Gains in evolutionary dynamics
unifying rational framework for dynamic stability

Dai ZUSAI

TEMPLE UNIVERSITY

Hitotsubashi University (Econ Dept.)
Economic Theory Workshop October 27, 2016

1 Introduction
- Motivation and literature Review
- Outline of this paper

2 Set up
- Population games
- Evolutionary dynamics

3 Properties
- Basic properties
- Gains

4 Equilibrium stability
- Stability theorems
- Extensions
Abstract

We investigate gains from strategy revisions in deterministic evolutionary dynamics.

- To clarify the gain from revision, we propose a framework to reconstruct an evolutionary dynamic from optimal decision with stochastic (possibly restricted) available action set and switching cost. Many of major non-imitative dynamics can be constructed in this framework.
- We formally define **net gains from revisions** and obtain several general properties of the gain function, which leads to **Nash stability of contractive games**—generalization of concave potential games—and **local asymptotic stability of a regular evolutionary stable state**.
- The unifying framework allows us to apply the Nash stability to mixture of heterogeneous populations, whether heterogeneity is observable or unobservable or whether heterogeneity is in payoffs or in revision protocols.
- This extends the known positive results on **evolutionary implementation of social optimum through Pigouvian pricing** to the presence of heterogeneity and non-aggregate payoff perturbations.
- While the analysis here is confined to general strategic-form games, we finally discuss that the idea of reconstructing evolutionary dynamics from optimization with switching costs and focusing on net revision gains for stability is promising for **further applications** to more complex situations.

**Keywords**: evolutionary dynamics, equilibrium stability, Lyapunov function, contractive/negative definite/ stable games, passivity, evolutionary implementation
1 Introduction
   • Motivation and literature Review
   • Outline of this paper

2 Set up

3 Properties

4 Equilibrium stability

Gains in evolutionary dynamics | Zusai | 3
Evolutionary dynamics

In (deterministic) evolutionary game theory, we study various evolutionary dynamics

- These differential equations define different dynamics of the social state (distribution of actions) in large population (continuum of agents) in continuous time.

- We investigate off-equilibrium processes in which agents occasionally switch their actions in response to the change in the social state and the payoffs.

- The variety reflects a variety in economic agents’ decision rules: optimization, imitation, etc.

- In the study of deterministic dynamics, we examine if a well-known equilibrium concept (e.g. Nash, ESS) can be justified by dynamic stability under these dynamics.

\[
\begin{align*}
\dot{x}_i &= x_i \hat{F}_i(x) \\
\dot{x} &\in M(F(x)) - x \\
\dot{x}_i &= \frac{\exp(\eta^{-1} F_i(x))}{\sum_{k \in S} \exp(\eta^{-1} F_k(x))} - x_i \\
\dot{x}_i &= [\hat{F}_i(x)]_+ - x_i \sum_{j \in S} [\hat{F}_j(x)]_+ \\
\dot{x}_i &= \sum_{j \in S} x_j [F_i(x) - F_j(x)]_+ - x_i \sum_{j \in S} [F_j(x) - F_i(x)]_+
\end{align*}
\]

Source: Sandholm 2015, Table 13.1 (Handbook of Game Theory, Elsevier)
Different evolutionary dynamics may behave differently; in particular, they might yield different results on equilibrium stability.

**Theorem  Lyapunov stability theorem**

Let $A$ be a closed subset of a compact space $\mathcal{X}$ and $A'$ be a neighborhood of $A$. Suppose that two continuous functions $W : \mathcal{X} \to \mathbb{R}$ and $\tilde{W} : \mathcal{X} \to \mathbb{R}$ satisfy (i) $W(x) \geq 0$ and $\tilde{W}(x) \leq 0$ for all $x \in \mathcal{X}$ and (ii) $W^{-1}(0) = \tilde{W}^{-1}(0) = A$. In addition, assume that $W$ is Lipschitz continuous in $x \in \mathcal{X}$ with Lipschitz constant $K \in (0, \infty)$. If any Carathéodory solution $\{x_t\}$ starting from $A'$ satisfies

$$\dot{W}(x_t) \leq \tilde{W}(x_t)$$

for almost all $t \in [0, \infty)$, then $A$ is asymptotically stable and $A'$ is its basin of attraction.

- I call $W$ the **Lyapunov function** and $\tilde{W}$ the **decaying rate function**.
- This is a general version for a differential inclusion (set-value dynamic to allow multiple transition vectors: e.g. best response dynamic).
- If it is a differential equation, it is sufficient to verify that

$$[\dot{W}(x) \geq 0 \text{ for all } x \in A'] \quad \text{and} \quad [\dot{W}(x) = 0 \iff x \in A].$$

No need to specify the decaying rate function $\tilde{W}$.

In general, a Lyapunov function may vary with games and it also depends on specification of the dynamic.
In potential games, stability of Nash eqm is commonly guaranteed for major dynamics thanks to positive correlation.

\[ \frac{\partial f}{\partial x_a}(x) = F_a(x) - \bar{F}(x) \text{ for all } a \in \mathcal{A}, \quad \text{i.e., } \left[ \frac{df}{dx}(x) \right]^T = F(x) - \bar{F}(x)1. \]

\( f : \mathbb{R}^A \rightarrow \mathbb{R} \) s.t.

- Equivalent to externality symmetry: \( \frac{\partial F_a}{\partial x_b}(x) = \frac{\partial F_b}{\partial x_a}(x) \) for all \( a, b \in \mathcal{A}, x \in \Delta^\mathcal{A} \).
- e.g. Congestion games with payoff=−traveling time; Binary action games

Major evolutionary dynamics (BRD, pairwise payoff comparison, etc.) satisfy positive correlation:

\[ z \cdot F(x) \geq 0 \quad \forall z \in V(x), \quad = 0 \Leftrightarrow x \in \text{NE}(F). \]

Then, local max of \( f \) in \( \Delta^\mathcal{A} \Rightarrow \) a Nash eqm, and

\[ \dot{f} = (df/dx)\dot{x} = F(x)\dot{x} \geq 0 \quad \text{by positive correlation.} \]

Potential game: immune to small changes in payoffs that break externality symmetry.
Nash stability of contractive game

Contractive/stable game: a generalization of a concave potential game

no scalar-valued summary measure of the state (action distribution) of the game.

---

**Definition  Contractive game**

A population game $F$ is a **contractive game** if

$$
(y - x) \cdot (F(y) - F(x)) \leq 0
$$

for all $x, y \in \Delta^A$.

or (if $F$ is $C^1$) equivalent to

$$
z \cdot \frac{dF}{dx}(x)z \leq 0
$$

for any $z \in \mathbb{R}^A$.

- e.g. Congestion games with payoff heterogeneity; Two-player zero sum games; War of attrition
- (regular) ESS: local version (see later).

HS verified Nash stability under major dynamics by finding a Lyapunov function for each.

---

<table>
<thead>
<tr>
<th>Dynamic</th>
<th>Lyapunov function ($p = 1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BNN</td>
<td>$\Gamma(x) = \frac{1}{2} \sum_{i \in S} [\hat{F}<em>i(x)]</em>+^2$</td>
</tr>
<tr>
<td>Best response</td>
<td>$G(x) = \max_{i \in S} \hat{F}_i(x)$</td>
</tr>
<tr>
<td>Logit</td>
<td>$\tilde{G}(x) = \max_{y \in \text{int}(X)} (y' \hat{F}(x) - \eta \sum_{i \in S} y_i \log y_i) + \eta \sum_{i \in S} x_i \log x_i$</td>
</tr>
<tr>
<td>Smith</td>
<td>$\Psi(x) = \frac{1}{2} \sum_{i \in S} \sum_{j \in S} x_i [F_j(x) - F_i(x)]_+^2$</td>
</tr>
<tr>
<td>Replicator</td>
<td>$H_{x^<em>}(x) = \sum_{i \in S(x^</em>)} x_i^* \log \frac{x_i^*}{x_i}$</td>
</tr>
<tr>
<td>Projection</td>
<td>$E_{x^*}(x) =</td>
</tr>
</tbody>
</table>

Source: Hofbauer & Sandholm (JET, 2009: Table 2).
δ-passivity

Is there any general condition to guarantee Nash stability of contractive games?

- General economic intuition behind this commonality of Nash stability over different dynamics.
- Robustness to perturbation/mis-identification of dynamics (individuals’ decision rules).

**Definition δ-passivity**

Micheal J. Fox & Jeff Shamma (2013, Games)

Evolutionary dynamic $\dot{x} = V(x, \pi)$ is **δ-passive**, if there exists a storage function $L$ s.t.

$$
\frac{\partial L}{\partial x} \Delta x + \frac{\partial L}{\partial \pi} \Delta \pi \leq \Delta x \cdot \Delta \pi \quad \text{for any } \Delta \pi \text{ (with } \Delta x = V(x, \pi)).
$$

If we allow differential inclusion, a decay function $\tilde{W}$ should be added to the RHS.

If the game $F$ is contractive and the dynamic $V$ is δ-passive, then we have

$$
\dot{L} = \frac{\partial L}{\partial x} \dot{x} + \frac{\partial L}{\partial \pi} \dot{\pi} \leq \dot{x} \cdot \dot{\pi} = \dot{x} \cdot \frac{dF}{dx}(x) \dot{x} \leq 0 \quad \left( \text{with } \dot{\pi} = \frac{dF}{dx}(x) \dot{x} \right).
$$

Taking $L$ as a Lyapunov function, we can confirm global stability of $L^{-1}(0)$.

δ-passivity points out exactly the property that should be satisfied for evolutionary dynamics, **separately from games**, in order to guarantee Nash stability of contractive games.

But, it looks purely mathematical. (Passivity is originally a concept in electric engineering.)

- What behavioral/economical aspects of those well-behaved dynamics result in δ-passivity?
- Isn’t there any general principle to find a storage function?
Basic idea: Track the gain from revision, i.e.,
Gross gain = [payoff from new action] − [payoff from old action].

- At Nash eqm, “gain” = 0 for all agents.
- If not at Nash eqm, “gain” > 0 for some agents.

However, except BRD, agents do not choose an action to achieve the greatest payoff.
  e.g. Pairwise comparison dynamics (Smith dynamic):
  switch to a sampled action as long as it is better than the current action

So, agents may leave some of possible gains not wholly exploited;
  Wouldn’t it be possible that the gain may not steadily decrease over time
  before reaching eqm or may not vanish to zero in some games?
  Indeed so, if it is not a potential game.

We need to rationalize such irrationality and refine the notion of “gains from revision.”
To rationalize the irrationality, I reconstruct an evolutionary dynamic
by optimization protocol
with randomly limited available action set and stochastic switching costs.

That is, a revising agent is supposed
to have a random draw of the available set and the switching cost
and then to choose
\[
\begin{cases}
\text{the best among the available actions} & \text{if the gross gain} > \text{switching cost} \\
\text{keeping the current action} & \text{if the gross gain} < \text{switching cost}.
\end{cases}
\]

Then, the net gain is defined as
\[
\text{Net gain} = [\text{payoff from new action}] - [\text{payoff from old action}] - [\text{switching cost}].
\]

The aggregate gain function works generally as a scalar-valued function to measure how far the current state is from an eqm. We use it to prove $\delta$-passivity of such a dynamic and Nash stability in contractive games; this extends to local stability of a (regular) ESS.

Generality of our idea allows to make it robust to any mixture of such dynamics (whether heterogeneity is observable or not).
Introduction

Set up
- Population games
- Evolutionary dynamics

Properties

Equilibrium stability
For now, start from a single-population game.

- A continuum of agents; total mass=1
- Each agent chooses an action from $\mathcal{A} = \{1, \ldots, A\}$.
- $x \in \Delta^A$: the action distribution (the social state).
- $\Delta^A := \{x \in \mathbb{R}^A \mid \sum_{a \in A} x_a = 1\}$.
- $F: \Delta^A \to \mathbb{R}^A$: payoff function. $F_a(x) \in \mathbb{R}$: payoff of action $a$ in state $x$.

**Assumption** Assumption on the game

(A-F1) $F$ is continuously differential.
### Standard formulation of an evolutionary dynamic

<table>
<thead>
<tr>
<th>Sandholm 2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Each agent occasionally receives an opportunity to revise the action. Revision opportunities follow a Poisson process; the arrival rate is fixed at 1.</td>
</tr>
<tr>
<td>- An agent uses a certain <strong>revision protocol</strong> to choose a new action, depending on the current payoff vector $\pi \in \mathbb{R}^A$ and the social state $x \in \Delta A$.</td>
</tr>
<tr>
<td>- Evolutionary dynamic $\dot{x} \in V(x)$ is obtained as the aggregation of such individual revision processes in large population.</td>
</tr>
</tbody>
</table>

- e.g. **BRD**: optimization revision protocol: pick an action that yields the greatest payoff.
- e.g. **Smith**: randomly pick another action $b$ with equal probability and switch to it (from $a$) with prob proportional to the payoff difference, as long as $b$ is better than $a$. 
To rationalize deviation from exact optimization, I propose to reconstruct evolutionary dynamics from optimization protocol with switching cost and restricted action set.

General reconstruction by optimization-based revision protocol

- Upon receipt of a revision opportunity, a revising agent compares payoffs from the current action $a$ and from other available actions.
- The available action set $A'_a$ is a subset of $A \setminus \{a\}$.
- If the revising agent chooses to switch an action, he needs to pay switching cost $q$. The switching cost is common to all the available actions.
- Given payoff vector $\pi \in \mathbb{R}^A$, the revising agent chooses an action to optimize his net payoff within the available set:

$$\max \{ \pi_a, \max_{b \in A'_a} \pi_b - q \}$$

N.B. In other words, the current action $a$ is status quo in the decision of new action and $q$ is the bias to prefer the status quo.

Allowing the randomness

- $A'_a$ is drawn from from probability distribution $\mathbb{P}_{Aa}$ over the power set of $A \setminus \{a\}$.
- $q$ is drawn from probability distribution $\mathbb{P}_Q$ over $\mathbb{R}$.
- Assume that these random draws are independent of each other, the agent’s current action and the social state and also their probability distributions are invariant over time.
\( \mathcal{V}_a \): change in the action distribution (mixed str) of agents who have been taking \( a \) so far:

\[
\mathcal{V}_a[\pi] := \sum_{A'_a \subset A \backslash \{a\}} P_{Aa}[A'_a] Q(\pi_*[A'_a] - \pi_a) (\Delta^A(b_*[\pi; A'_a]) - e_a),
\]

where \( b_*[\pi; A'_a] = \arg\max_{a \in A'_a} \pi_a \) and \( \Delta^A(A') = \{ x \in \Delta^A \mid x_a > 0 \Rightarrow a \in A' \} \).

\( \mathcal{V} \): the dynamic of the social state, induced from the constrained optimization protocol:

\[
\dot{x} \in \mathcal{V}[\pi] := \sum_{a \in A} x_a \mathcal{V}_a[\pi].
\]

When the payoff function is specified as \( F \), we denote \( \mathcal{V}[F(x)] \) by \( \mathcal{V}^F(x) \).

- **Standard BRD**: always switch to the current optimal action
  \[ \Leftrightarrow P_{Aa}(A \backslash \{a\}) = 1 \text{ for any } a \in A \text{ and } Q(q) = 1 \text{ for all } q > 0 \text{ while } Q(0) = 0. \]

- **Tempered BRD**: switch to the current optimal action with prob \( Q(\pi_* - \pi_a) \)
  \[ \Leftrightarrow P_{Aa}(A \backslash \{a\}) = 1 \text{ for any } a \in A \text{ and this } Q \text{ as c.d.f. of } P_Q. \]

- **Symmetric pairwise payoff comparison**: randomly pick another action \( b \) and switch to it with prob \( Q([\pi_b - \pi_a]_+) \) (Smith: \( Q(\check{\pi}) = \check{\pi} \)).
  \[ \Leftrightarrow P_{Aa}(\{b\}) = 1/(A - 1) \text{ for all } b \in A \backslash \{a\} \text{ and this } Q. \]

- **Smoothed BRD** (e.g. logit dyn) can be seen as aggregate BRD under payoff heterogeneity.

- **Excess payoff target dynamics** (e.g. Brown-von Neumann-Nash dyn) do not fall into this framework as it is. But, the protocol can be interpreted as a constrained optimization protocol by taking the mixed strategy \( x \) (random assignment based on the current action distribution) as the status quo and charging the cost on switching from this status quo. Then, the gain function satisfies the properties that we will see.
**Assumption**  Assumptions on the revision protocol

(A-Q1) **Nonnegativity of q:**
\[ Q(0) = \mathbb{P}_Q((\neg\infty, 0)) = 0, \quad \text{i.e., } q \geq 0 \text{ almost surely.} \]

(A-Q2) **Infinitesimally small positive q:** \( \forall q > 0, \)
\[ Q(q) = \mathbb{P}_Q((\neg\infty, q)) > 0. \]

(A-A1) **Ex ante availability of all actions:** \( \forall a \in \mathcal{A}, b \in \mathcal{A} \setminus \{a\}, \)
\[ \mathbb{P}_{Aa}(\{A'_a | b \in A'_a\}) > 0 \]
\( b \text{ is switchable from } a \)

(A-A2) **Symmetry of available action set distribution:** \( \forall a, b \in \mathcal{A}, A^0 \subset \mathcal{A} \setminus \{a, b\}, \)
\[ \mathbb{P}_{Aa}(\{A'_a | A'_a \cap A^0 \neq \emptyset\}) = \mathbb{P}_{Ab}(\{A'_b | A'_b \cap A^0 \neq \emptyset\}). \]
\( \exists \text{ an action in } A^0 \text{ switchable from } a \quad \exists \text{ an action in } A^0 \text{ switchable from } b \)

(A-A3) **Invariance of } A'_a \text{ to the social state:**
\[ \mathbb{P}_{Aa} \text{ is independent of } x. \]

The first mentioned classes of unobservatory dynamics (BRD, pairwise payoff comparison) satisfy these assumptions.
The second (perturbed BRD, excess payoff) need some modifications. Observatory dynamics (imitative dynamics like replicator) do not meet A3.
Introduction

Set up

Properties

Basic properties

Gains

Equilibrium stability
BR stationarity and positive correlation

Let $b^*_\pi = \arg\max_{a \in \mathcal{A}} \pi_a$, the set of optimal actions given payoff vector $\pi$.

**Theorem** Best response/Nash stationarity

Assume Assumptions Q1, Q2 and A1. Then,

i) $x \in \Delta^A(b^*_\pi) \implies \mathcal{V}(x)[\pi] = \{0\}$;  
   ii) $x \notin \Delta^A(b^*_\pi) \implies 0 \notin \mathcal{V}(x)[\pi]$.

If $F$ is $C^1$, these imply Nash stationarity:

$\mathcal{V}^F(x) = \{0\} \iff x \in \text{NE}(F)$.

**Theorem** Positive correlation and Nash stability of potential games

Assume Assumptions Q1, Q2 and A1. Then,

i) $\Delta x \cdot \pi \geq 0$ for any $\Delta x \in \mathcal{V}(x)[\pi]$;
   
   ii) $[\Delta x \cdot \pi = 0$ with some $\Delta x \in \mathcal{V}(x)[\pi] ] \iff x \in \Delta^A(b^*_\pi)$.

If $F$ is a potential game, these imply that i) $\text{NE}(F)$ is globally attracting, 
and ii) each local maximizer of $f$ is Lyapunov stable.

- Nash stability of potential games is immediate from positive correlation:
  
  $\dot{f} = \frac{df}{dx} \cdot x = F(x) \cdot x \geq 0$.

- Note that $\Delta x \cdot \pi$ is the aggregate gross gain:
  
  Let $\Delta x \in \sum_{a \in \mathcal{A}} x_a(y_a - e_a)$ where $y_a$ is the distribution of new actions for those who have been taking action $a$ before the revision. Then, we have
  
  $\Delta x \cdot \pi = \sum_{a \in \mathcal{A}} x_a(y_a \cdot \pi - \pi_a)$.
Individual expected net gains

The **expected (first-order) net gain** for an action-$a$ player

$$g_a^*[\pi] := \sum_{A'_a \subset A \setminus \{a\}} P_{Aa}[A'_a] \mathbb{E}_Q[\pi^* [A'_a] - \pi_a - q] + \text{net gain of switch from } a \text{ to the best in } A'_a$$

**Theorem**  Properties of first-order net gain function

Assume Assumptions Q1, Q2 and A1.

**(g0)** i) $g^* [\cdot] \geq 0$. ii) $g_a^* [\pi] = 0$ if and only if $a \in b^*(\pi)$.

**(g1)** Further assume Assumption A2. Consider arbitrary two actions $a, b$;

i) $\pi_a \leq \pi_b \Rightarrow g_a^* [\pi] \geq g_b^* [\pi]$; \hspace{1cm} ii) $\pi_a < \pi_b \Rightarrow g_a^* [\pi] > g_b^* [\pi]$.

**(g2)** For each $a \in A$, function $g_a^* : \mathbb{R}^A \rightarrow \mathbb{R}$ is differentiable almost everywhere in $\mathbb{R}^A$.

If it is differentiable at $\pi \in \mathbb{R}^A$,

$$\frac{dg_a^*}{d\pi} [\pi] \Delta \pi = \sum_{A'_a \subset A \setminus \{a\}} P_{Aa}[A'_a] Q(\pi^* [A'_a] - \pi_a) (\Delta \pi^* [A'_a] - \Delta \pi_a) \quad \text{for any } \Delta \pi \in \mathbb{R}^A$$

**(g0)** $g^* [\cdot]$ surely represents the gain from revision:

[Net gain] $\geq 0$ always; \hspace{1cm} [Net gain]=0 $\iff$ the current action is optimal.

**(g1)** Ranking of actions by $g^* [\cdot]$ reversed payoff ranking by $\pi$.

**(g2)** Differentiability of $g^* [\cdot]$ and derivative formula (better to see the next slide).
Individual expected net second-order gains

Note that (g1) implies that any of the best actions in available action set $A'_a$ yield the smallest gain among actions in this set:

$$b \in b_*[\pi] \implies g_{b*}[\pi] = \min_{c \in A'_a} g_{c*}[\pi] =: g^{**}[\pi; A'_a].$$

The expected second-order net gain for an action-$a$ player is defined as

$$h_{a*}[\pi] := \sum_{A'_a \subset A \setminus \{a\}} \text{Prob of switch} \cdot \text{the change in gain by the switch} \cdot P_{Aa}[A'_a] Q(\pi_*[A'_a] - \pi_a) (g^{**}[\pi; A'_a] - g_{a*}[\pi]).$$

Let $g_*[\pi] = (g_{a*}[\pi])_{a \in A}$ be the vector that collects the expected gains of all the actions, and similarly $h_*[\pi] = (h_{a*}[\pi])_{a \in A}.$

**Theorem** Properties of second-order net gain function

Assume that $g_*$ satisfies g1.

**(gh) For any $a \in A$ and $\pi \in \mathbb{R}^A,$**

$$h_{a*}[\pi] = z_a \cdot g_*[\pi] \quad \text{for any } z_a \in \mathcal{V}_a[\pi].$$

**(h) i) $h_*[\cdot] \leq 0.$ ii) $h_{a*}[\pi] = 0$ if and only if $a \in b_*(\pi).$**

**(gh) $h_{a*}$=expected change in gain $g_*$ by a revision for an agent who currently takes $a$**

**(h) Gain $g_*$ can only decrease after a revision;**

Gain $g_{a*}$ remains unchanged $\iff$ action $a$ is optimal given payoff vector $\pi.$
Aggregate gains and δ-passivity

Aggregate first and second order gains:

\[ G(x)[\pi] = x \cdot g_*(\pi) \quad \text{and} \quad H(x)[\pi] = x \cdot h_*(\pi). \]

**Theorem** Properties of aggregate gain functions

Assume that \( g_* \) and \( h_* \) satisfy properties \( g0, g2, h \) and \( gh \).

Then, their aggregates \( G \) and \( H \) satisfy the followings.

**(G)** Property \( g0 \) implies i) \( G \geq 0 \) and, ii) \( G(x, \pi) = 0 \iff x \in \Delta^A(b_*[\pi]) \).

**(H)** Property \( h \) implies i) \( H \leq 0 \) and, ii) \( H(x, \pi) = 0 \iff x \in \Delta^A(b_*[\pi]) \).

**(GH-0)** Property \( gh \) implies

\[ \forall x \in \Delta^A, \pi \in \mathbb{R}^A \quad H(x)[\pi] = \Delta x \cdot g_*(\pi) \quad \text{for any} \Delta x \in \mathcal{V}(x)[\pi]. \]

**(GH-1)** i) Property \( g2 \) implies

\[ \forall x \in \Delta^A, \pi \in \mathbb{R}^A \quad \frac{\partial G}{\partial \pi}(x, \pi) \Delta \pi = \Delta x \cdot \Delta \pi \quad \text{for any} \Delta x \in \mathcal{V}(x)[\pi], \Delta \pi \in \mathbb{R}^A. \]

ii) Further assume Assumption (A-A3). Then, properties \( gh \) and \( g2 \) imply

\[ \forall x \in \Delta^A, \pi \in \mathbb{R}^A \quad \frac{\partial G}{\partial x}(x, \pi) \Delta x = H(x, \pi) \quad \text{for any} \Delta x \in \mathcal{V}(x)[\pi]. \]

These properties of \( G, H \) imply δ-passivity:

\[ \dot{G} = \frac{\partial G}{\partial x}(x, \pi) \dot{x} + \frac{\partial G}{\partial \pi}(x, \pi) \dot{\pi} = H(x, \pi) + \dot{x} \cdot \dot{\pi} \leq \dot{x} \cdot \dot{\pi}. \]
Equilibrium stability

1 Introduction

2 Set up

3 Properties

4 Equilibrium stability
   - Stability theorems
   - Extensions
Nash stability of contractive games

Further, with $\hat{\pi} = (dF/dx)\dot{x}$,

$$\dot{G} = \frac{\partial G}{\partial x}(x, \pi) \dot{x} + \frac{\partial G}{\partial \pi}(x, \pi) \dot{\pi} = H(x, \pi) + \dot{x} \cdot \dot{\pi} \leq \dot{x} \cdot \dot{\pi} = \dot{x} \cdot \frac{dF}{dx} \dot{x} \leq 0.$$

Use $G$ for $W$ and $H$ for $\tilde{W}$ in the Lyapunov stability theorem.

**Theorem** Nash stability in contractive games

Consider a contractive game with continuously differentiable payoff function $F$. Assume an evolutionary dynamic that generate the total expected first/second-order gain functions $G, H$ satisfying properties $G, H, \text{and } GH-0,1$. Then, NE($F$) is asymptotically stable under the evolutionary dynamic.
Note that these are not equivalent relations. Even if a dynamic does not fit into this reformulation or satisfy the assumptions, it may retain $\delta$-passivity and Nash stability of contractive games; e.g. Excess payoff target dynamic.
Evolutionary stability: robustly optimal to small mutation

Any sufficiently small mutation that was equally optimal at the equilibrium worse off the mutants themselves compared to the incumbents.

**Definition**  Regular ESS

$x^*$ is a **regular (Taylor) evolutionary stable state** if it satisfies both of the following two conditions:

i) $x^*$ is a quasi-strict equilibrium:

$$F_b(x^*) = F_*(x^*) > F_a(x^*)$$

whenever $x^*_b > 0 = x^*_a$;

ii) $DF(x^*)$ is negative definite with respect to $\mathbb{R}^A \setminus \{0\}$:

$$z \cdot DF(x^*)z < 0 \quad \text{for all } z \in \mathbb{R}^A \setminus \{0\}.$$ 

Here $S$ is the set of actions used in $x^*$, i.e., $S := \{b \in A \mid x^*_b > 0\}$; and, let

$$\mathbb{R}^A_{S_0} := \left\{ z \in \mathbb{R}^A \mid [z_b = 0 \text{ for any } b \notin S] \text{ and } \sum_{a \in A} z_a = 0 \right\} = \left\{ z \in \mathbb{R}^A_0 \mid z_b = 0 \text{ if } x^*_b = 0 \right\}.$$

**Theorem**  Local stability of a regular ESS

Consider a game with a regular ESS $x^*$ and an evolutionary dynamic with Assumptions Q1, Q2, and A1. Then, $x^*$ is (locally) asymptotically stable.

We modify the Lyapunov function:

$$G^*(x) = G(x) + \sum_{a \notin S} x_a.$$
Robustness of contractiveness

(Fox & Shamma) The following modifications to payoffs $\pi$ keep the payoff dynamic anti-passive and thus retains stability of Nash equilibria:

- **Smoothed payoff modification**: a revising agent’s decision is based on an exponentially weighted moving average of past payoffs. That is, given the payoff stream $\{\pi_t\}_{t \geq 0}$ and discount rate $\lambda > 0$, the perceived payoff vector at time $t$ is

$$
e^{-\lambda t} \pi_0 + \int_0^t e^{-\lambda (t-\tau)} \pi_\tau d\tau.$$

- **Anticipatory payoff modification**: there is a predictor function $\pi^e$ of near-future payoffs such as $\dot{\pi}^e = \lambda (\pi - \pi^e)$. With weight $k$ on the predictor, a revising agent’s decision is based on the perceived payoff such as $\pi + k\dot{\pi}^e$. So, if the actual payoff for an action is greater than the predicted payoff, an agent becomes more favor of that action.
Extension to multi-population games

Extending to a multi-population game

- Finitely many populations $\mathcal{P}$
  - Each population $p$ consists of a continuous mass $m^p$ of agents. $\sum_{p \in \mathcal{P}} m^p = 1$.
- Each population $p$ may have
  - a different action set $A^p$;
  - a different payoff function $F^p : \mathcal{X}^\mathcal{P} \rightarrow \mathbb{R}^{A^p}$; and/or
  - a different revision protocol.

$\text{(}\mathcal{X}^\mathcal{P} = \times_{p \in \mathcal{P}} m^p \Delta A^p :$ the set of the profiles of action distributions.$)\text{.$}$

**Theorem  Aggregability of $\delta$-passivity**

Suppose that, for each population $p \in \mathcal{P}$, the aggregate first and second order gain functions $G^p, H^p : \Delta A^p \times \mathbb{R}^{A^p} \rightarrow \mathbb{R}$ satisfy the properties G, H, and GH-0,1.

Then, the total of them $G^\mathcal{P}, H^\mathcal{P} : \mathcal{X}^\mathcal{P} \times \mathbb{R}^{A^\mathcal{P}} \rightarrow \mathbb{R}$ given by

$$G^\mathcal{P}(x^\mathcal{P}, \pi^\mathcal{P}) := \sum_{p \in \mathcal{P}} G^p(x^p, \pi^p), \quad \text{and} \quad H^\mathcal{P}(x^\mathcal{P}, \pi^\mathcal{P}) := \sum_{p \in \mathcal{P}} H^p(x^p, \pi^p)$$

also satisfy the properties G, H, and GH-0,1.
Unobservable heterogeneity with additive separability in payoffs

- Aggregate game: Payoff $F^p$ depends only on aggregate state $\bar{x} := \sum_{q \in P} x^q \in \Delta^A$.
- Additive separability: each population’s payoff function $F^p : \chi^P \rightarrow \mathbb{R}^A$ consists of common payoff function $F^0 : \Delta^A \rightarrow \mathbb{R}^A$ and constant payoff perturbation $\theta^p \in \mathbb{R}^A$:
  $$F^p(x^P) = F^0(\sum_{q \in P} x^q) + \theta^p.$$ 

**Theorem** Robustness of contractiveness to unobservable payoff heterogeneity

Suppose that the common payoff function $F^0 : \Delta^A \rightarrow \mathbb{R}^A$ is contractive.

$\Rightarrow$ the extended payoff function $F^P = (F^p)_{p \in P} : \chi^P \rightarrow \mathbb{R}^{AP}$ is also contractive.

**Corollary** Robustness of Nash stability to heterogeneity

Nash stability of a contractive game is robust to heterogeneity in payoffs and revision protocols, as long as

- Payoff heterogeneity is additively separable, and
- Heterogeneity in revision protocols keeps the properties G and H.
Evolutionary implementation

Social optimum in an aggregate game

- The aggregate game with additively separable payoff heterogeneity:
  \[ F^p(x^p) = F^0(\sum_{q \in P} x^q) + \theta^p. \]

  Here, the base game \( F^0 \) can be anything.

- The social optimum: maximum of the total payoff
  \[ \bar{F}^p(x^p) = \sum_{q \in P} x^q \cdot F^q(x^p) = \bar{x} \cdot F^0(\bar{x}) + \sum_{q \in P} x^q \cdot \theta^q. \]

Pigouvian pricing

- Adding a monetary payment \( T : \Delta^A \rightarrow \mathbb{R}^A \) to the payoff: \( F^p(\bar{x}) + T(\bar{x}) \) for each \( p \).

- Dynamic Pigouvian pricing:
  \[ T_a(\bar{x}) := \bar{x} \cdot \frac{\partial F^0}{\partial \bar{x}_a}(\bar{x}) = \sum_{b \in A} \bar{x}_b \frac{\partial F^0_b}{\partial \bar{x}_a}(\bar{x}). \]

- \( F^p + T \) is a potential game with potential \( \bar{F}^p \).

Evolutionary implementation (Sandholm, '03,'05 RES)

(A-F1) Assume that the common payoff function \( F^0 \) is \( C^1 \) and its Hessian is negative definite on the tangent space of \( \Delta^A \) (negative externality).

\[ \Rightarrow \] The total payoff function \( \bar{F}^p \) is strictly concave.

\[ \Rightarrow \] The social optimum is globally stable under any dynamic that satisfies Nash stationarity and positive correlation.
Equilibrium stability | Extensions

## Gains in evolutionary dynamics

### Zusai

30

---

### Evolutionary implementation: perturbation

**Perturbation to the game**

- \( \mathbf{F}_\delta^p(\mathbf{x}^p) = \mathbf{F}^p(\mathbf{x}^p) + \delta \tilde{\mathbf{F}}^p(\mathbf{x}^p) \) with \( \delta \in \mathbb{R} \).
- Imagine that the central planner does not know the perturbation \( \delta \tilde{\mathbf{F}}^p \) and thus designs the Pigouvian pricing scheme based on \( \mathbf{F}^p \).
- Then the game is no longer a potential game.

### Robustness of evolutionary implementation

(A-F2) Assume that \( \tilde{\mathbf{F}}^p \) is \( C^1 \) and there exists an upper bound on \( |\partial \tilde{\mathbf{F}}^p / \partial \mathbf{x}^p| \).

⇒ With strict concavity of \( \mathbf{F}^0 \) (A-F1), this implies that the perturbed game \( \mathbf{F}_\delta^p \) is still a contractive game as long as \( \delta \) is sufficiently small.

⇒ Further, the total payoff maximization is approximately achieved at the limit state under any evolutionary dynamics that guarantee Nash stability in contractive games.

---

**Theorem** Robustness of evolutionary implementation

Consider the perturbed game as above that satisfies (A-F1) and (A-F2).

For any \( \varepsilon > 0 \) there is \( \tilde{\delta} (\varepsilon) > 0 \) such that

\[
|\delta| < \tilde{\delta} (\varepsilon) \quad \implies \quad | \lim_{t \to \infty} \tilde{F}_\delta^p (\mathbf{x}_{\delta,t}^p) - \max_{\mathbf{x}^p \in \mathcal{X}^p} \tilde{F}_\delta^p (\mathbf{x}^p) | < \varepsilon,
\]

where \( \{x_{\delta,t}^p\}_{t \geq 0} \) is the trajectory starting from an arbitrary initial state \( x_{\delta,0}^p \in \mathcal{X}^p \) in the population game \( \mathbf{F}_\delta^p \) under an evolutionary dynamic that satisfies all the assumptions.
Summary
• We reconstruct evolutionary dynamics from optimization-based protocol
• A variety of dynamics can be retrieved by random restriction to available actions and random switching costs.
• Then, we can quantitatively identify the net gain from revision.
• While the aggregate gross gain serves as a Lyapunov function for Nash stability of potential games, the aggregate net gain works generally for Nash stability of contractive games.
• With small modification, it extends to local stability of a regular ESS.
• Generality of these stability results allow us to mix different dynamics and to add perturbation; they are robust to heterogeneity.
• With Nash stability of contractive games, these robust stability results imply robustness of evolutionary implementation of social optimum by dynamic Pigouvian pricing.

Extension
• I believe that this idea is generally applicable to a various (possibly more complex) dynamics even if the assumptions do not hold or the dynamic may not fit exactly into the framework just as presented here.
• e.g. Sawa & Zusai (2016) consider the “multitasking BRD” in which an agent engages simultaneously in multiple games but can change action in only one of them. The mBRD does not exactly fit into our framework. But, applying the same idea, we define the gain function and use it to prove Nash stability.