Homogenization in an optimal design problem with quadratic weakly discontinuous objective functional

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Abstract

In this paper we consider an optimal design problem, where the goal is to find the layout of two conductors that minimize a given quadratic objective functional. The most important feature of the objective functional is that it is weakly discontinuous. In that case the tools of homogenization that were traditionally used in order to study such problems, are helpful but not sufficient. In this paper we present an example where additional tools are required to analyze the problem.

1 Introduction

The stationary process of heat or electric conduction in a solid body Ω can be described by an elliptic PDE of the form

$$\nabla \cdot \boldsymbol{a}(\boldsymbol{x})\nabla\phi = f(\boldsymbol{x}), \tag{1.1}$$

where $\phi(\boldsymbol{x})$ has the physical meaning either of temperature or of electrostatic potential. The function $f(\boldsymbol{x})$ corresponds to the external heat source, and the symmetric positive definite matrix field $\boldsymbol{a}(\boldsymbol{x})$ describes local properties of the conducting medium at the point $\boldsymbol{x}$.

Any layout of two conductors with conductivities $\alpha\boldsymbol{I}$ and $\beta\boldsymbol{I}$ can be described by a characteristic function $\chi(\boldsymbol{x})$ of the set occupied by the material with conductivity $\alpha\boldsymbol{I}$. The local conductivity of the medium $\boldsymbol{a}(\boldsymbol{x})$ then has the form $\boldsymbol{a}(\boldsymbol{x}) = a(\boldsymbol{x})\boldsymbol{I}$, where

$$a(\boldsymbol{x}) = \alpha\chi(\boldsymbol{x}) + \beta(1-\chi(\boldsymbol{x})). \tag{1.2}$$

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Let us assume that $\alpha < \beta$ and that the boundary condition supplementing (1.1) is

$$\phi|_{\partial\Omega} = 0. \tag{1.3}$$

Let $J(\boldsymbol{x}, \boldsymbol{e}) : \Omega \times \mathbb{R}^3 \rightarrow \mathbb{R}_+$ be a coercive function in the sense that

$$\inf_{\boldsymbol{x}\in\Omega} \liminf_{|\boldsymbol{e}| \to \infty} \frac{J(\boldsymbol{x}, \boldsymbol{e})}{|\boldsymbol{e}|^p} > 0,$$

for some $p > 1$. We will consider a problem of finding a geometric arrangement of materials $\chi(\boldsymbol{x})$ that minimizes

$$I(\chi) = \int_\Omega J(\boldsymbol{x}, \nabla\phi(\boldsymbol{x}))d\boldsymbol{x} \tag{1.4}$$

These problems ultimately require a numerical solution. The existence of a solution is, however, a fundamental issue that needs to be addressed first. If $\chi^\varepsilon(\boldsymbol{x})$ is a minimizing sequence and ϕ_ε is a corresponding sequence of solutions of (1.1)–(1.3), then we can assert only the existence of the weakly convergent subsequence

$$\nabla\phi_\varepsilon \rightharpoonup \nabla\phi_0 \text{ weakly in } L^p$$

There are two separate difficulties that have to be dealt with. The first one is that there is no reason why the potential ϕ_0 should solve an equation (1.1) with $a(\boldsymbol{x})$ given by (1.2). The second difficulty is that the value $\int_\Omega J(\boldsymbol{x}, \nabla\phi_0(\boldsymbol{x}))d\boldsymbol{x}$ may have little to do with the limit points of the sequence $I(\chi^\varepsilon)$. The presence or absence of the latter difficulty is what distinguishes the two types of optimal design problems: the problems with weakly continuous and the problems with weakly discontinuous objective functionals.

The optimal design problems with weakly continuous objective functionals have received a lot of attention in recent years, see for example a survey paper [19] and the books [3, 5] for extensive review and references. By contrast the optimal design problems with weakly discontinuous objective functionals and partial differential constraints of the type (1.1)–(1.3) have not been treated as extensively. Pedregal [17, 16] showed that these problems are intimately related to Young measures [2] and to quasiconvexification [13, 14]. Pedregal also pointed out the almost complete absence of particular examples. The notable exceptions are [1, 6, 10, 21], all of which except the first paper treat one and the same example of a weakly discontinuous functional

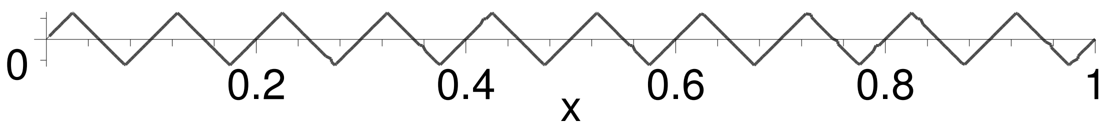
$$I(\chi) = \int_\Omega |\nabla\phi(\boldsymbol{x}) - \boldsymbol{\eta}(\boldsymbol{x})|^2 d\boldsymbol{x}, \tag{1.5}$$

where $\boldsymbol{\eta}(\boldsymbol{x}) \in L^2(\Omega; \mathbb{R}^3)$ is a given function. This example was suggested by Tartar [21] as the simplest non-trivial example with all the requisite features. We will follow the example of our predecessors and consider the functional (1.5) as well. We will discuss relations between contributions mentioned above a little later, after we address the first issue of non-admissibility of the limit potential.

We conclude this section by a discussion of a simple one-dimensional example whose basic idea is probably due to L. C. Young [22]. Suppose we need to minimize

$$I(u) = \int_0^1 \left\{ u^2(x) + ((u'(x))^2 - 1)^2 \right\} dx \tag{1.6}$$

over all C^1 functions $u(x)$ with $u'(0) = 1$. In order to make $I(u)$ as small as possible we would like to have u' take values of only ± 1 and have u be as small as possible. Obviously, we cannot have both terms zero, but we can approach our goal as close as we like. Indeed a sequence u^ϵ that looks like this



will make the value of $I(u^\epsilon)$ very close to zero. We see that the limiting function $u_0 = 0$ is not admissible: $u_0'(0) \neq 1$. This illustrates the first difficulty mentioned above. In addition

$$I(u_0) = 1 \neq 0 = \lim_{\epsilon \to 0} I(u^\epsilon).$$

This illustrates the weak discontinuity of the functional I.

We can try to resolve the first difficulty by enlarging the set of admissible functions to a weak closure of the original set. In our example this weak closure is just a Sobolev space $W^{1,4}([0,1])$. We can resolve the second difficulty by changing or *relaxing* the objective functional in a special way. The new relaxed functional must have the property that every minimizing sequence for the original functional converges to a minimizer of the new functional and every minimizer of the new functional is a weak limit of a minimizing sequence for the original functional. In this example the function $F(u') = ((u')^2 - 1)^2$ has to be replaced by its convexification $F^*(u') = ((u')^2 - 1)_+^2$, where $(a)_+ = \max(a, 0)$.

2 Composites as generalized controls

Let $(\chi^\varepsilon, \phi_\varepsilon)$ be a minimizing sequence for (1.4). Can we extract a convergent subsequence? The obvious answer is yes, we can extract a $W^{1,p}$ weakly convergent subsequence $\phi_\varepsilon \rightharpoonup \phi_0$ and an L^∞ weak-* convergent subsequence $\chi^\varepsilon(\boldsymbol{x}) \overset{*}{\rightharpoonup} \theta(\boldsymbol{x})$, where $\theta(\boldsymbol{x})$ can be any measurable function with values in the interval $[0,1]$. However, the limit potential ϕ_0 may no longer satisfy the constraints (1.1)–(1.3). Therefore, we need a little different notion of convergence that is better suited to the nature of our constraints. Such a notion has been put forward by Murat and Tartar [15] under the name *H-convergence*. Let $\boldsymbol{a}^\varepsilon(\boldsymbol{x})$ be a sequence of conductivity tensors and ϕ_ε be the sequence of solutions to the Dirichlet problem

$$\begin{cases} \nabla \cdot \boldsymbol{a}^\varepsilon(\boldsymbol{x}) \nabla \phi_\varepsilon = f, \\ \phi_\varepsilon|_{\partial\Omega} = 0, \end{cases}$$

Definition 1 *The sequence of conductivity tensors* $\boldsymbol{a}^\varepsilon(\boldsymbol{x})$ *H-converges to the conductivity tensor* $\boldsymbol{a}^*(\boldsymbol{x})$ *if for all* $f \in L^2(\Omega)$

1. $\phi_\varepsilon \rightharpoonup \phi_0$ *weakly in* $W_0^{1,2}(\Omega)$
2. $\boldsymbol{a}^\varepsilon \nabla \phi_\varepsilon \rightharpoonup \boldsymbol{a}^* \nabla \phi_0$ *weakly in* $L^2(\Omega)$,

where ϕ_0 *solves*

$$\begin{cases} \nabla \cdot \boldsymbol{a}^*(\boldsymbol{x}) \nabla \phi_0 = f, \\ \phi_0|_{\partial\Omega} = 0. \end{cases} \tag{2.1}$$

Physically, the H-limit $\boldsymbol{a}^*(\boldsymbol{x})$ can be understood as the effective conductivity of the composite material described mathematically by a sequence $\boldsymbol{a}^\varepsilon(\boldsymbol{x})$. The weak-* limit $\theta(\boldsymbol{x})$ of the sequence χ^ε can be interpreted as a local volume fraction of the material α in a composite placed at the point $\boldsymbol{x} \in \Omega$. To give a non-trivial example of an H-convergent sequence $\boldsymbol{a}^\varepsilon(\boldsymbol{x})$ consider a periodic composite [4] described by the sequence

$$\boldsymbol{a}^\varepsilon(\boldsymbol{x}) = \boldsymbol{a}_0(\boldsymbol{x}/\varepsilon), \quad \boldsymbol{a}_0(\boldsymbol{y}) = \alpha \boldsymbol{I} \chi(\boldsymbol{y}) + \beta \boldsymbol{I}(1 - \chi(\boldsymbol{y})),$$

where $\chi(\boldsymbol{y})$ is a triply periodic characteristic function with parallelepiped of periods Q. Then there exists a constant tensor $\boldsymbol{a}^*$ such that $\boldsymbol{a}^\varepsilon(\boldsymbol{x}) \overset{H}{\rightharpoonup} \boldsymbol{a}^*$. The tensor $\boldsymbol{a}^*$ is defined in terms of the unique (up to an additive constant) Q-periodic solution $\psi^{\boldsymbol{\xi}}$ of the *cell problem*:

$$\nabla \cdot \Big(\boldsymbol{a}_0(\boldsymbol{y})(\nabla \psi + \boldsymbol{\xi}) \Big) = 0, \tag{2.2}$$

where $\boldsymbol{\xi}$ is an arbitrary constant vector in $\mathbb{R}^3$. Notice that $\psi^{\boldsymbol{\xi}}$ depends linearly on $\boldsymbol{\xi}$. Therefore,

$$\boldsymbol{a}^* \boldsymbol{\xi} = \int_Q \boldsymbol{a}_0(\boldsymbol{y})(\nabla \psi^{\boldsymbol{\xi}} + \boldsymbol{\xi}) d\boldsymbol{y}$$

defines a 3 by 3 matrix $\boldsymbol{a}^*$, which is exactly the H-limit of the sequence $\boldsymbol{a}^\varepsilon(\boldsymbol{x})$.

The explicit form of the H-limit $\boldsymbol{a}^*$ for a periodic composite justifies the following definition

Definition 2 *The G-closure* $G(\mathcal{S})$ *of the set of materials* $\mathcal{S} \subset \mathrm{Sym}^+(\mathbb{R}^3)$ *is the set of all effective tensors* $\boldsymbol{a}^*$ *of periodic composites made of materials from the set* $\mathcal{S}$.

We can also define the G_θ-closure for composites made with two materials α and β.

Definition 3 *The* G_θ*-closure of two materials* α *and* β *is the set of all effective tensors* $\boldsymbol{a}^*$ *of periodic composites made with materials* α *and* β *taken in volume fractions* θ *and* $1-\theta$ *respectively.*

The G_θ-closure of two materials α and β is known exactly [11, 12, 20]. It consists of all symmetric 3 by 3 matrices $\boldsymbol{a}^*$ satisfying the following inequalities

$$h\boldsymbol{I} \leq \boldsymbol{a}^* \leq m\boldsymbol{I} \tag{2.3}$$

in the sense of quadratic forms, where m and h are arithmetic and harmonic means of α and β respectively:

$$m = \alpha\theta + \beta(1-\theta), \qquad h = (\theta/\alpha + (1-\theta)/\beta)^{-1}. \tag{2.4}$$

Also,

$$\mathrm{Tr}\,(\boldsymbol{a}^* - \alpha\boldsymbol{I})^{-1} \le \frac{d}{m-\alpha} + \frac{\theta}{\alpha(1-\theta)} \tag{2.5}$$

$$\mathrm{Tr}\,(\beta\boldsymbol{I} - \boldsymbol{a}^*)^{-1} \le \frac{d}{\beta-m} - \frac{1-\theta}{\beta\theta} \tag{2.6}$$

The following two fundamental theorems explain why the notion of H-convergence is so useful and why the G-closure sets are important.

THEOREM **2.1 (Kohn and Dal Maso)** *The tensor field $\boldsymbol{a}^*(\boldsymbol{x})$ is an H-limit of a sequence of tensors $\boldsymbol{a}^\varepsilon(\boldsymbol{x})$ taking values in the set $\mathcal{S}$ if and only if $\boldsymbol{a}^*(\boldsymbol{x}) \in G(\mathcal{S})$ for almost all $\boldsymbol{x} \in \Omega$.*

THEOREM **2.2 (Murat and Tartar)** *The topology of sequential H-convergence is locally compact: Any bounded sequence $\boldsymbol{a}^\varepsilon(\boldsymbol{x})$ has an H-convergent subsequence.*

Theorem 2.1 is coming from unpublished work of Kohn and Dal Maso. It's proof can be found in [18]. Theorem 2.2 is proved in [15]. For our particular case of two isotropic conductors, Tartar [20] showed explicitly that if $\chi^\varepsilon(\boldsymbol{x}) \overset{*}{\rightharpoonup} \theta(\boldsymbol{x})$ and $\boldsymbol{a}^\varepsilon(\boldsymbol{x}) \overset{H}{\rightharpoonup} \boldsymbol{a}^*(\boldsymbol{x})$ then for almost every $\boldsymbol{x} \in \Omega$ we have $\boldsymbol{a}^*(\boldsymbol{x}) \in G_{\theta(\boldsymbol{x})}$. And conversely, if the pair of measurable fields $(\theta(\boldsymbol{x}), \boldsymbol{a}^*(\boldsymbol{x}))$ satisfies $0 \le \theta(\boldsymbol{x}) \le 1$ and $\boldsymbol{a}^*(\boldsymbol{x}) \in G_{\theta(\boldsymbol{x})}$ then there is a sequence of characteristic functions $\chi^\varepsilon(\boldsymbol{x})$ and the corresponding (via (1.2)) sequence $\boldsymbol{a}^\varepsilon(\boldsymbol{x})$ such that $\chi^\varepsilon(\boldsymbol{x}) \overset{*}{\rightharpoonup} \theta(\boldsymbol{x})$ and $\boldsymbol{a}^\varepsilon(\boldsymbol{x}) \overset{H}{\rightharpoonup} \boldsymbol{a}^*(\boldsymbol{x})$.

Now we can approach our optimal design problem again applying the notion of H-convergence. Let $\chi^\varepsilon(\boldsymbol{x})$ be a minimizing sequence for (1.4). Then by Theorem 2.2 we can extract a subsequence still labeled $\chi^\varepsilon(\boldsymbol{x})$ such that $\chi^\varepsilon(\boldsymbol{x}) \overset{*}{\rightharpoonup} \theta(\boldsymbol{x})$ and $\boldsymbol{a}^\varepsilon(\boldsymbol{x}) \overset{H}{\rightharpoonup} \boldsymbol{a}^*(\boldsymbol{x})$. Then $\phi_\varepsilon \rightharpoonup \phi_0$ weakly in $W^{1,p}$ and ϕ_0 satisfies (2.1). That way we can say that we have enlarged the set of admissible structural elements in a layout from the two pure materials α and β to the set of all possible composites made with α and β. Therefore, the answer to the optimal design problem will be specifying which *composites* rather than materials are occupying each point $\boldsymbol{x} \in \Omega$.

3 Relaxation

Now we would like to point out that knowledge of the weak limit data such as $\phi_0(\boldsymbol{x})$, $\boldsymbol{a}^*(\boldsymbol{x})$ and $\theta(\boldsymbol{x})$ is not enough to describe the limit of $I(\chi^\varepsilon)$ as $\varepsilon \to 0$. The reason is that there may be many very different composites, even periodic composites, with the same value of $\boldsymbol{a}^*$. At the same time, the functional I is sensitive to the particular kind of composite used. A similar effect was present in Axell's work, where he computed bounds for the variances of the field in each phase of a two-phase isotropic composite. Axell's bounds also imply bounds on the L^2 norm of the field, relevant for the objective functional (1.5). They can be shown to be weaker than ours simply because Axell's goal was obtaining bounds for variances within each phase and not for the L^2 norm of the field in the whole composite. This conclusion should not be surprising, if one realizes that the fields in both phases are strongly correlated. Most recently Lipton [8, 9] obtained the optimal bounds on the variance of field fluctuations in the conducting composite. Our bounds could be derived from his, and in fact our approaches are in many respects similar.

Tartar [21] was the first to propose the form (1.5) of the objective functional. He proved a remarkable result that for $\boldsymbol{\eta}(\boldsymbol{x})$ in a dense G_δ subset of $L^2(\Omega;\mathbb{R}^3)$ the minimizing sequence ϕ_ε converges strongly in $W_0^{1,2}(\Omega)$. His was a Baire category argument and so it gave no indication of what that dense G_δ set might look like. Unfortunately, this result does not exclude pathological examples. For instance, the set $L^2(\Omega;\mathbb{R}^3)\setminus W^{1,2}(\Omega;\mathbb{R}^3)$ is a G_δ dense subset of $L^2(\Omega;\mathbb{R}^3)$ but contains no continuous, piecewise smooth functions. The strong convergence of the sequence ϕ_ε means that at almost every point $\boldsymbol{x}\in\Omega$ the corresponding composite is a rank-1 laminate (a composite with alternating layers of materials α and β) whose effective conductivity is the matrix $\boldsymbol{a}^*(\boldsymbol{x})$ with eigenvalues $(m(\boldsymbol{x}), m(\boldsymbol{x}), h(\boldsymbol{x}))$, where m and h are given by (2.4). Moreover, for almost all $\boldsymbol{x}\in\Omega$ the limit potential gradient $\nabla\phi_0(\boldsymbol{x})$ is an eigenvector of $\boldsymbol{a}^*(\boldsymbol{x})$ with eigenvalue $m(\boldsymbol{x})$:

$$\boldsymbol{a}^*(\boldsymbol{x})\nabla\phi_0(\boldsymbol{x}) = m(\boldsymbol{x})\nabla\phi_0(\boldsymbol{x}). \tag{3.1}$$

In other words the field $\nabla\phi_0(\boldsymbol{x})$ is always directed parallel to the layers.

Dvořák, Haslinger and Miettinen [6] approached the optimal design problems from the numerical point of view. In their work they considered many different objective functionals, (1.5) among them, and showed how the knowledge of the relaxed formulation makes it possible to solve the problem numerically. Unfortunately, they were not able to compute the relaxation of (1.5).

A more recent investigation was done by Lipton and Velo [10] who studied the optimization problem numerically using only simply layered microstructures. According to Tartar's result the solution to Lipton and Velo's problem exists for a G_δ dense set of fields $\boldsymbol{\eta}$. They derive the formula for the relaxed functional for "good" choices of $\boldsymbol{\eta}$ that agrees with our formula (3.5) below. Their numerical experiments suggest a conjecture that Tartar's dense G_δ set may be the whole space.

We show elsewhere [7] that in fact the optimal composites at almost every $\boldsymbol{x}\in\Omega$ must be rank-1 laminates. However, we were not able to show that (3.1) holds, which is necessary (and sufficient) to settle the conjecture about Tartar's dense G_δ set. We also note that introduction of the volume fraction constraints considered by other authors invalidate our arguments that led to the last conclusion about rank-1 laminates being the optimal composites at almost every point $\boldsymbol{x}\in\Omega$.

Our idea is to fix $\boldsymbol{a}^*$ and to minimize explicitly over all possible composites with the same value of $\boldsymbol{a}^*$. More precisely, we denote

$$\mathcal{A}(\boldsymbol{a}^*) = \{\{\chi^\varepsilon(\boldsymbol{x}):\ \varepsilon>0\}|\ \boldsymbol{a}^\varepsilon \xrightarrow{H} \boldsymbol{a}^*(\boldsymbol{x})\}$$

and we define

$$I^*(\boldsymbol{a}^*) = \inf_{\chi^\varepsilon \in \mathcal{A}(\boldsymbol{a}^*)} \liminf_{\varepsilon \to 0} I(\chi^\varepsilon) \tag{3.2}$$

It is remarkable that it is possible to compute I^* explicitly for our example (1.5).

THEOREM **3.1** *The relaxed formulation of the optimal design problem (1.5) with constraints (1.1)–(1.3) is the following problem*

$$\inf_{\chi} I(\chi) = \inf_{\boldsymbol{a}^*(\boldsymbol{x}) \in G(\{\alpha,\beta\})} I^*(\boldsymbol{a}^*), \tag{3.3}$$

where I^* is given by

$$I^*(\boldsymbol{a}^*) = \int_\Omega \left\{ |\boldsymbol{\eta}|^2 - 2\boldsymbol{\eta} \cdot \nabla\phi_0 - \frac{1}{\beta} f\phi_0 + \psi(\boldsymbol{a}^*)|(\beta \boldsymbol{I} - \boldsymbol{a}^*)\nabla\phi_0|^2 \right\} d\boldsymbol{x}, \tag{3.4}$$

ϕ_0 solves (2.1) and

$$\psi(\boldsymbol{a}^*) = \frac{\mathrm{Tr}\,(\beta \boldsymbol{I} - \boldsymbol{a}^*)^{-1} - 1/\beta}{\beta + \alpha}.$$

Proof. Let

$$J^*(\theta, \boldsymbol{a}^*) = \inf_{\chi^\varepsilon \in \mathcal{A}(\boldsymbol{a}^*, \theta)} \{\liminf_{\varepsilon \to 0} I(\chi^\varepsilon),$$

where

$$\mathcal{A}(\theta, \boldsymbol{a}^*) = \{\{\chi^\varepsilon(\boldsymbol{x}) : \ \varepsilon > 0\} |\ \boldsymbol{a}^\varepsilon \stackrel{H}{\rightharpoonup} \boldsymbol{a}^*(\boldsymbol{x}),\ \chi^\varepsilon(\boldsymbol{x}) \stackrel{*}{\rightharpoonup} \theta(\boldsymbol{x})\}.$$

In [7] we obtain

$$I^*(\theta, \boldsymbol{a}^*) = \int_\Omega \left\{ |\boldsymbol{\eta}|^2 - 2\boldsymbol{\eta} \cdot \nabla\phi_0 - \frac{1}{\beta} f\phi_0 + \frac{|(\beta \boldsymbol{I} - \boldsymbol{a}^*)\nabla\phi_0|^2}{\beta(\beta - \alpha)\theta} \right\}. \tag{3.5}$$

Observe that the admissible set of pairs $(\theta(\boldsymbol{x}), \boldsymbol{a}^*(\boldsymbol{x}))$

$$ad = \{(\theta(\boldsymbol{x}), \boldsymbol{a}^*(\boldsymbol{x})) : \ 0 \le \theta(\boldsymbol{x}) \le 1,\ \ \boldsymbol{a}^*(\boldsymbol{x}) \in G_{\theta(\boldsymbol{x})}\}$$

allows some freedom in the choice of the field $\theta(\boldsymbol{x})$ when the field $\boldsymbol{a}^*(\boldsymbol{x})$ is fixed. We can represent $I^*(\boldsymbol{a}^*)$ as

$$I^*(\boldsymbol{a}^*) = \min_{\theta \in ad_1} I^*(\theta, \boldsymbol{a}^*), \tag{3.6}$$

where ad_1 is the projection of the set ad onto the first component.

The functional $I^*(\theta, \boldsymbol{a}^*)$ is monotone decreasing in θ. The minimum in (3.6), therefore, is attained at the largest possible value of θ. That value is easily found from inequality (2.6) describing the upper boundary of the G_θ set:

$$\theta^*(\boldsymbol{x}) = \frac{1}{\beta(\beta - \alpha)\psi(\boldsymbol{a}^*)}.$$

Substituting it in (3.5) we obtain the statement of the theorem.

We remark that the form (3.3) of the relaxed problem is less convenient that the alternative formulation

$$\inf_{\chi} I(\chi) = \inf_{(\theta, \boldsymbol{a}^*) \in ad} I^*(\theta, \boldsymbol{a}^*) \tag{3.7}$$

obtained by combining (3.3) with (3.6). The reason for this is that the functional $I^*(\boldsymbol{a}^*)$ is *not* convex in $\boldsymbol{a}^*$, while the functional $I^*(\theta, \boldsymbol{a}^*)$ is. This allows us to include constraints (2.1) in (3.7) with a Lagrange multiplier and then use saddle point theorems of convex analysis.

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