly more complex) dynamics even if the assumptions do not hold or the dynamic may not fit exactly into the framework just as presented here.

- e.g. In the joint paper "BRD in multitasking environments" with Ryoji Sawa, we consider the “multitasking BRD” in which an agent engages in multiple games concurrently but can change action in only one of them at a single revision opportunity. The mBRD does not exactly fit into our framework. But, applying the same idea, we define the gain function and use it to prove Nash stability.
Apdx: Gains

Apdx: Extensions
Say, an agent has been taking action $a$ so far and receives a revision opportunity now, with available action set $A'_a$ and switching cost $q$. Let the current payoff vector be $\pi$.

If this agent chooses to switch action at this revision opportunity, the net gain from the next revision becomes

$$g^{**}[\pi; A'_a] := \min_{b \in A'_a} g_b[\pi].$$

By the **second-order net gain**, we refer to the change in the first-order net gain when an agent switches action, i.e.,

$$g^{**}[\pi; A'_a] - g_a[\pi].$$

Note that the available action set is $A'_a$ with prob $P_{Aa}[A'_a]$ and then the switch occurs with prob $Q(\pi_\ast[A'_a] - \pi_a)$.

The expected second-order net gain for an action-$a$ player is defined as

$$h_{a\ast}[\pi] := \sum_{A'_a \subset A \setminus \{a\}} P_{Aa}[A'_a] Q(\pi_\ast[A'_a] - \pi_a) (g^{**}[\pi; A'_a] - g_a[\pi]).$$

The **total expected second order gain** is

$$H(x)[\pi] := x \cdot h_\ast[\pi].$$
Apdx: Extensions

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Our continuous population model can be considered as a limit case of finite population model, especially in which
a) the limit continuous-population dynamic is represented by a differential inclusion with a non-empty, convex-valued and upper semicontinuous correspondence,
b) and the Lyapunov function is obtained for equilibrium stability.
From these properties, we can utilize findings of preceding literature on finite-population dynamics:
   a) Roth & Sandholm (2013, SIAM J of Control & Optim.): The continuous-population dynamic indeed approximates the middle-run behavior and the limit states of the finite-population dynamic.
   b) Ellison, Fudenberg & Imhof (2016, JET): The expected convergence time is bounded.
Robustness of contractiveness

(Fox & Shamma) The following modifications to payoffs $\pi$ keep the payoff dynamic anti-passive and thus retains stability of Nash equilibria:

- **Smoothed payoff modification:** a revising agent’s decision is based on an exponentially weighted moving average of past payoffs. That is, given the payoff stream $\{\pi_t\}_{t \geq 0}$ and discount rate $\lambda > 0$, the perceived payoff vector at time $t$ is
  \[ e^{-\lambda t} \pi_0 + \int_0^t e^{-\lambda (t-\tau)} \pi_\tau d\tau. \]

- **Anticipatory payoff modification:** there is a predictor function $\pi^e$ of near-future payoffs such as $\dot{\pi}^e = \lambda (\pi - \pi^e)$. With weight $k$ on the predictor, a revising agent’s decision is based on the perceived payoff such as $\pi + k\dot{\pi}^e$. So, if the actual payoff for an action is greater than the predicted payoff, an agent becomes more favor of that action.
Finitely many populations $\mathcal{P}$
Each population $p$ consists of a continuous mass $m^p$ of agents. $\sum_{p \in \mathcal{P}} m^p = 1$.
Each population $p$ may have
- a different action set $\mathcal{A}^p$;
- a different payoff function $\mathbf{F}^p : \mathcal{X}^\mathcal{P} \rightarrow \mathbb{R}^{\mathcal{A}^p}$; and/or
- a different revision protocol.

$(\mathcal{X}^\mathcal{P} = \times_{p \in \mathcal{P}} m^p \Delta^{\mathcal{A}^p} :$ the set of the profiles of action distributions.$)$

**Theorem**  Aggregability of $\delta$-passivity

Suppose that, for each population $p \in \mathcal{P}$, the aggregate first and second order gain functions $G^p, H^p : \Delta^{\mathcal{A}^p} \times \mathbb{R}^{\mathcal{A}^p} \rightarrow \mathbb{R}$ satisfy the properties G, H, and GH-0,1.
Then, the total of them $G^\mathcal{P}, H^\mathcal{P} : \mathcal{X}^\mathcal{P} \times \mathbb{R}^{\mathcal{A}^\mathcal{P}} \rightarrow \mathbb{R}$ given by

$$G^\mathcal{P}(x^\mathcal{P}, \pi^\mathcal{P}) := \sum_{p \in \mathcal{P}} G^p(x^p, \pi^p), \quad \text{and} \quad H^\mathcal{P}(x^\mathcal{P}, \pi^\mathcal{P}) := \sum_{p \in \mathcal{P}} H^p(x^p, \pi^p)$$
also satisfy the properties G, H, and GH.
Unobservable heterogeneity with additive separability in payoffs

- Aggregate game: Payoff $F^p$ depends only on aggregate state $\bar{x} := \sum_{q \in P} x^q \in \Delta^A$.
- Additive separability: each population’s payoff function $F^p : \mathcal{X}^P \to \mathbb{R}^A$ consists of common payoff function $F^0 : \Delta^A \to \mathbb{R}^A$ and constant payoff perturbation $\theta^p \in \mathbb{R}^A$:
  $$F^p(x^P) = F^0(\sum_{q \in P} x^q) + \theta^p.$$

**Theorem** Robustness of contractiveness to unobservable payoff heterogeneity

Suppose that the common payoff function $F^0 : \Delta^A \to \mathbb{R}^A$ is contractive.

$\Rightarrow$ the extended payoff function $F^P = (F^p)_{p \in P} : \mathcal{X}^P \to \mathbb{R}^{A^P}$ is also contractive.

**Corollary** Robustness of Nash stability to heterogeneity

Nash stability of a contractive game is robust to heterogeneity in payoffs and revision protocols, as long as
- Payoff heterogeneity is additively separable, and
- Heterogeneity in revision protocols keeps the properties G and H.
Evolutionary implementation

Social optimum in an aggregate game

- The aggregate game with additively separable payoff heterogeneity:
  \[ F^p(x^p) = F^0(\sum_{q \in P} x^q) + \theta^p. \]
  Here, the base game \( F^0 \) can be anything.

- The social optimum: maximum of the total payoff
  \[ \bar{F}^p(x^p) = \sum_{q \in P} x^q \cdot F^q(x^p) = \bar{x} \cdot F^0(\bar{x}) + \sum_{q \in P} x^q \cdot \theta^q. \]

Pigouvian pricing

- Adding a monetary payment \( T : \Delta^A \rightarrow \mathbb{R}^A \) to the payoff: \( F^p(\bar{x}) + T(\bar{x}) \) for each \( p \).
- Dynamic Pigouvian pricing:
  \[ T_a(\bar{x}) := \bar{x} \cdot \frac{\partial F^0}{\partial \bar{x}_a}(\bar{x}) = \sum_{b \in A} \bar{x}_b \frac{\partial F^0}{\partial \bar{x}_a}(\bar{x}). \]

- \( F^p + T \) is a potential game with potential \( \bar{F}^p \).

Evolutionary implementation (Sandholm, '03,'05 RES)

(A-F1) Assume that the common payoff function \( F^0 \) is \( C^1 \) and its Hessian is negative definite on the tangent space of \( \Delta^A \) (negative externality).

\[ \Rightarrow \] The total payoff function \( \bar{F}^p \) is strictly concave.

\[ \Rightarrow \] The social optimum is globally stable under any dynamic that satisfies Nash stationarity and positive correlation.
Perturbation to the game

- \( F_\delta^p(x^P) = F^p(x^P) + \delta \tilde{F}^p(x^P) \) with \( \delta \in \mathbb{R} \).

- Imagine that the central planner does not know the perturbation \( \delta \tilde{F}^p \) and thus designs the Pigouvian pricing scheme based on \( F^p \).

- Then the game is no longer a potential game.

Robustness of evolutionary implementation

(A-F2) Assume that \( \tilde{F}^p \) is \( C^1 \) and there exists an upper bound on \( |\partial \tilde{F}^p / \partial x^P| \).

\[ \implies \text{With strict concavity of } F^0 \text{ (A-F1), this implies that} \]

the perturbed game \( F_\delta^p \) is still a contractive game as long as \( \delta \) is sufficiently small.

\[ \implies \text{Further, the total payoff maximization is approximately achieved at the limit state} \]

under any evolutionary dynamics that guarantee Nash stability in contractive games.

**Theorem** Robustness of evolutionary implementation

Consider the perturbed game as above that satisfies (A-F1) and (A-F2).

For any \( \varepsilon > 0 \) there is \( \tilde{\delta}(\varepsilon) > 0 \) such that

\[ |\delta| < \tilde{\delta}(\varepsilon) \implies \lim_{t \rightarrow \infty} \bar{F}_\delta^p(x^P_{\delta,t}) - \max_{x^P \in \mathcal{X}^P} \bar{F}_\delta^p(x^P) | < \varepsilon, \]

where \( \{x^P_{\delta,t}\}_{t \geq 0} \) is the trajectory starting from an arbitrary initial state \( x^P_{\delta,0} \in \mathcal{X}^P \) in the population game \( F_\delta^p \) under an evolutionary dynamic that satisfies all the assumptions.