Exact relations for effective conductivity of fiber-reinforced conducting composites with Hall effect via a general theory

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Abstract

In this paper we apply the general theory of exact relations to derive all microstructure-independent relations for effective conductivity of fiber-reinforced composites with Hall effect. We also derive all possible links between effective conductivities of two composites that have the same microstructure but are built using different materials. Our results hold for any number of constituents with any anisotropy. This paper is a record of the work of 14 undergraduate students in the NSF-sponsored REU (Research Experience for Undergraduates) program led by the author.

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1 Introduction

We consider fiber-reinforced conducting composites, i.e. three-dimensional composite materials, whose microstructure is fully described by any two-dimensional cross-section transversal to the fibers. Mathematically, such a microstructure can be modeled by an oscillatory sequence of 3×3 positive definite conductivity tensors $\boldsymbol{L}^{\epsilon}(\boldsymbol{x})$, where $\boldsymbol{x}=(x_1,x_2)$ is the transversal variable. The conductivities $\boldsymbol{L}^{\epsilon}(\boldsymbol{x})$ are assumed to be non-symmetric, i.e. exhibiting Hall effect—a skew-symmetric correction to a symmetric conductivity tensor in the presence of a stationary magnetic field. We would like to emphasize here that the direction of the magnetic field is not assumed to be aligned in any way either with the crystallographic axes of the constituent materials or with the directions of fibers. The effective tensor of the composite with microstructure described by $\boldsymbol{L}^{\epsilon}(\boldsymbol{x})$ is understood mathematically as the H-limit $\boldsymbol{L}^{*}(\boldsymbol{x})$ of $\boldsymbol{L}^{\epsilon}(\boldsymbol{x})$, as $\epsilon\to0$, [19].

Our goal is to identify those functions of the effective tensor $\boldsymbol{L}^{*}$ that do not depend on the microstructure. For example, in the context of two-dimensional conductivity, if the conductivity tensors $\boldsymbol{\sigma}_j$, $j=1,\dots N$ of all N constituents satisfy

$$\det\boldsymbol{\sigma}_j=d_0,\qquad j=1,\dots N, \tag{1.1}$$

then, *regardless of the microstructure*, $\det \boldsymbol{\sigma}^* = d_0$, [7, 16]. In addition to exact relations like (1.1) we are also interested in volume fraction relations. For example, in the context of fiber-reinforced conducting composites, any mixture of any number of isotropic materials $\sigma_1\boldsymbol{I}, \ldots, \sigma_N\boldsymbol{I}$ has an effective conductivity of the form

$$\boldsymbol{L}^* = \begin{bmatrix} \boldsymbol{\sigma}^* & \boldsymbol{0} \\ \boldsymbol{0} & \alpha^* \end{bmatrix}, \tag{1.2}$$

where $\boldsymbol{\sigma}^*$ is a 2×2 sub-block of $\boldsymbol{L}^*$ and

$$\alpha^* = \sum_{i=1}^{N} \theta_i \sigma_i. \tag{1.3}$$

where θ_i is the volume fraction of the material $\sigma_i\boldsymbol{I}$, $i = 1, \ldots, N$.

Finally, we are also interested in links between the uncoupled problems. The links relate effective conductivities of composites with the same microstructure but different composition. Again, taking the example of two-dimensional conductivity, there is a link due to Mendelson [17]. If

$$\boldsymbol{\sigma}_2(\boldsymbol{x}) = \frac{\boldsymbol{\sigma}_1(\boldsymbol{x})}{\det \boldsymbol{\sigma}_1(\boldsymbol{x})}, \tag{1.4}$$

then

$$\boldsymbol{\sigma}_2^* = \frac{\boldsymbol{\sigma}_1^*}{\det \boldsymbol{\sigma}_1^*}. \tag{1.5}$$

In general, links carry more information about effective tensors than exact relations. For example, the link (1.4)–(1.5) implies the exact relation (1.1). Indeed, if for all $\boldsymbol{x}$ we have $\det \boldsymbol{\sigma}_1(\boldsymbol{x}) = d_0$ then, the effective tensor $\boldsymbol{\sigma}_2^*$ of $\boldsymbol{\sigma}_2(\boldsymbol{x}) = \boldsymbol{\sigma}_1(\boldsymbol{x})/d_0$, will be equal to $\boldsymbol{\sigma}_1^*/d_0$. Now the formula (1.5) implies that $\det \boldsymbol{\sigma}_1^* = d_0$.

The general theory of exact relations [9, 12] (see also [18, Chapter 17]) gives a recipe of how to compute *all* exact relations, all volume fraction relations and all links, missing none. We have applied our theory in many physical contexts, giving complete lists of exact relations for polycrystals [8, 12, 13], as well as 2D Hall effect [9]. This paper gives a complete solution to the problem of exact relations and links in the case of fiber reinforced conducting composites with Hall effect. The exact relations for effective conductivity in our context are of interest to physicists (see [1, 3, 5, 4, 6, 2, 22, 20, 21] and many others by the same group of researchers). Our results will allow to simplify and unify many of their results.

We prefer to think about exact relations in a geometric way. The microstructure-independent combinations of components of the tensor $\boldsymbol{L}^*$ can be thought of as defining a surface in the 9-dimensional space of 3×3 matrices, while the links can be represented by surfaces in the 18-dimensional space of pairs of 3×3 matrices. The general theory of exact relations and links converts a question about solutions of partial differential equations, defining the effective conductivity of a composite, into a problem from matrix algebra. In this paper we formulate and solve this problem in the context of the fiber-reinforced conducting composites with Hall effect. After the algebraic problem is solved, one still needs to translate the results back into physical variables and present the results in as compact a form as possible. We accomplish all these steps and obtain a complete and non-redundant list of all the exact relations and links.

2 Summary of exact relations and links

The following is a complete set of microstructure-independent relations for effective conductivity of fiber-reinforced conducting composites with Hall effect. In order to state them we write both the local conductivity $\boldsymbol{L} = \boldsymbol{L}(\boldsymbol{x})$ and the effective conductivity $\boldsymbol{L}^*$ in the block form

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{\Lambda} & \boldsymbol{p} \\ \boldsymbol{q} & \alpha \end{bmatrix}.$$

1. Let

$$\boldsymbol{\Psi}(\boldsymbol{\Lambda}) = \frac{(\boldsymbol{\Lambda} - r_0\boldsymbol{S})^T}{\det(\boldsymbol{\Lambda} - r_0\boldsymbol{S})}, \qquad \boldsymbol{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.1}$$

The following relation is a link

$$\begin{aligned} \boldsymbol{\Lambda}' &= \tau_0\boldsymbol{\Psi}(\boldsymbol{\Lambda}) + r_0'\boldsymbol{S}, \\ \boldsymbol{p}' &= \boldsymbol{p}_0' + \mu_0\boldsymbol{\Psi}(\boldsymbol{\Lambda})(\boldsymbol{p} - \boldsymbol{p}_0)^\perp, \\ \boldsymbol{q}' &= \boldsymbol{q}_0' + \nu_0\boldsymbol{\Psi}(\boldsymbol{\Lambda})^T(\boldsymbol{q} - \boldsymbol{q}_0)^\perp, \\ \alpha' &= \alpha_0 + \tfrac{\mu_0\nu_0}{\tau_0}\left\{(\boldsymbol{\Psi}(\boldsymbol{\Lambda})(\boldsymbol{p} - \boldsymbol{p}_0)^\perp, (\boldsymbol{q} - \boldsymbol{q}_0)^\perp) - \alpha\right\}, \end{aligned} \tag{2.2}$$

where $\boldsymbol{a}^\perp = \boldsymbol{S}\boldsymbol{a}$ and r_0, r_0', τ_0, μ_0, ν_0, α_0 are arbitrary constants and $\boldsymbol{p}_0$, $\boldsymbol{q}_0$, $\boldsymbol{p}_0'$, $\boldsymbol{q}_0'$ are arbitrary vectors in $\mathbb{R}^2$.

2. The following relation is a link

$$\begin{aligned} \boldsymbol{\Lambda}' &= \tau_0\boldsymbol{\Lambda} + r_0'\boldsymbol{S}, \\ \boldsymbol{p}' &= \boldsymbol{\Lambda}\boldsymbol{p}_0 + \mu_0\boldsymbol{p} + \boldsymbol{p}_0', \\ \boldsymbol{q}' &= \boldsymbol{\Lambda}^T\boldsymbol{q}_0 + \nu_0\boldsymbol{q} + \boldsymbol{q}_0', \\ \alpha' &= \tau_0^{-1}\{\mu_0\nu_0\alpha + \mu_0(\boldsymbol{p}, \boldsymbol{q}_0) + \nu_0(\boldsymbol{q}, \boldsymbol{p}_0) + (\boldsymbol{\Lambda}\boldsymbol{p}_0, \boldsymbol{q}_0)\} + \alpha_0, \end{aligned} \tag{2.3}$$

The link (2.3) is a limiting case of the link (2.2), when some of the parameters in (2.2) go to infinity.

3. The effective tensor $\boldsymbol{L}^*$ enjoys the “transpose symmetry”

$$(\boldsymbol{L}^T)^* = (\boldsymbol{L}^*)^T. \tag{2.4}$$

4. Let us write $\boldsymbol{L}^* = \mathbb{H}(\boldsymbol{L}(\boldsymbol{x}))$, thinking of $\mathbb{H}$ as a "homogenization operator". Then we have

$$\mathbb{H}\left(\begin{bmatrix} \boldsymbol{\Lambda} & \boldsymbol{p} \\ \boldsymbol{q} & \alpha \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{H}(\boldsymbol{\Lambda}) & \boldsymbol{h}(\boldsymbol{\Lambda}, \boldsymbol{p}) \\ \boldsymbol{h}(\boldsymbol{\Lambda}^T, \boldsymbol{q}) & \eta(\boldsymbol{\Lambda}, \boldsymbol{p}, \boldsymbol{q}, \alpha) \end{bmatrix}. \tag{2.5}$$

Formula (2.5) expresses compactly the facts that $\boldsymbol{\Lambda}^*$ depends only on $\boldsymbol{\Lambda}(\boldsymbol{x})$, but not $\boldsymbol{p}$, $\boldsymbol{q}$ or α, while $\boldsymbol{p}^*$ depends only on $\boldsymbol{\Lambda}(\boldsymbol{x})$ and $\boldsymbol{p}(\boldsymbol{x})$. The transposition link (2.4) then implies that the dependence of $\boldsymbol{q}^*$ on $\boldsymbol{\Lambda}(\boldsymbol{x})^T$ and $\boldsymbol{q}(\boldsymbol{x})$ is the same as the dependence of $\boldsymbol{p}^*$ on $\boldsymbol{\Lambda}(\boldsymbol{x})$ and $\boldsymbol{p}(\boldsymbol{x})$. The transpose symmetry also implies

$$\boldsymbol{H}(\boldsymbol{\Lambda})^T = \boldsymbol{H}(\boldsymbol{\Lambda}^T), \qquad \eta(\boldsymbol{\Lambda}, \boldsymbol{p}, \boldsymbol{q}, \alpha) = \eta(\boldsymbol{\Lambda}^T, \boldsymbol{q}, \boldsymbol{p}, \alpha). \tag{2.6}$$

5. If the constituents of a fiber-reinforced composite do not exhibit Hall effect, i.e. $\boldsymbol{\Lambda}(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x})$ is symmetric, then

$$\begin{bmatrix} \boldsymbol{\sigma} & \boldsymbol{p}+\boldsymbol{q} \\ \boldsymbol{q} & \alpha \end{bmatrix}^* = \begin{bmatrix} \boldsymbol{\sigma}^* & \boldsymbol{p}^*+\boldsymbol{q}^* \\ \boldsymbol{q}^* & \alpha^* \end{bmatrix}, \tag{2.7}$$

where

$$\begin{bmatrix} \boldsymbol{\sigma} & \boldsymbol{p} \\ \boldsymbol{q} & \alpha \end{bmatrix}^* = \begin{bmatrix} \boldsymbol{\sigma}^* & \boldsymbol{p}^* \\ \boldsymbol{q}^* & \alpha^* \end{bmatrix}.$$

6. If $\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{p}_0$ then

$$\boldsymbol{h}(\boldsymbol{\Lambda}, \boldsymbol{p}_0) = \boldsymbol{p}_0, \qquad \eta(\boldsymbol{\Lambda}, \boldsymbol{p}_0, \boldsymbol{q}, \alpha) = \langle \alpha \rangle, \tag{2.8}$$

or in other words

$$\boldsymbol{p}^* = \boldsymbol{p}_0, \qquad \alpha^* = \langle \alpha \rangle.$$

Here, as usual $\langle \cdot \rangle$ denotes the average of indicated components of the local tensor of a periodic composite over the period cell.

7. If $\boldsymbol{\Lambda}(\boldsymbol{x})\boldsymbol{e}_0 = \boldsymbol{j}_0$ then

$$\boldsymbol{H}(\boldsymbol{\Lambda}(\boldsymbol{x}))\boldsymbol{e}_0 = \boldsymbol{j}_0, \qquad (\boldsymbol{h}(\boldsymbol{\Lambda}^T, \boldsymbol{q}), \boldsymbol{e}_0) = (\langle \boldsymbol{q} \rangle, \boldsymbol{e}_0), \tag{2.9}$$

or in other words

$$\boldsymbol{\Lambda}^* \boldsymbol{e}_0 = \boldsymbol{j}_0, \qquad (\boldsymbol{q}^*, \boldsymbol{e}_0) = (\langle \boldsymbol{q} \rangle, \boldsymbol{e}_0).$$

8. If $\boldsymbol{\Lambda}(\boldsymbol{x}) = \boldsymbol{\Lambda}_0$ then

$$\boldsymbol{H}(\boldsymbol{\Lambda}_0) = \boldsymbol{\Lambda}_0, \qquad \boldsymbol{h}(\boldsymbol{\Lambda}_0, \boldsymbol{p}) = \langle \boldsymbol{p} \rangle, \tag{2.10}$$

or in other words

$$\boldsymbol{\Lambda}^* = \boldsymbol{\Lambda}_0, \qquad \boldsymbol{p}^* = \langle \boldsymbol{p} \rangle, \qquad \boldsymbol{q}^* = \langle \boldsymbol{q} \rangle.$$

It is instructive to compare our results with what one would get by a direct approach via the definition of the effective moduli for periodic composites. The formulas for the block-components of the effective tensor, as well as the discussion below can be found in [10]. We give it here for the sake of completeness.

Let us define an operator $\Gamma_{\boldsymbol{\Lambda}}$ on $L^2(Q;\mathbb{R}^2)$, given by

$$\Gamma_{\boldsymbol{\Lambda}}\boldsymbol{p} = \nabla\psi,$$

where $\psi(\boldsymbol{x})$ is the unique (up to an additive constant) Q-periodic solution of

$$-\nabla\cdot\boldsymbol{\Lambda}(\boldsymbol{x})\nabla\psi = \nabla\cdot\boldsymbol{p}(\boldsymbol{x}).$$

We may also extend the definition of $\Gamma_{\boldsymbol{\Lambda}}$ from $L^2(Q;\mathbb{R}^2)$ to $L^2(Q;\mathrm{End}(\mathbb{R}^2))$ by

$$(\Gamma_{\boldsymbol{\Lambda}}\boldsymbol{P})\boldsymbol{e} = \Gamma_{\boldsymbol{\Lambda}}(\boldsymbol{P}\boldsymbol{e})$$

for any $\boldsymbol{e}\in\mathbb{R}^2$. The effective conductivity $\boldsymbol{L}^*$ can be expressed in terms of $\Gamma_{\boldsymbol{\Lambda}}$ as follows

$$\begin{aligned} \boldsymbol{\Lambda}^* &= \langle\boldsymbol{\Lambda}\rangle + \langle\boldsymbol{\Lambda}\Gamma_{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}\rangle, \qquad & \boldsymbol{p}^* &= \langle\boldsymbol{p}\rangle + \langle\boldsymbol{\Lambda}\Gamma_{\boldsymbol{\Lambda}}\boldsymbol{p}\rangle,\\ \alpha^* &= \langle\alpha\rangle + \langle(\Gamma_{\boldsymbol{\Lambda}}\boldsymbol{p},\boldsymbol{q})\rangle, \qquad & \boldsymbol{q}^* &= \langle\boldsymbol{q}\rangle + \langle\boldsymbol{\Lambda}^T\Gamma_{\boldsymbol{\Lambda}^T}\boldsymbol{q}\rangle. \end{aligned} \tag{2.11}$$

The properties (2.2) and (2.3) are not immediately readable off the formulas (2.11), even though (2.3) can be proved by substituting the expressions for $\boldsymbol{\Lambda}'$, $\boldsymbol{p}'$, $\boldsymbol{q}'$ and α' into (2.11). The remaining microstructure-independent relations are more or less immediate consequences of (2.11).

The transpose symmetry (2.4) is a consequence of the easily proved property of the operator $\Gamma_{\boldsymbol{\Lambda}}$:

$$\langle(\Gamma_{\boldsymbol{\Lambda}}\boldsymbol{p},\boldsymbol{q})\rangle = \langle(\Gamma_{\boldsymbol{\Lambda}^T}\boldsymbol{q},\boldsymbol{p})\rangle.$$

The properties (2.5) and (2.7) are clearly readable off (2.11), however, (2.11) contains a bit more microstructure-independent information than (2.5) and (2.7). Namely, it is the linear dependence of $\boldsymbol{p}^*$ on $\boldsymbol{p}$, in the case when $\boldsymbol{\Lambda}(\boldsymbol{x})$ is not symmetric and the dependence of α^* on $\langle\alpha\rangle$ that do not follow from (2.5) and (2.7). In order for our theory of exact relations to pick up these properties, we should have considered links between *three* uncoupled problems.

Finally, the properties (2.8) and (2.9) follow from (2.11) and the fact that $\Gamma_{\boldsymbol{\Lambda}}\boldsymbol{p}_0 = \boldsymbol{0}$, while the property (2.10) follows from the relation $\langle\Gamma_{\boldsymbol{\Lambda}}\boldsymbol{p}\rangle = 0$. In [10] the author has applied the exact relations above to derive the explicit formulas for effective conductivity of polycrystals with statistically isotropic microstructure (see Section 3.1 for a brief summary) and transversely-isotropic two-phase composites with fibers in the shape of Vigdergauz inclusions [11, 23, 24, 25]. Also in [10] it was shown (see Section 3.2 for a brief summary) how our results could be used to reduce the computation of the effective conductivity of general two-phase fiber-reinforced composites with Hall effect to the computation of the effective conductivity of 2D composites without Hall effect. The formulas equivalent to ours have also been obtained and applied in [2, 3, 4, 20, 21].

3 Physical consequences

Our theory can be used in many ways. It identifies all the special cases when microstructure-independent information is available. The links are then used to map the composite in question to a different composite that falls into one or more of these special cases. Once explicit answers are available for a specific composite, they can be further studied for physically interesting effects. This has been done in a long series of papers by Bergman, Strelniker and others cited in the introduction. For comparison the reader may want to see [10], where the author has shown how one can apply the exact relations in the form given here to obtain explicit results for polycrystals and two-phase media. For the sake of completeness we restate some of these results here. For details the reader is referred to [10].

3.1 A polycrystal

Suppose that the conductivity of a single crystal is

$$\boldsymbol{L}_0 = \begin{bmatrix} \boldsymbol{\Lambda}_0 & \boldsymbol{p}_0 \\ \boldsymbol{q}_0 & \alpha_0 \end{bmatrix}.$$

Suppose that the magnetic field is directed along the fibers and the 2D texture in the transversal plane is isotropic. Then

$$\boldsymbol{L}^* = \begin{bmatrix} \sqrt{\det \boldsymbol{\sigma}_0}\boldsymbol{I} + r_0\boldsymbol{S} & \boldsymbol{0} \\ \boldsymbol{0} & \alpha_0 - \dfrac{(\boldsymbol{p}_0, \boldsymbol{q}_0)}{\operatorname{Tr} \boldsymbol{\sigma}_0} \end{bmatrix},$$

where $\boldsymbol{\Lambda}_0 = \boldsymbol{\sigma}_0 + r_0\boldsymbol{S}$ is the decomposition of $\boldsymbol{\Lambda}_0$ into the symmetric and antisymmetric parts.

3.2 Two-phase composites

The most obvious application of our exact relations is for understanding of how the effective Hall conductivity depends on the microstructure. Generically, the 3×3 effective tensor of a two-phase fiber-reinforced composite can be expressed in terms of a 2×2 symmetric conductivity tensor of a two-dimensional composite that has the same microstructure.

Suppose, for simplicity, that the $\boldsymbol{\Lambda}$ block of the constituent conductivities of a two-phase composite is isotropic, in other words,

$$\boldsymbol{L}_i = \begin{bmatrix} \sigma_i\boldsymbol{I} + r_i\boldsymbol{S} & \boldsymbol{p}_i \\ \boldsymbol{q}_i & \alpha_i \end{bmatrix}, \quad i = 1, 2.$$

Suppose in addition that the microstructure in the plane transversal to fibers is also isotropic or has a square symmetry. Then all components of the effective tensor $\boldsymbol{L}^*$ can be expressed in terms of a single scalar microstructure-dependent function $\mathfrak{S}^*(h)$, which is the effective conductivity of a 2D composite made with conductivities 1 and h. Assuming $\sigma_1 \neq \sigma_2$, we first need to solve

$$\frac{r_0 - r_1}{(r_0 - r_1)^2 + \sigma_1^2} = \frac{r_0 - r_2}{(r_0 - r_2)^2 + \sigma_2^2} \tag{3.1}$$

for r_0. If $r_1 \neq r_2$ this equation has two real roots. Choosing either root for r_0 does not affect the final answer. Next we compute

$$\Sigma^* = \frac{\sigma_1}{(r_0 - r_1)^2 + \sigma_1^2} \mathfrak{S}^* \left(\frac{\sigma_2(r_0 - r_1)}{\sigma_1(r_0 - r_2)} \right).$$

Then

$$\sigma^* = \frac{\Sigma^*}{(r_0')^2 + (\Sigma^*)^2}, \qquad r^* = r_0 - \frac{r_0'}{(r_0')^2 + (\Sigma^*)^2} \qquad r_0' = \frac{r_0 - r_1}{(r_0 - r_1)^2 + \sigma_1^2}.$$

Let

$$\boldsymbol{\Lambda}_i = \sigma_1 \boldsymbol{I} + r_i \boldsymbol{S}, \quad i = 1, 2, \qquad \boldsymbol{\Lambda}^* = \sigma^* \boldsymbol{I} + r^* \boldsymbol{S}.$$

Then

$$\boldsymbol{p}^* = (\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}^*)(\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2)^{-1}(\boldsymbol{p}_2 - \boldsymbol{p}_1) + \boldsymbol{p}_1, \tag{3.2}$$

$$\boldsymbol{q}^* = (\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}^*)^T(\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2)^{-T}(\boldsymbol{q}_2 - \boldsymbol{q}_1) + \boldsymbol{q}_1, \tag{3.3}$$

$$\alpha^* = \langle \alpha \rangle - ((\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2)^{-1}(\langle \boldsymbol{\Lambda} \rangle - \boldsymbol{\Lambda}^*)(\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2)^{-1}(\boldsymbol{p}_2 - \boldsymbol{p}_1), (\boldsymbol{q}_2 - \boldsymbol{q}_1)). \tag{3.4}$$

These relations together with 2D Mendelson's duality (1.4) (see [17]) have been derived in a different but equivalent form in [2, 3, 4, 20, 21].

3.3 Dependence of Hall conductivity on the magnetic field

Here we apply our exact relations and links to the question of possible constitutive dependence of the conductivity tensor on the magnetic field. In our opinion the observed magnetic field dependence should be preserved under homogenization, since most materials in nature are in one sense or another composite materials.

The symmetry considerations imply that symmetric part of $\boldsymbol{L}$ is an even function of the magnetic field, while the antisymmetric part is odd. Our formulas for the two-phase composite show that r_0 and r_0' are odd functions of the magnetic field while Σ^* is an even function. Therefore, σ^* is even, while r^* is odd. For the $\boldsymbol{p}$ and $\boldsymbol{q}$ blocks we know that $\boldsymbol{p} + \boldsymbol{q}$ is even, while $\boldsymbol{p} - \boldsymbol{q}$ is odd. Therefore,

$$\boldsymbol{p}(-\boldsymbol{h}) + \boldsymbol{q}(-\boldsymbol{h}) = \boldsymbol{p}(\boldsymbol{h}) + \boldsymbol{q}(\boldsymbol{h}), \qquad \boldsymbol{p}(-\boldsymbol{h}) - \boldsymbol{q}(-\boldsymbol{h}) = -\boldsymbol{p}(\boldsymbol{h}) + \boldsymbol{q}(\boldsymbol{h}).$$

Hence $\boldsymbol{p}(-\boldsymbol{h}) = \boldsymbol{q}(\boldsymbol{h})$ and $\boldsymbol{q}(-\boldsymbol{h}) = \boldsymbol{p}(\boldsymbol{h})$ and we see that the symmetric part of $\boldsymbol{L}^*$ is indeed even in $\boldsymbol{h}$, while the anti-symmetric part is odd.

Next we may ask about the behavior of $r(\boldsymbol{h})$ and $\sigma(\boldsymbol{h})$ for large values of $|\boldsymbol{h}|$. Let us assume that both materials in a composite have the same power law asymptotics: $\sigma_i \sim |\boldsymbol{h}|^\alpha$ and $r_i \sim |\boldsymbol{h}|^\beta$, $i = 1, 2$, when $|\boldsymbol{h}|$ is large. Then our formulas tell us the asymptotics of $\sigma^*(\boldsymbol{h})$

and $r^*(\boldsymbol{h})$. If $\beta \leq \alpha$ then $\mathfrak{S}^* = O(1)$, $r_0' \sim |\boldsymbol{h}|^{\beta-2\alpha}$ and $\sigma^* \sim |\boldsymbol{h}|^{\alpha}$, while $r^* \sim |\boldsymbol{h}|^{\beta}$. If $\beta > \alpha$ then

$$\frac{\sigma_2}{\sigma_1} \cdot \frac{r_0 - r_1}{r_0 - r_2} \sim |\boldsymbol{h}|^{2\alpha - 2\beta}.$$

Therefore, depending upon the microstructure, $\mathfrak{S}^*$ can have any asymptotics between $O(1)$ and $|\boldsymbol{h}|^{2\alpha-2\beta}$. Thus, in this case the microstructure may have a dramatic influence on the dependence of the ohmic conductivity on the magnetic field. We obtain that the asymptotics of σ^* can vary between $|\boldsymbol{h}|^{\alpha}$ and $|\boldsymbol{h}|^{\beta}$. At the same time, the asymptotics of $r^* \sim |\boldsymbol{h}|^{\beta}$ holds regardless of the microstructure. We conclude that for the power-law asymptotics to be stable with respect to making composites we need to require that $\alpha \geq \beta$.

In a free electron model, the ohmic resistivity is not altered by the magnetic field, while the skew-symmetric part of the resistivity tensor (Hall part) is homogeneous of degree 1 in $\boldsymbol{h}$ as in the formula for the Lorentz force. Hence, computing the conductivity tensor, we get the values $\alpha = 0$ and $\beta = -1$. This asymptotics is stable with respect to making composites, since $\alpha > \beta$. Therefore, we expect that the ohmic conductivity will saturate in a strong magnetic field. A different kind of asymptotics is possible if the inclusions are either perfectly insulating or superconducting, (see e.g. [5]).

4 Notions and notations

Algebra	a vector space with multiplication satisfying distributive law.
Euclidean space	A vector space with an inner product. For example $\mathbb{R}^3$ with a dot product is a Euclidean space.
End(V)	the algebra of all linear maps from a vector space V into itself. Linear combinations of maps are defined in the usual way and multiplication is a composition of maps. Hence the space End($\mathbb{R}^3$) is a space of 3×3 matrices with usual matrix addition and multiplication. If V is a Euclidean space then End(V) is also a Euclidean space with inner product $(\boldsymbol{A}, \boldsymbol{B}) = \mathrm{Tr}\,(\boldsymbol{A}^T\boldsymbol{B})$, where $\boldsymbol{A}^T$ is the transpose (Euclidean adjoint) of $\boldsymbol{A}$.
End$^+$(V)	the space of all positive definite linear maps on V, when V is a Euclidean space
Span($\mathcal{S}$)	the set of all finite linear combinations of elements of the set $\mathcal{S}$. When $\mathcal{S}$ is a subset of a vector space V, Span($\mathcal{S}$) is the smallest subspace of V containing $\mathcal{S}$.
$\boldsymbol{a} \otimes \boldsymbol{b}$	the matrix $\boldsymbol{A}$ with components $A_{ij} = a_i b_j$. Such a matrix always has rank 1. It has the properties $\boldsymbol{A}^T = \boldsymbol{b} \otimes \boldsymbol{a}$ and $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{a}(\boldsymbol{b}, \boldsymbol{x})$, where $(\boldsymbol{b}, \boldsymbol{x})$ is the dot (inner) product of $\boldsymbol{b}$ and $\boldsymbol{x}$.

$V_1 \oplus V_2$	direct sum of two vector spaces. It consists of all pairs $[\boldsymbol{v}_1, \boldsymbol{v}_2]$, where $\boldsymbol{v}_1 \in V_1$ and $\boldsymbol{v}_2 \in V_2$. The vector space operations are defined component-wise.
$\mathcal{L}^\perp$	the orthogonal complement of a subspace $\mathcal{L}$ in a specified larger Euclidean space V: $\mathcal{L}^\perp = \{\boldsymbol{v} \in V : (\boldsymbol{v}, \boldsymbol{w}) = 0 \text{ for all } \boldsymbol{w} \in \mathcal{L}\}$.
$\ker(\boldsymbol{A})$	kernel or null-space of a linear map $\boldsymbol{A}$: $\ker(\boldsymbol{A}) = \{\boldsymbol{x} \in V : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$.
Ideals $\mathcal{I}$, Π^{*2}	the subalgebra of the algebra Π such that $\boldsymbol{P} * \boldsymbol{N} \in \mathcal{I}$ for all $\boldsymbol{P} \in \Pi$ and all $\boldsymbol{N} \in \mathcal{I}$, where $\boldsymbol{P} * \boldsymbol{N}$ denotes the (commutative) multiplication of elements in algebra Π. $\Pi^{*2} = \mathrm{Span}\{\boldsymbol{P} * \boldsymbol{P} : \boldsymbol{P} \in \Pi\}$ is called the derived ideal of the (commutative) algebra.
$\Pi/\mathcal{I}$	factor algebra consists of equivalence classes with respect to the equivalence relation $\boldsymbol{P}_1 \sim \boldsymbol{P}_2$ if $\boldsymbol{P}_1 - \boldsymbol{P}_2 \in \mathcal{I}$. The product of two equivalence classes is defined to be the equivalence class containing the product of any two representatives from the equivalence classes being multiplied. The product is independent of the choice of the representative of each class.
$[\![\boldsymbol{P}]\!]$	the equivalence class in $\Pi/\mathcal{I}$ containing element $\boldsymbol{P} \in \Pi$.
Isomorphism	An invertible linear map between two algebras preserving multiplication.
Automorphism	An Jordan isomorphism of an algebra onto itself.
Homomorphism	A linear map (not necessarily invertible) between two algebras preserving multiplication. A kernel of a Jordan homomorphism is always an ideal.

5 A general theory

Exact relations are functions of $\boldsymbol{L}^*$ that do not depend on the microstructure. For example the function $\boldsymbol{\sigma}^* \mapsto \det \boldsymbol{\sigma}^*$ is independent of the microstructure in the context of 2D conductivity. It is helpful to think about exact relations in a geometric way. The "level surfaces" of these functions, i.e. sets of tensors $\boldsymbol{L}$, where the microstructure-independent functions are constant represent the exact relations geometrically. For example, the exact relation responsible for (1.2) is a 4-dimensional surface in a 9-dimensional space of all 3×3 matrices. Links can also be thought of as surfaces, but in the 18-dimensional space of pairs of 3×3 matrices. In the context of 2D conductivity, the equation (1.4) defines a 3D surface in a 6 dimensional space of pairs of 2×2 symmetric matrices. The definitions below formalize our geometric point of view.

Definition 5.1. *A surface* $\mathbb{M} \subset \mathrm{End}^+(\mathbb{R}^3)$ *is an exact relation if the effective tensor* $\boldsymbol{L}^*$ *of* any *composite made with materials taken from* $\mathbb{M}$ *belongs to* $\mathbb{M}$.

The surface $\mathbb{M}$ can be given by a system of equations constraining components of a 3×3 matrix. If all conductivity tensors of materials in a mixture satisfy these equations then the components of the effective tensor will necessarily satisfy the same system of equations, provided $\mathbb{M}$ is an exact relation. In most cases the system of equations above is not written as a collection of individual equations, but rather given as one or more matrix identities, as in Section 2.

Definition 5.2. *A function* $\boldsymbol{f}(\boldsymbol{L})$ *is called a volume fraction relation if* $\boldsymbol{f}(\boldsymbol{L}^*) = \langle \boldsymbol{f}(\boldsymbol{L}(\boldsymbol{x}))\rangle$.

For example, $\boldsymbol{f}(\boldsymbol{L}) = [\boldsymbol{p}, \boldsymbol{q}]$ in item 8 in Section 2.

Definition 5.3. *A surface* $\widehat{\mathbb{M}} \subset \mathbb{E} = \mathrm{End}^+(\mathbb{R}^3) \times \mathrm{End}^+(\mathbb{R}^3)$ *is called a link if the effective tensors* $\boldsymbol{L}_1^*$ *and* $\boldsymbol{L}_2^*$ *of two composites are related via* $[\boldsymbol{L}_1^*, \boldsymbol{L}_2^*] \in \widehat{\mathbb{M}}$ *provided their local tensors* $\boldsymbol{L}_1(\boldsymbol{x})$ *and* $\boldsymbol{L}_2(\boldsymbol{x})$ *are related by* $[\boldsymbol{L}_1(\boldsymbol{x}), \boldsymbol{L}_2(\boldsymbol{x})] \in \widehat{\mathbb{M}}$ *for all* $\boldsymbol{x}$ *in a period cell* Q.

For example, $\widehat{\mathbb{M}}$ can be a set of pairs $\{[\boldsymbol{L}, \boldsymbol{L}^T] : \boldsymbol{L} \in \mathrm{End}^+(\mathbb{R}^3)\}$. It corresponds to the link (2.4). Another example is $\widehat{\mathbb{M}} = \{[\boldsymbol{L}_1, \boldsymbol{L}_2] \in \mathbb{E} : \boldsymbol{\Lambda}_1 = \boldsymbol{\Lambda}_2\}$. It corresponds to the fact in item 4 in Section 2 that $\boldsymbol{\Lambda}^*$ depends only on $\boldsymbol{\Lambda}(\boldsymbol{x})$.

General theory [18, Chapter 17] (see also [9, 12]) tells us that in order to find all exact relation surfaces $\mathbb{M}$ passing through a given tensor $\boldsymbol{L}_0$ we need to find all subspaces $\Pi \subset \mathrm{End}(\mathbb{R}^3)$ with additional property:

$$\boldsymbol{P}\boldsymbol{A}\boldsymbol{P} \in \Pi \tag{5.1}$$

for all $\boldsymbol{P} \in \Pi$ and all $\boldsymbol{A} \in \mathcal{A}$, where

$$\mathcal{A} = \mathrm{Span}\left\{ \frac{\boldsymbol{L}_0\boldsymbol{n} \otimes \boldsymbol{n}}{(\boldsymbol{L}_0\boldsymbol{n}, \boldsymbol{n})} - \frac{\boldsymbol{L}_0\boldsymbol{n}_0 \otimes \boldsymbol{n}_0}{(\boldsymbol{L}_0\boldsymbol{n}_0, \boldsymbol{n}_0)} : \boldsymbol{n} = [n_1, n_2, 0],\ |\boldsymbol{n}| = 1 \right\}. \tag{5.2}$$

It is an easy exercise to prove that $\mathcal{A}$ is independent of the choice of the unit vector $\boldsymbol{n}_0$. The subspace $\mathcal{A}$ can be computed explicitly for each choice of $\boldsymbol{L}_0$. We will see shortly that we only need to compute $\mathcal{A}$ and solve (5.1) for one choice of $\boldsymbol{L}_0$ (we will choose $\boldsymbol{L}_0 = \boldsymbol{I}$). Then all solutions Π of (5.1) for any choice of $\boldsymbol{L}_0$ can be easily reconstructed from the already computed solutions. We remark here that exact relations in other physical contexts, such as elasticity or piezo-electricity can be identified by solving (5.1) with a different choice of $\mathcal{A}$ and $\mathbb{R}^3$ replaced with an appropriate higher dimensional space. We refer to [12, 18] for details.

Substituting $\boldsymbol{P} = \boldsymbol{P}_1 + \boldsymbol{P}_2$ into (5.1) and expanding we obtain that

$$\boldsymbol{P}_1\boldsymbol{A}\boldsymbol{P}_2 + \boldsymbol{P}_2\boldsymbol{A}\boldsymbol{P}_1 \in \Pi$$

for all $\{\boldsymbol{P}_1, \boldsymbol{P}_2\} \subset \Pi$ and all $\boldsymbol{A} \in \mathcal{A}$. The subspace Π therefore becomes endowed with *a family* of multiplications parameterized by $\boldsymbol{A} \in \mathcal{A}$

$$\boldsymbol{P}_1 *_{\boldsymbol{A}} \boldsymbol{P}_2 = \frac{1}{2}(\boldsymbol{P}_1\boldsymbol{A}\boldsymbol{P}_2 + \boldsymbol{P}_2\boldsymbol{A}\boldsymbol{P}_1). \tag{5.3}$$

Vector spaces with multiplications are ubiquitous in both pure mathematics and theoretical physics (quantum groups is a prominent example). They are called *algebras*. Hence, the property (5.1) of a subspace Π corresponding to an exact relation makes it into an algebra. The multiplication $*_{\boldsymbol{A}}$ defined by (5.3) is obviously commutative, but not associative. Such multiplications have been encountered before in algebra and in quantum mechanics[1] (see [14, 15]). The algebras with such a multiplication are called *Jordan algebras* after German theoretical physicist Pascual Jordan. The new twist here is that Π is a Jordan algebra with respect to an infinite family[2] of multiplications parameterized by $\boldsymbol{A} \in \mathcal{A}$.

According to the general theory [9, 12, 18], the points of Π parameterize an exact relation surface $\mathbb{M}$ via

$$\mathbb{M} \ni \boldsymbol{L} = \boldsymbol{L}_0 - [\boldsymbol{I} + \boldsymbol{P}\boldsymbol{M}]^{-1}\boldsymbol{P}\boldsymbol{L}_0, \quad \boldsymbol{P} \in \Pi \tag{5.4}$$

where $\boldsymbol{M}$ is any 3×3 matrix that has the property

$$\boldsymbol{P}(\boldsymbol{\Gamma}(\boldsymbol{n}_0) - \boldsymbol{M})\boldsymbol{P} \in \Pi \tag{5.5}$$

for any $\boldsymbol{P} \in \Pi$, and where

$$\boldsymbol{\Gamma}(\boldsymbol{n}) = \frac{\boldsymbol{L}_0\boldsymbol{n} \otimes \boldsymbol{n}}{(\boldsymbol{L}_0\boldsymbol{n}, \boldsymbol{n})}.$$

The surface $\mathbb{M}$ is independent of the choice of $\boldsymbol{n}_0$ or $\boldsymbol{M}$, satisfying (5.5). Changing $\boldsymbol{n}_0$ and $\boldsymbol{M}$ will only produce another parameterization of $\mathbb{M}$. Obviously, $\boldsymbol{M} = \boldsymbol{\Gamma}(\boldsymbol{n}_0)$ satisfies (5.5). However, in some cases (quite often, in our experience) $\boldsymbol{M} = \boldsymbol{0}$ satisfies (5.5) as well. This choice will then simplify (5.4) tremendously.

In addition to exact relations we have a way of recognizing when the relations involving volume averages are also present. These additional relations appear whenever the subspace Π^{*2} defined by

$$\Pi^{*2} = \mathrm{Span}\{\boldsymbol{P}\boldsymbol{A}\boldsymbol{P} : \boldsymbol{P} \in \Pi,\ \boldsymbol{A} \in \mathcal{A}\}$$

is strictly smaller than Π. In this case these additional relations can be described as follows. Let $\mathcal{P}_{(\Pi^{*2})^\perp}$ be the orthogonal projection onto the orthogonal complement of Π^{*2} in Π. Then

$$\mathcal{P}_{(\Pi^{*2})^\perp}\langle \boldsymbol{P}(\boldsymbol{L}(\boldsymbol{x}))\rangle = \mathcal{P}_{(\Pi^{*2})^\perp}\boldsymbol{P}(\boldsymbol{L}^*), \tag{5.6}$$

where

$$\boldsymbol{P}(\boldsymbol{L}) = \left[\boldsymbol{I} - \boldsymbol{M}\left(\boldsymbol{I} - \boldsymbol{L}\boldsymbol{L}_0^{-1}\right)\right]^{-1}\left(\boldsymbol{I} - \boldsymbol{L}\boldsymbol{L}_0^{-1}\right) \tag{5.7}$$

and $\boldsymbol{M}$ is any 3×3 matrix satisfying (5.5). Here (5.7) is just an inverse transformation of (5.4).

Now let us turn to the links between exact relations. The links are nothing more than the exact relations for the two uncoupled Hall-conductivity problems. As such they can be thought of as surfaces in the 18-dimensional space $V = \mathrm{End}(\mathbb{R}^3) \oplus \mathrm{End}(\mathbb{R}^3)$. Similarly to

[1] Jordan algebras are no longer used in modern description.

[2] Equivalently, Π is an algebra with respect to finitely many multiplications generated by any basis of $\mathcal{A}$.

exact relations, the links $\widehat{\mathbb{M}}$ passing through the point $[\boldsymbol{L}_1^0, \boldsymbol{L}_2^0] \in V$, can be parameterized by subspaces $\widehat{\Pi} \subset V$, satisfying

$$\widehat{\boldsymbol{P}}\widehat{\boldsymbol{A}}\widehat{\boldsymbol{P}} \in \widehat{\Pi} \tag{5.8}$$

for all $\widehat{\boldsymbol{P}} \in \widehat{\Pi}$ and all $\widehat{\boldsymbol{A}} \in \widehat{\mathcal{A}}$, where

$$\widehat{\mathcal{A}} = \mathrm{Span}\{[\boldsymbol{A}(\boldsymbol{L}_1^0, \boldsymbol{n}, \boldsymbol{n}_0), \boldsymbol{A}(\boldsymbol{L}_2^0, \boldsymbol{n}, \boldsymbol{n}_0)]\}, \quad \boldsymbol{A}(\boldsymbol{L}, \boldsymbol{n}, \boldsymbol{n}_0) = \frac{\boldsymbol{L}\boldsymbol{n} \otimes \boldsymbol{n}}{(\boldsymbol{L}\boldsymbol{n}, \boldsymbol{n})} - \frac{\boldsymbol{L}\boldsymbol{n}_0 \otimes \boldsymbol{n}_0}{(\boldsymbol{L}\boldsymbol{n}_0, \boldsymbol{n}_0)}. \tag{5.9}$$

The correspondence between the subspace $\widehat{\Pi}$ and the link $\widehat{\mathbb{M}}$ passing through $\widehat{\boldsymbol{L}}_0 = [\boldsymbol{L}_1^0, \boldsymbol{L}_2^0] \in V$ is given by the formula, obtained from (5.4) by replacing $\boldsymbol{P}$ with $\widehat{\boldsymbol{P}}$ and $\boldsymbol{L}_0$ with $\widehat{\boldsymbol{L}}_0$:

$$\widehat{\boldsymbol{L}} = \widehat{\boldsymbol{L}}_0 - \left[\widehat{\boldsymbol{I}} + \widehat{\boldsymbol{P}}\widehat{\boldsymbol{M}}\right]^{-1} \widehat{\boldsymbol{P}}\widehat{\boldsymbol{L}_0}, \tag{5.10}$$

where $\widehat{\boldsymbol{M}} = [\boldsymbol{M}_1, \boldsymbol{M}_2]$ has the property

$$\widehat{\boldsymbol{P}}(\widehat{\boldsymbol{\Gamma}}(\boldsymbol{n}_0) - \widehat{\boldsymbol{M}})\widehat{\boldsymbol{P}} \in \widehat{\Pi} \tag{5.11}$$

for any $\widehat{\boldsymbol{P}} \in \widehat{\Pi}$, and where

$$\widehat{\boldsymbol{\Gamma}}(\boldsymbol{n}) = \left[\frac{\boldsymbol{L}_1^0\boldsymbol{n} \otimes \boldsymbol{n}}{(\boldsymbol{L}_1^0\boldsymbol{n}, \boldsymbol{n})}, \frac{\boldsymbol{L}_2^0\boldsymbol{n} \otimes \boldsymbol{n}}{(\boldsymbol{L}_2^0\boldsymbol{n}, \boldsymbol{n})}\right].$$

We can construct all subspaces $\widehat{\Pi}$ satisfying (5.8) from the list of all solutions Π of (5.1), provided we also understand their Jordan algebra structure, see [9, 12]. We explain how this is done here for the sake of completeness.

For any link $\widehat{\Pi}$ let Π_1 and Π_2 be its canonical projections onto the first and the second copy of $\mathrm{End}(\mathbb{R}^3)$ in V, respectively. In other words, Π_1 is simply the set of all matrices $\boldsymbol{P}_1$ that occur in the first position in points $[\boldsymbol{P}_1, \boldsymbol{P}_2] \in \widehat{\Pi}$. Similarly, Π_2 is the set of all matrices $\boldsymbol{P}_2$ that occur in the second position in points of $\widehat{\Pi}$. If we denote by $\mathcal{P}_1$ and $\mathcal{P}_2$ the projections onto the first and the second component in V, respectively, then $\Pi_i = \mathcal{P}_i(\widehat{\Pi})$, $i = 1, 2$. It is easy to check that both $\mathcal{P}_1$ and $\mathcal{P}_2$ are Jordan homomorphisms. Therefore, both Π_1 and Π_2 must solve (5.1). However, the knowledge of Π_1 and Π_2 is insufficient for reconstruction of $\widehat{\Pi}$. In order to recover $\widehat{\Pi}$ we also need to know the sets $\mathcal{I}_1 = \{\boldsymbol{P}_1 \in \Pi_1 : [\boldsymbol{P}_1, \boldsymbol{0}] \in \widehat{\Pi}\}$ and $\mathcal{I}_2 = \{\boldsymbol{P}_2 \in \Pi_2 : [\boldsymbol{0}, \boldsymbol{P}_2] \in \widehat{\Pi}\}$. Clearly, $\mathcal{I}_1 = \mathcal{P}_1(\ker(\mathcal{P}_2))$ and $\mathcal{I}_2 = \mathcal{P}_2(\ker(\mathcal{P}_1))$. Therefore, $\mathcal{I}_i$ is an ideal in Π_i, $i = 1, 2$. In order to reconstruct $\widehat{\Pi}$ from Π_i and $\mathcal{I}_i$, $i = 1, 2$ we observe that $\widehat{\Pi}$ induces a bijective linear map Φ from $\Pi_1/\mathcal{I}_1$ onto $\Pi_2/\mathcal{I}_2$ according to the rule $[\![\boldsymbol{P}_2]\!] = \Phi([\![\boldsymbol{P}_1]\!])$, provided $[\boldsymbol{P}_1, \boldsymbol{P}_2] \in \widehat{\Pi}$, where $[\![\boldsymbol{P}_i]\!]$ denotes the equivalence class in $\Pi_i/\mathcal{I}_i$ containing $\boldsymbol{P}_i$, $i = 1, 2$. It is easy to check that the map Φ also preserves factor-algebra multiplication. Therefore, Φ is a Jordan isomorphismbetween $\Pi_1/\mathcal{I}_1$ and $\Pi_2/\mathcal{I}_2$. This analysis gives a procedure for computing all links $\widehat{\Pi}$. We first identify all ideals $\mathcal{I}$ in all algebras Π. We then classify distinct factors $\Pi/\mathcal{I}$ as either isomorphic to one of the Jordan algebras we have encountered before or as not isomorphic to anything we've seen so far. This permits us to identify all Jordan isomorphisms

$$\Phi:\Pi_1/\mathcal{I}_1\to\Pi_2/\mathcal{I}_2.$$

Then

$$\widehat{\Pi}=\{[\boldsymbol{P}_1,\boldsymbol{P}_2]\in\Pi_1\oplus\Pi_2:[\![\boldsymbol{P}_2]\!]=\Phi([\![\boldsymbol{P}_1]\!])\}. \tag{5.12}$$

The equivalence classes in $\Pi_i/\mathcal{I}_i$ can be labeled by elements of $\mathcal{I}_i^{\perp}$—the orthogonal complement of $\mathcal{I}_i$ in Π_i, $i=1,2$. Then the map Φ can be thought of as a linear isomorphism between $\mathcal{I}_1^{\perp}$ and $\mathcal{I}_2^{\perp}$. Therefore, the link corresponding to $\widehat{\Pi}$ can also be represented as

$$\widehat{\mathbb{M}}=\{[\boldsymbol{L}_1,\boldsymbol{L}_2]\in\mathbb{E}:\Phi(\mathcal{P}_{\mathcal{I}_1^{\perp}}\boldsymbol{P}(\boldsymbol{L}_1))=\mathcal{P}_{\mathcal{I}_2^{\perp}}\boldsymbol{P}(\boldsymbol{L}_2)\}, \tag{5.13}$$

where $\mathcal{P}_{\mathcal{I}_i^{\perp}}$ is the orthogonal projection onto the orthogonal complement of $\mathcal{I}_i$ in Π_i, $i=1,2$ and $\boldsymbol{P}(\boldsymbol{L})$ is given by (5.7).

6 Simplifying the problem

The first step in the application of our general theory is finding all solutions Π of (5.1). However, the subspace $\mathcal{A}$ depends on the 9 components of the reference conductivity $\boldsymbol{L}_0$. This dependence is not essential, however, and can be eliminated. Exactly the same problem afflicts the equation (5.8) needed to compute links.

6.1 Simplifying subspaces $\mathcal{A}$ and $\widehat{\mathcal{A}}$

Let $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ be invertible 3×3 matrices. Let

$$\mathcal{A}_0=\boldsymbol{C}_2\mathcal{A}\boldsymbol{C}_1,\quad \Pi_0=\boldsymbol{C}_1^{-1}\Pi\boldsymbol{C}_2^{-1}. \tag{6.1}$$

Then Π_0 satisfies (5.1) with $\mathcal{A}$ replaced by $\mathcal{A}_0$. This simple observation allows us to simplify the subspace $\mathcal{A}$ by choosing matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$.

Claim **6.1.** *There exist invertible 3×3 matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ such that*

$$\mathcal{A}_0=\left\{\begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & 0\end{bmatrix}:\boldsymbol{A}^T=\boldsymbol{A},\ \mathrm{Tr}\,\boldsymbol{A}=0\right\}, \tag{6.2}$$

where $\mathcal{A}_0$ is given by (6.1).

Proof. Suppose that the matrix $\boldsymbol{L}_0$ used in (5.2) has the form

$$\boldsymbol{L}_0=\begin{bmatrix}\boldsymbol{\Lambda}_0 & \boldsymbol{p}_0\\ \boldsymbol{q}_0 & \alpha_0\end{bmatrix}.$$

Let $\boldsymbol{\sigma}_0 = (\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_0^T)/2$ be the symmetric part of $\boldsymbol{\Lambda}_0$. Let $\boldsymbol{n} = [\boldsymbol{n}', 0]$, where $\boldsymbol{n}' = [n_1, n_2]$. Then

$$\boldsymbol{\Gamma}(\boldsymbol{n}) = \frac{1}{(\boldsymbol{\sigma}_0\boldsymbol{n}', \boldsymbol{n}')}\begin{bmatrix} \boldsymbol{\Lambda}_0\boldsymbol{n}' \otimes \boldsymbol{n}' & \boldsymbol{0} \\ (\boldsymbol{q}_0, \boldsymbol{n}')\boldsymbol{n}' & 0 \end{bmatrix}.$$

Choosing

$$\boldsymbol{C}_1 = \begin{bmatrix} \boldsymbol{\sigma}_0^{1/2} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}, \qquad \boldsymbol{C}_2 = \begin{bmatrix} \boldsymbol{\sigma}_0^{1/2}\boldsymbol{\Lambda}_0^{-1} & \boldsymbol{0} \\ -\boldsymbol{\Lambda}_0^{-T}\boldsymbol{q}_0 & 1 \end{bmatrix}, \tag{6.3}$$

we obtain

$$\boldsymbol{C}_2\boldsymbol{\Gamma}(\boldsymbol{n})\boldsymbol{C}_1 = \begin{bmatrix} \boldsymbol{m} \otimes \boldsymbol{m} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix}, \qquad \boldsymbol{m} = \frac{\boldsymbol{\sigma}_0^{1/2}\boldsymbol{n}'}{|\boldsymbol{\sigma}_0^{1/2}\boldsymbol{n}'|}.$$

Hence $\mathcal{A}_0$ is given by (6.2). □

Let us now turn to the subspace $\widehat{\mathcal{A}}$, given by (5.9). We apply the simplifying transformation (6.1) to each component of the link:

$$[\boldsymbol{P}, \boldsymbol{P}'] \mapsto [\boldsymbol{C}_1^{-1}\boldsymbol{P}\boldsymbol{C}_2^{-1}, \boldsymbol{C}_1'^{-1}\boldsymbol{P}'\boldsymbol{C}_2'^{-1}].$$

Then, according to Claim 6.1, $\widehat{\mathcal{A}}$ transforms into

$$\widehat{\mathcal{A}}_0 = \{[\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{A}}'] : \widehat{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix},\ \widehat{\boldsymbol{A}}' = \begin{bmatrix} \boldsymbol{A}' & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix},\ [\boldsymbol{A}, \boldsymbol{A}'] \in \widehat{\mathcal{A}}_{2d}\},$$

where

$$\widehat{\mathcal{A}}_{2d} = \text{Span}\left\{\left[\boldsymbol{m} \otimes \boldsymbol{m} - \boldsymbol{m}_0 \otimes \boldsymbol{m}_0, \frac{\boldsymbol{Q}\boldsymbol{m} \otimes \boldsymbol{Q}\boldsymbol{m}}{|\boldsymbol{Q}\boldsymbol{m}|^2} - \frac{\boldsymbol{Q}\boldsymbol{m}_0 \otimes \boldsymbol{Q}\boldsymbol{m}_0}{|\boldsymbol{Q}\boldsymbol{m}_0|^2}\right] : |\boldsymbol{m}| = 1\right\}$$

and where $\boldsymbol{Q} = \boldsymbol{\sigma}_2^{1/2}\boldsymbol{\sigma}_1^{-1/2}$.

Claim **6.2.** *If $\boldsymbol{\sigma}_2 = s\boldsymbol{\sigma}_1$ for some scalar $s > 0$, then $\widehat{\mathcal{A}}_0 = \{[\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{A}}] : \widehat{\boldsymbol{A}} \in \mathcal{A}_0\}$. If $\boldsymbol{\sigma}_2 \neq s\boldsymbol{\sigma}_1$ then $\widehat{\mathcal{A}}_0 = \mathcal{A}_0 \oplus \mathcal{A}_0$.*

This claim was stated without proof in [9, Theorem 15]. To prove this claim we will need a complex variable formalism that we have used in [9]. To a complex number $z = a + ib$ we associate a vector $\boldsymbol{z} = (a, b) \in \mathbb{R}^2$ and two 2×2 matrices:

$$\phi(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ and } \psi(z) = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}. \tag{6.4}$$

The functions ϕ and ψ enjoy many special properties. For example,

$$\phi(z)\boldsymbol{u} = \boldsymbol{v} \Rightarrow v = zu, \qquad \psi(z)\boldsymbol{u} = \boldsymbol{w} \Rightarrow w = z\overline{u}.$$

As a corollary we have the following multiplicative identities:

$$\phi(z_1)\phi(z_2)=\phi(z_1z_2),\ \phi(z_1)\psi(z_2)=\psi(z_1z_2),\ \psi(z_1)\phi(z_2)=\psi(z_1\overline{z_2}),\ \psi(z_1)\psi(z_2)=\phi(z_1\overline{z_2}).$$

We may also represent every 2×2 matrix $\boldsymbol{Q}$ in the form

$$\boldsymbol{Q}=\phi(q_1)+\psi(q_2) \tag{6.5}$$

in a unique way for some $\{q_1,q_2\}\subset\mathbb{C}$.

Proof. If $\boldsymbol{\sigma}_2=s\boldsymbol{\sigma}_1$ then $\boldsymbol{Q}=\sqrt{s}\boldsymbol{I}$ and the statement of the theorem easily follows. Conversely, if $\boldsymbol{\sigma}_2\neq s\boldsymbol{\sigma}_1$ then the matrix $\boldsymbol{Q}$ is not a multiple of the identity. In that case, $\boldsymbol{Q}$ is a diagonalizable 2×2 matrix with two positive and distinct eigenvalues.[3] For a 2×2 matrix this is equivalent to

$$\mathrm{Tr}\,\boldsymbol{Q}>0,\qquad 0<\det\boldsymbol{Q}<\frac{1}{4}(\mathrm{Tr}\,\boldsymbol{Q})^2. \tag{6.6}$$

If we represent $\boldsymbol{Q}$ in the form (6.5), then conditions (6.6) can be written as

$$\mathfrak{Re}(q_1)>0,\qquad |\mathfrak{Im}(q_1)|<|q_2|<|q_1|. \tag{6.7}$$

Our goal is to prove that there are no linear relations between the two block-components in the definition of $\widehat{\mathcal{A}}_{2d}$. Suppose, that there is a linear relation. In other words, assume that there exist $\{a,b\}\subset\mathbb{C}$ such that

$$(\psi(a)\boldsymbol{n},\boldsymbol{n})-(\psi(a)\boldsymbol{n}_0,\boldsymbol{n}_0)=\frac{(\psi(b)\boldsymbol{Q}\boldsymbol{n},\boldsymbol{Q}\boldsymbol{n})}{|\boldsymbol{Q}\boldsymbol{n}|^2}-\frac{(\psi(b)\boldsymbol{Q}\boldsymbol{n}_0,\boldsymbol{Q}\boldsymbol{n}_0)}{|\boldsymbol{Q}\boldsymbol{n}_0|^2}.$$

Formulated in another way, the expression

$$f(\boldsymbol{n})=(\psi(a)\boldsymbol{n},\boldsymbol{n})-\frac{(\psi(b)\boldsymbol{Q}\boldsymbol{n},\boldsymbol{Q}\boldsymbol{n})}{|\boldsymbol{Q}\boldsymbol{n}|^2}$$

does not depend on the choice of the unit vector $\boldsymbol{n}\in\mathbb{R}^2$. Rewriting everything using complex variable formalism we get that

$$f(\boldsymbol{n})=\mathfrak{Re}(ae^{-2i\alpha})-\frac{\mathfrak{Re}(b(\bar{q}_1e^{-i\alpha}+\bar{q}_2e^{i\alpha})^2)}{|q_1e^{i\alpha}+q_2e^{-i\alpha}|^2},$$

where $\boldsymbol{n}$ is represented by $e^{i\alpha}$. Simplifying, we get

$$f(\boldsymbol{n})=\mathfrak{Re}\left(ae^{-2i\alpha}-b\frac{\bar{q}_1e^{-i\alpha}+\bar{q}_2e^{i\alpha}}{q_1e^{i\alpha}+q_2e^{-i\alpha}}\right).$$

[3] An $n\times n$ matrix is a product of two symmetric positive definite matrices if and only if it is diagonalizable and has all positive eigenvalues. Indeed, $\boldsymbol{AB}$ is similar to $\boldsymbol{B}^{1/2}\boldsymbol{AB}^{1/2}$ and $\boldsymbol{CDC}^{-1}=\boldsymbol{CDC}^T(\boldsymbol{CC}^T)^{-1}$.

Let $z = e^{-i\alpha}$. Then our requirement of the constancy of $f(\boldsymbol{n})$ can be stated as the constancy of $\mathfrak{Re}(F(z))$ on the unit circle $|z| = 1$, where

$$F(z) = az^2 - b\frac{\bar{q}_1 z^2 + \bar{q}_2}{q_1 + q_2 z^2}.$$

The function $F(z)$ is analytic on the closed unit disk, since $|q_1| > |q_2|$. Thus, $F(z)$ has to be constant on $\mathbb{C}$. It follows that $a = 0$, because $q_2 \neq 0$, due to (6.7). But then $b = 0$, because $|q_1| \neq |q_2|$. Hence, $\widehat{\mathcal{A}}_{2d}$ has a maximal dimension of 4. □

We will see later that there is no need to consider links for which $\widehat{\mathcal{A}}_0 = \mathcal{A}_0 \oplus \mathcal{A}_0$. Hence we will focus exclusively on the case $\boldsymbol{\sigma}_2 = s\boldsymbol{\sigma}_1$.

6.2 Simplifying the inversion formulas for exact relations and links

The ultimate goal of the theory is to compute all of the exact relations passing through a given reference tensor $\boldsymbol{L}_0$ and all links passing through a pair of given reference tensors $[\boldsymbol{L}_1^0, \boldsymbol{L}_2^0]$. These are given by (5.4) for exact relations and by (5.10) for links, in terms of the Jordan algebras Π and $\widehat{\Pi}$. However, as explained in Section 6.1, we actually compute $\Pi_0 = \boldsymbol{C}_1^{-1}\Pi\boldsymbol{C}_2^{-1}$ and $\widehat{\Pi}_0 = \widehat{\boldsymbol{C}}_1^{-1}\widehat{\Pi}_0\widehat{\boldsymbol{C}}_2^{-1}$, instead of Π and $\widehat{\Pi}$, where $\widehat{\boldsymbol{C}}_j = [\boldsymbol{C}_j, \boldsymbol{C}_j']$, $j = 1, 2$. For this reason, we need to write the inversion formulas in terms of Π_0 and $\widehat{\Pi}_0$. Let $\boldsymbol{L}_1^0$ and $\boldsymbol{L}_2^0$ be such that $\boldsymbol{\sigma}_1^0 = s_0\boldsymbol{\sigma}_2^0$. It is easy choose parameters r_0, r_0', τ_0, μ_0, ν_0, α_0, $\boldsymbol{p}_0$, $\boldsymbol{q}_0$, $\boldsymbol{p}_0'$, $\boldsymbol{q}_0'$ defining the link (2.3), for example in such a way that the link passes through $\widehat{\boldsymbol{L}}_0 = [\boldsymbol{L}_1^0, \boldsymbol{L}_2^0]$. Thus, we may use this link to map any exact relation passing through $\begin{bmatrix} \boldsymbol{\Lambda}_0 & \boldsymbol{p}_0 \\ \boldsymbol{q}_0 & \alpha_0 \end{bmatrix}$ into an exact relation passing through $\boldsymbol{L}_0' = \begin{bmatrix} \boldsymbol{\sigma}_0 & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}$, where $\boldsymbol{\sigma}_0 = (\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_0^T)/2$. Similarly, we may map any link passing through $\widehat{\boldsymbol{L}}_0$, as above, into a link passing through $\widehat{\boldsymbol{L}}_0' = \left[\begin{bmatrix} \boldsymbol{\sigma}_1^0 & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\sigma}_1^0 & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}\right]$. Indeed, let $\widehat{\mathbb{M}}_1$ be the link of the form (2.3) that passes through $\left[\boldsymbol{L}_1^0, \begin{bmatrix} \boldsymbol{\sigma}_1^0 & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}\right]$. The link $\widehat{\mathbb{M}}_1$ can be represented as $\widehat{\mathbb{M}}_1 = \{[\boldsymbol{L}, \Phi_1(\boldsymbol{L})] : \boldsymbol{L} \in \mathrm{End}^+(\mathbb{R}^3)\}$ for some bijective function $\Phi_1 : \mathrm{End}(\mathbb{R}^3) \to \mathrm{End}(\mathbb{R}^3)$. Similarly, there is a link $\widehat{\mathbb{M}}_2 = \{[\boldsymbol{L}, \Phi_2(\boldsymbol{L})] : \boldsymbol{L} \in \mathrm{End}^+(\mathbb{R}^3)\}$ passing through $\left[\boldsymbol{L}_2^0, \begin{bmatrix} \boldsymbol{\sigma}_1^0 & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}\right]$, since $\boldsymbol{\sigma}_2 = s\boldsymbol{\sigma}_1$. The maps Φ_1 and Φ_2 induce a "link of links":

$$\widehat{\Phi} = \Phi_1 \oplus \Phi_2 : V \to V, \qquad \widehat{\Phi}([\boldsymbol{L}_1, \boldsymbol{L}_2]) = [\Phi_1(\boldsymbol{L}_1), \Phi_2(\boldsymbol{L}_2)].$$

The surface

$$\widehat{\mathbb{M}}' = \widehat{\Phi}(\widehat{\mathbb{M}})$$

passes through $\widehat{\boldsymbol{L}}_0'$. Let us show that it is also a link. Suppose that the local tensors $\boldsymbol{L}_1'(\boldsymbol{x})$ and $\boldsymbol{L}_2'(\boldsymbol{x})$ are such that $[\boldsymbol{L}_1'(\boldsymbol{x}), \boldsymbol{L}_2'(\boldsymbol{x})] \in \widehat{\mathbb{M}}'$ for all $\boldsymbol{x}$. Then there exist $[\boldsymbol{L}_1(\boldsymbol{x}), \boldsymbol{L}_2(\boldsymbol{x})] \in \mathbb{M}$ such that $\boldsymbol{L}_1'(\boldsymbol{x}) = \Phi_1(\boldsymbol{L}_1(\boldsymbol{x}))$ and $\boldsymbol{L}_2'(\boldsymbol{x}) = \Phi_2(\boldsymbol{L}_2(\boldsymbol{x}))$. But then $(\boldsymbol{L}_1')^* = \Phi_1(\boldsymbol{L}_1^*)$ and $(\boldsymbol{L}_2')^* = \Phi_2(\boldsymbol{L}_2^*)$. In addition, since $\widehat{\mathbb{M}}$ is a link, $[\boldsymbol{L}_1^*, \boldsymbol{L}_2^*] \in \widehat{\mathbb{M}}$. Therefore, $[(\boldsymbol{L}_1')^*, (\boldsymbol{L}_2')^*] = [\Phi_1(\boldsymbol{L}_1^*), \Phi_2(\boldsymbol{L}_2^*)] \in \widehat{\mathbb{M}}'$.

If the exact relation passes through the simplified tensor $\boldsymbol{L}_0'$, the inversion formula (5.4) can be made especially simple in terms of Π_0, given by (6.1). We have

$$\boldsymbol{L} = \boldsymbol{C}_1(\boldsymbol{I} - [\boldsymbol{I} + \boldsymbol{P}\boldsymbol{M}_0]^{-1}\,\boldsymbol{P})\boldsymbol{C}_1, \qquad \boldsymbol{P} \in \Pi_0, \tag{6.8}$$

where $\boldsymbol{C}_1$ is given by (6.3) and $\boldsymbol{M}_0$ satisfies

$$\boldsymbol{P}(\boldsymbol{n}_0 \otimes \boldsymbol{n}_0 - \boldsymbol{M}_0)\boldsymbol{P} \in \Pi_0 \tag{6.9}$$

for all $\boldsymbol{P} \in \Pi_0$. In particular $\boldsymbol{M}_0 = \boldsymbol{e} \otimes \boldsymbol{e}$, where $\boldsymbol{e} = (e_1, e_2, 0)$ is a unit vector, works. In that case the inversion formula (6.8) written in block form, becomes

$$\begin{aligned} \boldsymbol{\Lambda} &= \boldsymbol{\sigma}_0^{1/2}\left(\boldsymbol{I} - \boldsymbol{K} + \frac{\boldsymbol{K}\boldsymbol{e} \otimes \boldsymbol{K}^T\boldsymbol{e}}{1 + (\boldsymbol{K}\boldsymbol{e}, \boldsymbol{e})}\right)\boldsymbol{\sigma}_0^{1/2}, \qquad \boldsymbol{p} = \boldsymbol{\sigma}_0^{1/2}\left(\frac{(\boldsymbol{u}, \boldsymbol{e})\boldsymbol{K}\boldsymbol{e}}{1 + (\boldsymbol{K}\boldsymbol{e}, \boldsymbol{e})} - \boldsymbol{u}\right), \\ \alpha &= 1 - \rho + \frac{(\boldsymbol{u}, \boldsymbol{e})(\boldsymbol{v}, \boldsymbol{e})}{1 + (\boldsymbol{K}\boldsymbol{e}, \boldsymbol{e})}, \qquad \boldsymbol{q} = \boldsymbol{\sigma}_0^{1/2}\left(\frac{(\boldsymbol{v}, \boldsymbol{e})\boldsymbol{K}^T\boldsymbol{e}}{1 + (\boldsymbol{K}\boldsymbol{e}, \boldsymbol{e})} - \boldsymbol{v}\right), \end{aligned} \tag{6.10}$$

where $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}$.

In many cases, however, a far simpler choice $\boldsymbol{M}_0 = \boldsymbol{0}$ will satisfy (6.9). In that case (6.8) becomes:

$$\boldsymbol{\Lambda} = \boldsymbol{\sigma}_0^{1/2}\,(\boldsymbol{I} - \boldsymbol{K})\,\boldsymbol{\sigma}_0^{1/2}, \qquad \boldsymbol{p} = -\boldsymbol{\sigma}_0^{1/2}\boldsymbol{u}, \qquad \boldsymbol{q} = -\boldsymbol{\sigma}_0^{1/2}\boldsymbol{v}, \qquad \alpha = 1 - \rho. \tag{6.11}$$

If formulas (6.11) can be used, the volume fraction relations still have the form (5.6), where $\boldsymbol{P}(\boldsymbol{L})$ is now taken to be the inverse of (6.11).

The inversion formulas for links are completely analogous to the ones for exact relations. The choice $\widehat{\boldsymbol{M}_0} = [\boldsymbol{e} \otimes \boldsymbol{e}, \boldsymbol{e} \otimes \boldsymbol{e}]$ is always appropriate, while $\widehat{\boldsymbol{M}_0} = [\boldsymbol{0}, \boldsymbol{0}]$ will be used, if possible. In either case, the inversion formulas are given by two copies of (6.10) or (6.11), one for each block-component of $\widehat{\boldsymbol{P}}$, and where $\boldsymbol{e}$ is the same in both copies of (6.10), if formulas (6.10) have to be used. Equivalently, the links can be computed using the formula (5.13), where the transformation $\boldsymbol{P}(\boldsymbol{L})$ uses the simplified values for $\boldsymbol{L}_0$ and $\boldsymbol{M}$.

7 Finding all Jordan algebras

From now on when we refer to equation (5.1) we mean that $\mathcal{A}$ is replaced by $\mathcal{A}_0$ in that equation. Consider a projection $\pi : \mathrm{End}(\mathbb{R}^3) \to \mathrm{End}(\mathbb{R}^2)$ given by

$$\pi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}\right) = \boldsymbol{K}.$$

The key observation is that if Π is a Jordan algebra then so is $\Pi_{2d} = \pi(\Pi)$, satisfying $\boldsymbol{KAK} \in \Pi_{2d}$ for all $\boldsymbol{K} \in \Pi_{2d}$ and all $\boldsymbol{A} \in \mathcal{A}_{2d}$, where

$$\mathcal{A}_{2d} = \{\boldsymbol{A} : \boldsymbol{A}^T = \boldsymbol{A},\ \mathrm{Tr}\,\boldsymbol{A} = 0\}.$$

The projection π is then a Jordan algebra homomorphism, or a Jordan homomorphism for short. In [9] we found all Jordan algebras corresponding to two-dimensional conducting composites with Hall effect. Here we present a much shorter proof of that result than in [9].

Theorem **7.1.** *If a subspace* $\Pi \subset \mathrm{End}(\mathbb{R}^2)$ *satisfies* $\boldsymbol{KAK} \in \Pi$ *for all* $\boldsymbol{K} \in \Pi$ *and all* $\boldsymbol{A} \in \mathcal{A}_{2d}$*, then* Π *is one of the following subspaces:*

1. $\{\boldsymbol{0}\}$
2. $\Pi_{\boldsymbol{a},\boldsymbol{b}} = \mathbb{R}(\boldsymbol{a} \otimes \boldsymbol{b})$
3. $\Pi_{\boldsymbol{a}} = \{\boldsymbol{u} \otimes \boldsymbol{a} : \boldsymbol{u} \in \mathbb{R}^2\}$
4. $\Pi_{\boldsymbol{a}}^T$
5. $\mathcal{A}_{2d}$
6. $\Pi_\beta = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = \beta(b - c) \right\}$, $\beta \in \mathbb{R} \cup \{\infty\}$
7. $\mathrm{End}(\mathbb{R}^2)$

Proof. The proof is based on the observation that for $\boldsymbol{A} \in \mathcal{A}_{2d}$, $\boldsymbol{A} \neq \boldsymbol{0}$, the inverse $\boldsymbol{A}^{-1}$ exists. We may then write, using Cayley-Hamilton theorem,

$$\boldsymbol{KAK} = (\boldsymbol{KA})^2\boldsymbol{A}^{-1} = \mathrm{Tr}\,(\boldsymbol{KA})\boldsymbol{K} - \det(\boldsymbol{KA})\boldsymbol{A}^{-1} = \mathrm{Tr}\,(\boldsymbol{KA})\boldsymbol{K} - \det(\boldsymbol{K})\mathrm{cof}(\boldsymbol{A}), \quad (7.1)$$

where $\mathrm{cof}(\boldsymbol{A}) = \boldsymbol{A}^{-T}\det\boldsymbol{A}$ is the matrix of co-factors of $\boldsymbol{A}$. Noticing that

$$\{\mathrm{cof}(\boldsymbol{A}) : \boldsymbol{A} \in \mathcal{A}_{2d}\} = \mathcal{A}_{2d},$$

we conclude that Π_{2d} is a Jordan algebra if and only if either $\mathcal{A}_{2d} \subset \Pi_{2d}$ or $\Pi_{2d} \subset \{\boldsymbol{K} : \det(\boldsymbol{K}) = 0\}$. Items 1–4 above list all subspaces satisfying the latter inclusion, while items 5–7 list all subspaces satisfying the former.[9] □

Now suppose that Π is a solution of (5.1). Let π_Π denote the restriction of the projection π to Π. Then elementary linear algebra tells us that there is a linear map $\varpi : \Pi_{2d} \to \Pi$ such that

$$\Pi = \varpi(\Pi_{2d}) \oplus \ker(\pi_\Pi), \qquad \pi(\varpi(\boldsymbol{K})) = \boldsymbol{K}, \quad (7.2)$$

where the sum is orthogonal with respect to the standard inner product $(\boldsymbol{A},\boldsymbol{B}) = \mathrm{Tr}\,(\boldsymbol{A}\boldsymbol{B}^T)$ on $\mathrm{End}(\mathbb{R}^3)$. Let w, q and λ be linear maps defined on $\Pi_{2d} = \pi(\Pi)$ with values is $\mathbb{R}^2$, $\mathbb{R}^2$ and $\mathbb{R}$ respectively, such that

$$\varpi(\boldsymbol{K}) = \begin{bmatrix} \boldsymbol{K} & w(\boldsymbol{K}) \\ q(\boldsymbol{K}) & \lambda(\boldsymbol{K}) \end{bmatrix}. \tag{7.3}$$

Suppose, $\begin{bmatrix} \boldsymbol{0} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix} \in \ker \pi_\Pi$. Then

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}^{*2} = (\boldsymbol{A}\boldsymbol{u},\boldsymbol{v}) \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}, \tag{7.4}$$

where

$$\boldsymbol{P}^{*2} = \boldsymbol{P} *_{\boldsymbol{A}} \boldsymbol{P} = \boldsymbol{P}\boldsymbol{A}\boldsymbol{P} \tag{7.5}$$

is a “Jordan square”. Let

$$\boldsymbol{U}_0 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix}. \tag{7.6}$$

From the formula (7.4) we already see that $\boldsymbol{U}_0$ plays a special role. Hor this reason we consider two cases $\boldsymbol{U}_0 \in \ker \pi_\Pi$ and $\boldsymbol{U}_0 \notin \ker \pi_\Pi$ separately.

7.1 Case I: $\boldsymbol{U}_0 \notin \ker \pi_\Pi$

Equation (7.4) implies that if $\boldsymbol{P} \in \Pi$ then either $\boldsymbol{u} = \boldsymbol{0}$ or $\boldsymbol{v} = \boldsymbol{0}$, or both. Thus, (up to a transpose) we may regard $\ker \pi_\Pi$ to be a subspace of a two-dimensional space

$$\mathcal{L}_{\boldsymbol{r}} = \left\{ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{v} & (\boldsymbol{v},\boldsymbol{r}) \end{bmatrix} : \boldsymbol{v} \in \mathbb{R}^2 \right\}$$

for some $\boldsymbol{r} \in \mathbb{R}^2$. Thus, we need to consider only three cases:

1. $\ker \pi_\Pi = \{\boldsymbol{0}\}$;
2. $\ker \pi_\Pi = \mathcal{L}_{\boldsymbol{r}}$;
3. $\dim \ker \pi_\Pi = 1$.

Case I.1. Suppose first that $\ker \pi_\Pi = \{\boldsymbol{0}\}$. Then $\Pi = \{\varpi(\boldsymbol{K}) : \boldsymbol{K} \in \Pi_{2d}\}$, where ϖ is given by (7.3). In that case equation (5.1) is equivalent to

$$w(\boldsymbol{K}\boldsymbol{A}\boldsymbol{K}) = \boldsymbol{K}\boldsymbol{A}w(\boldsymbol{K}), \quad q(\boldsymbol{K}\boldsymbol{A}\boldsymbol{K}) = \boldsymbol{K}^T\boldsymbol{A}q(\boldsymbol{K}), \quad \lambda(\boldsymbol{K}\boldsymbol{A}\boldsymbol{K}) = (\boldsymbol{A}w(\boldsymbol{K}), q(\boldsymbol{K})) \tag{7.7}$$

for all $\boldsymbol{K} \in \Pi_{2d}$. The first two equations in (7.7) are equivalent. Indeed, the transpose transformation $(\mathbb{T}w)(\boldsymbol{K}) = w(\boldsymbol{K}^T)$ reduces one equation to the other. The equations (7.7) can be solved explicitly.

THEOREM **7.2.** *If* $w : \Pi_{2d} \to \mathbb{R}^2$*, and* $w(\boldsymbol{KAK}) = \boldsymbol{KA}w(\boldsymbol{K})$ *for all* $\boldsymbol{K} \in \Pi_{2d}$*,* $\boldsymbol{A} \in \mathcal{A}_{2d}$*, then* $w(\boldsymbol{K}) = \boldsymbol{Kr}$ *for some fixed* $\boldsymbol{r} \in \mathbb{R}^2$*.*

The proof is given in Appendix A. By Theorem 7.2 $w(\boldsymbol{K}) = \boldsymbol{Kr}_1$ for some $\boldsymbol{r}_1 \in \mathbb{R}^2$ and $q(\boldsymbol{K}) = \boldsymbol{K}^T\boldsymbol{r}_2$ for some $\boldsymbol{r}_2 \in \mathbb{R}^2$. Therefore, $\lambda(\boldsymbol{K}^{*2}) = (\boldsymbol{K}^{*2}\boldsymbol{r}_1, \boldsymbol{r}_2)$. But, since $\Pi_{2d}^{*2} = \Pi_{2d}$ for all solutions Π_{2d}, we conclude that $\lambda(\boldsymbol{K}) = (\boldsymbol{Kr}_1, \boldsymbol{r}_2)$. Thus,

$$\varpi(\boldsymbol{K}) = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{Kr}_1 \\ \boldsymbol{K}^T\boldsymbol{r}_2 & (\boldsymbol{Kr}_1, \boldsymbol{r}_2) \end{bmatrix}, \tag{7.8}$$

and

$$\Pi = \left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{Kr}_1 \\ \boldsymbol{K}^T\boldsymbol{r}_2 & (\boldsymbol{Kr}_1, \boldsymbol{r}_2) \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d} \right\} \tag{7.9}$$

is the solution of (5.1) corresponding to $\ker \pi_\Pi = \{\boldsymbol{0}\}$.

Case I.2. Now assume that $\ker \pi_\Pi = \mathcal{L}_{\boldsymbol{r}}$. Then condition (7.2) becomes $q(\boldsymbol{K}) = -\lambda(\boldsymbol{K})\boldsymbol{r}$. Substituting this into (7.3), the equation (5.1) reduces to $w(\boldsymbol{KAK}) = \boldsymbol{KA}w(\boldsymbol{K})$ and

$$(1 + |\boldsymbol{r}|^2)\lambda(\boldsymbol{KAK}) + (\boldsymbol{Av}, \boldsymbol{Kr}) - \lambda(\boldsymbol{K})(\boldsymbol{Ar}, \boldsymbol{Kr}) = (\boldsymbol{A}w(\boldsymbol{K}), \boldsymbol{v} - \lambda(\boldsymbol{K})\boldsymbol{r}). \tag{7.10}$$

By Theorem 7.2 there exists $\boldsymbol{r}_0 \in \mathbb{R}^2$ such that $w(\boldsymbol{K}) = \boldsymbol{Kr}_0$ and the relation (7.10) implies $\boldsymbol{Kr} = \boldsymbol{Kr}_0$ and $\lambda(\boldsymbol{K}) = 0$. Therefore, the solution of (5.1) with $\ker \pi_\Pi = \mathcal{L}_{\boldsymbol{r}}$ is

$$\Pi = \left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{Kr} \\ \boldsymbol{v} & (\boldsymbol{v}, \boldsymbol{r}) \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ \boldsymbol{v} \in \mathbb{R}^2 \right\},\ \boldsymbol{r} \in \mathbb{R}^2. \tag{7.11}$$

Case I.3. Now assume that

$$\ker \pi_\Pi = \mathbb{R} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{a} & \mu \end{bmatrix},\ |\boldsymbol{a}| = 1,\ \mu \in \mathbb{R}.$$

The orthogonality relation (7.2) becomes

$$\mu\lambda(\boldsymbol{K}) + (q(\boldsymbol{K}), \boldsymbol{a}) = 0, \tag{7.12}$$

and

$$\Pi = \left\{ \begin{bmatrix} \boldsymbol{K} & w(\boldsymbol{K}) \\ q(\boldsymbol{K}) + t\boldsymbol{a} & \lambda(\boldsymbol{K}) + t\mu \end{bmatrix} : \boldsymbol{K} \in\in \Pi_{2d},\ t \in \mathbb{R} \right\}.$$

In that case equation (5.1) can be written as follows:

$$w(\boldsymbol{KAK}) = \boldsymbol{KA}w(\boldsymbol{K}) \tag{7.13}$$

$$t'\boldsymbol{a} + q(\boldsymbol{KAK}) = \boldsymbol{K}^T\boldsymbol{A}(t\boldsymbol{a} + q(\boldsymbol{K})) \tag{7.14}$$

$$t'\mu + \lambda(\boldsymbol{KAK}) = (\boldsymbol{A}w(\boldsymbol{K}), t\boldsymbol{a} + q(\boldsymbol{K})) \tag{7.15}$$

Equation (7.13) implies, via Theorem 7.2 that $w(\boldsymbol{K}) = \boldsymbol{K}\boldsymbol{r}_0$ for some $\boldsymbol{r}_0 \in \mathbb{R}^2$. Taking a dot product of (7.14) with $\boldsymbol{a}$, we can solve for t'. Substituting this into (7.15) together with formula for $w(\boldsymbol{K})$, we obtain

$$t\mu(\boldsymbol{A}\boldsymbol{a}, \boldsymbol{K}\boldsymbol{a}) + \mu(\boldsymbol{A}q(\boldsymbol{K}), \boldsymbol{K}\boldsymbol{a}) + (1+\mu^2)\lambda(\boldsymbol{K}\boldsymbol{A}\boldsymbol{K}) = (\boldsymbol{A}\boldsymbol{K}\boldsymbol{r}_0, t\boldsymbol{a} + \boldsymbol{q}(\boldsymbol{K})). \tag{7.16}$$

Equating the coefficients of t and using the fact that $\boldsymbol{A}\boldsymbol{a}$ can be any vector by a choice of $\boldsymbol{A} \in \mathcal{A}$, we conclude that $\boldsymbol{K}\boldsymbol{r}_0 = \mu\boldsymbol{K}\boldsymbol{a}$. Substituting this back into (7.16) we get $\lambda(\boldsymbol{K}\boldsymbol{A}\boldsymbol{K}) = 0$. Therefore, $\lambda(\boldsymbol{K}) = 0$ for any $\boldsymbol{K} \in \Pi_{2d}$. Substituting the value of t' back into the equation (7.14) and equating the coefficients at t we obtain

$$\boldsymbol{K}^T\boldsymbol{A}\boldsymbol{a} = (\boldsymbol{A}\boldsymbol{a}, \boldsymbol{K}\boldsymbol{a})\boldsymbol{a}, \qquad (\boldsymbol{A}q(\boldsymbol{K}), \boldsymbol{K}\boldsymbol{a})\boldsymbol{a} + q(\boldsymbol{K}\boldsymbol{A}\boldsymbol{K}) = \boldsymbol{K}^T\boldsymbol{A}q(\boldsymbol{K}). \tag{7.17}$$

Recalling that $\boldsymbol{A}\boldsymbol{a}$ can be any vector in $\mathbb{R}^2$, we conclude that $\boldsymbol{K} = \boldsymbol{K}\boldsymbol{a} \otimes \boldsymbol{a}$. Substituting this into the second equation in (7.17), we obtain $q(\boldsymbol{K}) = \boldsymbol{0}$. Therefore,

$$\Pi = \left\{ \begin{bmatrix} \boldsymbol{K} & \mu\boldsymbol{K}\boldsymbol{a} \\ t\boldsymbol{a} & t\mu \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ t \in \mathbb{R} \right\}, \tag{7.18}$$

where Π_{2d} can be either $\{\boldsymbol{0}\}$, $\Pi_{\boldsymbol{b},\boldsymbol{a}}$ or $\Pi_{\boldsymbol{a}}$. Hence formulas (7.9), (7.11), (7.18) and their transposes describe all solutions of (5.1) in Case I.

7.2 Case II: $\boldsymbol{U}_0 \in \ker \pi_\Pi$

In this case $\ker(\pi_\Pi)$ has the form

$$\ker \pi_\Pi = \left\{ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix} : (\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{L},\ \rho \in \mathbb{R} \right\},$$

where $\mathcal{L}$ is a subspace of $\mathbb{R}^4$. Recall that $\varpi(\Pi_{2d})$ must be orthogonal to $\ker \pi_\Pi$, thus, ϖ is given by (7.3), where $\lambda(\boldsymbol{K}) = 0$ and

$$(\boldsymbol{u}, w(\boldsymbol{K})) + (\boldsymbol{v}, q(\boldsymbol{K})) = 0 \tag{7.19}$$

for all $(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{L}$ and all $\boldsymbol{K} \in \Pi_{2d}$.

According to (7.2), if $\boldsymbol{P} \in \Pi$, then $\boldsymbol{P} = \varpi(\boldsymbol{K}) + \boldsymbol{N}$, where $\boldsymbol{K} \in \Pi_{2d}$ and $\boldsymbol{N} \in \ker \pi_\Pi$. Expanding the $\boldsymbol{P}^{*2} \in \Pi$, we obtain

$$\varpi(\boldsymbol{K}) *_{\boldsymbol{A}} \boldsymbol{N} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{K}\boldsymbol{A}\boldsymbol{u} \\ \boldsymbol{K}^T\boldsymbol{A}\boldsymbol{v} & * \end{bmatrix} \in \ker \pi_\Pi, \qquad \varpi(\boldsymbol{K})^{*2} - \varpi(\boldsymbol{K}^{*2}) \in \ker \pi_\Pi, \tag{7.20}$$

where $\boldsymbol{N} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}$. Thus, we get an equation for the subspace $\mathcal{L}$:

$$(\boldsymbol{K}\boldsymbol{A}\boldsymbol{u}, \boldsymbol{K}^T\boldsymbol{A}\boldsymbol{v}) \in \mathcal{L},\ \text{for all } (\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{L},\ \boldsymbol{K} \in \Pi_{2d},\ \boldsymbol{A} \in \mathcal{A}_{2d}. \tag{7.21}$$

Our strategy is the same as in Case I: For each subspace $\mathcal{L}$ of $\mathbb{R}^4$ of given dimension we identify all solutions of (5.1) for which $\ker \pi_\Pi = \mathcal{L} \oplus \mathbb{R}\boldsymbol{U}_0$. Therefore, we have to consider 5 cases distinguished by the dimension of $\mathcal{L}$. The easiest two cases when $\mathcal{L} = \{\mathbf{0}\}$ and $\mathcal{L} = \mathbb{R}^4$ can be resolved immediately. If $\mathcal{L} = \mathbb{R}^4$ then all solutions of (5.1) are given by

$$\Pi = \left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ \boldsymbol{u} \in \mathbb{R}^2, \boldsymbol{v} \in \mathbb{R}^2,\ \rho \in \mathbb{R} \right\}. \tag{7.22}$$

If $\mathcal{L} = \{\mathbf{0}\}$ then $\Pi = \{\varpi(\boldsymbol{K}) : \boldsymbol{K} \in \Pi_{2d}\} \oplus \mathbb{R}\boldsymbol{U}_0$, provided $\varpi(\boldsymbol{K})$, given by (7.3) satisfies $(7.20)_2$. In other words, $w(\boldsymbol{K})$ and $q(\boldsymbol{K})$ satisfy (7.7). Theorem 7.2 implies then that

$$\varpi(\boldsymbol{K}) = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{K}\boldsymbol{r}_1 \\ \boldsymbol{K}^T\boldsymbol{r}_2 & 0 \end{bmatrix}. \tag{7.23}$$

and

$$\Pi = \left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{K}\boldsymbol{r}_1 \\ \boldsymbol{K}^T\boldsymbol{r}_2 & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ \rho \in \mathbb{R} \right\}, \quad \{\boldsymbol{r}_1, \boldsymbol{r}_2\} \subset \mathbb{R}^2. \tag{7.24}$$

Another special case that can and needs to be treated separately is the case $\Pi = \{\mathbf{0}\}$. In that case the equation $(7.20)_2$ is satisfied for any choice of the subspace $\mathcal{L}$. Hence,

$$\Pi = \left\{ \begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix} : (\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{L},\ \rho \in \mathbb{R} \right\}, \quad \mathcal{L} \subset \mathbb{R}^4 \text{ is any subspace.} \tag{7.25}$$

The the remaining three cases are examined in Appendix B. Here we list solutions of (5.1) corresponding to those cases, ordered by the dimension of $\mathcal{L}$.

$\dim \mathcal{L} = 3.$

1. $\left\{ \begin{bmatrix} \boldsymbol{v} \otimes \boldsymbol{a} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix} : \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2,\ t, \rho \in \mathbb{R} \right\}$, $\boldsymbol{a} \neq \mathbf{0}$, and its transpose
2. $\left\{ \begin{bmatrix} s(\boldsymbol{a} \otimes \boldsymbol{b}) & t\boldsymbol{a} \\ \boldsymbol{v} & \rho \end{bmatrix} : s, t, \rho \in \mathbb{R}^2,\ \boldsymbol{v} \in \mathbb{R}^2 \right\}$, $\boldsymbol{a} \neq \mathbf{0}$, $\boldsymbol{b} \neq \mathbf{0}$, and its transpose
3. $\left\{ \begin{bmatrix} s(\boldsymbol{a} \otimes \boldsymbol{b}) & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix} : s, \rho \in \mathbb{R},\ \boldsymbol{u} \cdot \boldsymbol{a}^\perp + \boldsymbol{v} \cdot \boldsymbol{b}^\perp = 0 \right\}$, $\boldsymbol{a} \neq \mathbf{0}$, $\boldsymbol{b} \neq \mathbf{0}$

$\dim \mathcal{L} = 2.$

1. $\left\{ \begin{bmatrix} s(\boldsymbol{a} \otimes \boldsymbol{b}) & t\boldsymbol{a} \\ r\boldsymbol{b} & \rho \end{bmatrix} : s, t, r, \rho \in \mathbb{R} \right\}$, $\boldsymbol{a} \neq \mathbf{0}$, $\boldsymbol{b} \neq \mathbf{0}$
2. $\left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{K}\boldsymbol{r} \\ \boldsymbol{u} & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ \boldsymbol{u} \in \mathbb{R}^2,\ \rho \in \mathbb{R} \right\}$, $\boldsymbol{r} \in \mathbb{R}^2$, and its transpose

3. $\left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{v} \\ \boldsymbol{K}^T\boldsymbol{r} + \boldsymbol{M}\boldsymbol{v} & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_\beta,\ \boldsymbol{v} \in \mathbb{R}^2,\ \rho \in \mathbb{R} \right\},\ \boldsymbol{r} \in \mathbb{R}^2,\ \boldsymbol{M} = \lambda \begin{bmatrix} 1 & -1/\beta \\ 1/\beta & 1 \end{bmatrix}$

4. $\left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \phi(c)\boldsymbol{u} + \boldsymbol{K}\boldsymbol{r} & \rho \end{bmatrix} : \boldsymbol{K} \in \mathcal{A}_{2d},\ \boldsymbol{u} \in \mathbb{R}^2,\ \rho \in \mathbb{R} \right\},\ \boldsymbol{r} \in \mathbb{R}^2,\ c \in \mathbb{C},$

   where $\phi : \mathbb{C} \to \mathrm{End}(\mathbb{R}^2)$ is defined in (6.4).

5. $\left\{ \begin{bmatrix} t(\boldsymbol{a} \otimes \boldsymbol{b}) & \boldsymbol{u} \\ t\boldsymbol{b} + \phi(\overline{\boldsymbol{a}}\boldsymbol{b})\boldsymbol{u} & \rho \end{bmatrix} : \boldsymbol{u} \in \mathbb{R}^2,\ t, \rho \in \mathbb{R} \right\},\ \boldsymbol{a} \neq \boldsymbol{0},\ \boldsymbol{b} \neq \boldsymbol{0}$

$$\dim \mathcal{L} = 1.$$

1. $\left\{ \begin{bmatrix} s(\boldsymbol{a} \otimes \boldsymbol{b}) & t\boldsymbol{a} \\ s\boldsymbol{b} & \rho \end{bmatrix} : s, t, \rho \in \mathbb{R} \right\},\ \boldsymbol{a} \neq \boldsymbol{0},\ \boldsymbol{b} \neq \boldsymbol{0},$ and its transpose

2. $\left\{ \begin{bmatrix} \boldsymbol{v} \otimes \boldsymbol{a} & \boldsymbol{v} \\ t\boldsymbol{a} & \rho \end{bmatrix} : \boldsymbol{v} \in \mathbb{R}^2,\ t, \rho \in \mathbb{R} \right\},\ \boldsymbol{a} \neq \boldsymbol{0}$ and its transpose

3. $\left\{ \begin{bmatrix} \boldsymbol{v} \otimes \boldsymbol{a} & \boldsymbol{0} \\ t\boldsymbol{a} & \rho \end{bmatrix} : \boldsymbol{v} \in \mathbb{R}^2,\ t, \rho \in \mathbb{R} \right\},\ \boldsymbol{a} \neq \boldsymbol{0}$ and its transpose

4. $\left\{ \begin{bmatrix} s(\boldsymbol{a} \otimes \boldsymbol{b}) & t\boldsymbol{a} \\ (t+s)\boldsymbol{b} & \rho \end{bmatrix} : s, t, \rho \in \mathbb{R} \right\},\ \boldsymbol{a} \neq \boldsymbol{0},\ \boldsymbol{b} \neq \boldsymbol{0}$

5. $\left\{ \begin{bmatrix} s(\boldsymbol{a} \otimes \boldsymbol{b}) & t\boldsymbol{a} \\ t\boldsymbol{b} & \rho \end{bmatrix} : s, t, \rho \in \mathbb{R} \right\},\ \boldsymbol{a} \neq \boldsymbol{0},\ \boldsymbol{b} \neq \boldsymbol{0}$

Examining our *entrire* list of solutions, in both Case I and II, we see *a posteriori* that the map $\varpi : \Pi_{2d} \to \Pi$ introduced in (7.2), basically has one form in all cases. We formalize this observation as a theorem.

THEOREM **7.3.** *Let* Π *be a Jordan algebra. Let* $\Pi_{2d} = \pi(\Pi)$*. Let* $\varpi : \Pi_{2d} \to \Pi$ *be a linear map such that*

(i) $\pi(\varpi(\boldsymbol{K})) = \boldsymbol{K}$

(ii) $\varpi(\Pi_{2d}) \perp \ker \pi_\Pi$

(iii) $\varpi(\boldsymbol{K})^{*2} - \varpi(\boldsymbol{K}^{*2}) \in \ker \pi_\Pi$

Then ϖ *is given by (7.8), if* $\boldsymbol{U}_0 \notin \Pi$ *and by (7.23), if* $\boldsymbol{U}_0 \in \Pi$ *for some* $\{\boldsymbol{r}_1, \boldsymbol{r}_2\} \subset \mathbb{R}^2$*.*

7.3 The simplified list

In Sections 7.2 and 7.1 we have obtained a complete list of all solutions Π of (5.1). However, this list is long and contains families of subspaces with rather complicated structure. Moreover, the list itself is not the final result. It has to be used to solve equation (5.8) and ultimately obtain formulas of Section 2.

The new idea here is to exploit symmetries of (5.1) to simplify the following analysis. We begin by finding those linear transformations that leave the equation (5.1) invariant. These transformations then map one Jordan algebra Π into another bijectively, lumping many items in our long list of solutions of (5.1) into a few isomorphism classes. This is a very old idea from algebra and it is very effective here.

Definition 7.4. *A bijective linear map* $\boldsymbol{\Psi} : \mathrm{End}(\mathbb{R}^3) \to \mathrm{End}(\mathbb{R}^3)$ *is called a global Jordan algebra automorphism, or global Jordan automorphism if*

$$\boldsymbol{\Psi}(\boldsymbol{P}^{*2}) = \boldsymbol{\Psi}(\boldsymbol{P})^{*2} \tag{7.26}$$

for all $\boldsymbol{P} \in \mathrm{End}(\mathbb{R}^3)$.

THEOREM **7.5.** *If* $\boldsymbol{\Psi}$ *is a global Jordan automorphism, then either* $\boldsymbol{\Psi}(\boldsymbol{P}) = \boldsymbol{X}\boldsymbol{P}\boldsymbol{Y}$ *or* $\boldsymbol{\Psi}(\boldsymbol{P}) = \boldsymbol{X}\boldsymbol{P}^T\boldsymbol{Y}$, *where*

$$\boldsymbol{X} = \begin{bmatrix} \phi(e^{i\theta}) & \boldsymbol{0} \\ \boldsymbol{a} & \tau \end{bmatrix}, \quad \boldsymbol{Y} = \begin{bmatrix} \phi(e^{i\theta}) & \boldsymbol{b} \\ \boldsymbol{0} & \delta \end{bmatrix} \tag{7.27}$$

for some $\theta \in [0, 2\pi)$, $\{\boldsymbol{a}, \boldsymbol{b}\} \subset \mathbb{R}^2$ *and* $\{\tau, \delta\} \subset \mathbb{R}$, *and where* $\phi : \mathbb{C} \to \mathrm{End}(\mathbb{R}^2)$ *is defined in (6.4).*

Proof. Let

$$\varpi_0(\boldsymbol{K}) = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix}. \tag{7.28}$$

Then $\varpi_0 : \mathrm{End}(\mathbb{R}^2) \to \mathrm{End}(\mathbb{R}^3)$ is a Jordan homomorphism. Here $\mathrm{End}(\mathbb{R}^2)$ is understood as a Jordan algebra from Theorem 7.1 with multiplications $*_{\boldsymbol{A}}$ with $\boldsymbol{A} \in \mathcal{A}_{2d}$. The space $\mathrm{End}(\mathbb{R}^3)$ is also understood as a Jordan algebra with multiplications $*_{\boldsymbol{A}}$ with $\boldsymbol{A} \in \mathcal{A} = \mathcal{A}_0$, defined by (6.2). Let $T = \pi \circ \boldsymbol{\Psi} \circ \varpi_0$. Then T is a Jordan homomorphism from $\mathrm{End}(\mathbb{R}^2)$ into itself. Thus, $\ker T$ is an ideal in $\mathrm{End}(\mathbb{R}^2)$. We have shown in [9], that the Jordan algebra $\mathrm{End}(\mathbb{R}^2)$ is simple (i.e. has no proper ideals), so either $\ker T = \{\boldsymbol{0}\}$ or $\ker T = \mathrm{End}(\mathbb{R}^2)$. Let us show that the latter possibility cannot happen. Let us define $\boldsymbol{P}^{*3} = \boldsymbol{P} *_{\boldsymbol{A}} \boldsymbol{P}^{*2}$. Suppose $\ker T = \mathrm{End}(\mathbb{R}^2)$. Then $\boldsymbol{\Psi}(\varpi_0(\boldsymbol{K})) \in \ker \pi$ for any $\boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2)$. Therefore, $\boldsymbol{\Psi}(\varpi_0(\boldsymbol{K}^{*3})) = \boldsymbol{\Psi}(\varpi_0(\boldsymbol{K}))^{*3} = \boldsymbol{0}$, while $\boldsymbol{K}^{*3} \neq \boldsymbol{0}$ for any $\boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2) \setminus \{\boldsymbol{0}\}$. This is a contradiction. We conclude that T is a Jordan isomorphism. Thus, according to [9] there exists $\alpha \in [0, 2\pi)$ such that either $T(\boldsymbol{K}) = \phi(e^{i\alpha})\boldsymbol{K}\phi(e^{i\alpha})$ or $T(\boldsymbol{K}) = \phi(e^{i\alpha})\boldsymbol{K}^T\phi(e^{i\alpha})$. Without loss of generality assume that $T(\boldsymbol{K}) = \phi(e^{i\alpha})\boldsymbol{K}\phi(e^{i\alpha})$, otherwise, replace $\boldsymbol{\Psi}(\boldsymbol{P})$ with $\boldsymbol{\Psi}(\boldsymbol{P}^T)$.

Let $w(\boldsymbol{K})$, $q(\boldsymbol{K})$ and $\lambda(\boldsymbol{K})$ be linear maps, such that

$$\boldsymbol{\Psi}\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix}\right) = \begin{bmatrix} T(\boldsymbol{K}) & w(\boldsymbol{K}) \\ q(\boldsymbol{K}) & \lambda(\boldsymbol{K}) \end{bmatrix}.$$

Then, (7.26) is equivalent to

$$\begin{aligned} w(\boldsymbol{K}^{*2}) &= T(\boldsymbol{K})\boldsymbol{A}w(\boldsymbol{K}), \\ q(\boldsymbol{K}^{*2}) &= T(\boldsymbol{K})^T\boldsymbol{A}q(\boldsymbol{K}), \\ \lambda(\boldsymbol{K}^{*2}) &= (\boldsymbol{A}w(\boldsymbol{K}), q(\boldsymbol{K})). \end{aligned} \tag{7.29}$$

Recalling the formula for T and the fact that $\phi(e^{i\alpha})\boldsymbol{A}\phi(e^{i\alpha}) = \boldsymbol{A}$ for any $\boldsymbol{A} \in \mathcal{A}_{2d}$, we get

$$\begin{aligned} w_\alpha(\boldsymbol{K}^{*2}) &= \boldsymbol{K}\boldsymbol{A}w_\alpha(\boldsymbol{K}), \\ q_\alpha(\boldsymbol{K}^{*2}) &= \boldsymbol{K}^T\boldsymbol{A}q_\alpha(\boldsymbol{K}), \\ \lambda(\boldsymbol{K}^{*2}) &= (\boldsymbol{A}w_\alpha(\boldsymbol{K}), q_\alpha(\boldsymbol{K})), \end{aligned} \tag{7.30}$$

where $w_\alpha(\boldsymbol{K}) = \phi(e^{-i\alpha})w(\boldsymbol{K})$ and $q_\alpha(\boldsymbol{K}) = \phi(e^{i\alpha})q(\boldsymbol{K})$. Then, according to Theorem 7.2, there exist $\{\boldsymbol{r}_1, \boldsymbol{r}_2\} \subset \mathbb{R}^2$ such that $w_\alpha(\boldsymbol{K}) = \boldsymbol{K}\boldsymbol{r}_1$, $q_\alpha(\boldsymbol{K}) = \boldsymbol{K}^T\boldsymbol{r}_2$ and $\lambda(\boldsymbol{K}) = (\boldsymbol{K}\boldsymbol{r}_1, \boldsymbol{r}_2)$. Thus, we have computed the map $\boldsymbol{\Psi} \circ \varpi_0$. It remains to determine the map $\boldsymbol{\Psi}$ on $\ker\pi$.

First we show that $\boldsymbol{\Psi}$ maps $\ker\pi$ onto $\ker\pi$. Suppose, there exist $\boldsymbol{N} \in \ker\pi$ such that $\pi(\boldsymbol{\Psi}(\boldsymbol{N})) \neq \boldsymbol{0}$ then, $\boldsymbol{\Psi}(\boldsymbol{N})^{*3} \neq \boldsymbol{0}$, but $\boldsymbol{N}^{*3} = \boldsymbol{0}$. This is a contradiction. Hence, if

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix},$$

then

$$\boldsymbol{\Psi}(\boldsymbol{N}) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{u}^* \\ \boldsymbol{v}^* & \rho^* \end{bmatrix},$$

where $\boldsymbol{u}^*$, $\boldsymbol{v}^*$ and ρ^* are linear functions of $\boldsymbol{u}$, $\boldsymbol{v}$ and ρ. If we set $\boldsymbol{P} = \boldsymbol{N}$ in (7.26) we get

$$(\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v})\boldsymbol{\Psi}(\boldsymbol{U}_0) = (\boldsymbol{A}\boldsymbol{u}^*, \boldsymbol{v}^*)\boldsymbol{U}_0.$$

It follows that there exists $\beta \neq 0$ such that $\boldsymbol{\Psi}(\boldsymbol{U}_0) = \beta\boldsymbol{U}_0$ and

$$(\boldsymbol{A}\boldsymbol{u}^*, \boldsymbol{v}^*) = \beta(\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}). \tag{7.31}$$

It follows that $\rho^*(\boldsymbol{0}, \boldsymbol{0}, \rho) = \beta\rho$. If we convert vectors to complex numbers in (7.31), we obtain

$$u^*v^* = \beta uv. \tag{7.32}$$

The complex numbers u^*, v^* are complex linear combinations of u, $\overline{u}$, v, $\overline{v}$ and ρ (expressing the linearity of $\boldsymbol{\Psi}$). Equation (7.32) then becomes an equality of polynomials in u, $\overline{u}$, v, $\overline{v}$ and ρ. We conclude that there exist a complex number $c \neq 0$ such that either

$$u^* = cu, \qquad v^* = \frac{\beta v}{c}. \tag{7.33}$$

or

$$u^* = cv, \qquad v^* = \frac{\beta u}{c}. \tag{7.34}$$

Let us first assume that (7.33) holds. Then, the identity

$$\boldsymbol{\Psi}(\varpi_0(\boldsymbol{K}) *_{\boldsymbol{A}} \boldsymbol{N}) = \boldsymbol{\Psi}(\varpi_0(\boldsymbol{K})) *_{\boldsymbol{A}} \boldsymbol{\Psi}(\boldsymbol{N})$$

becomes

$$\begin{gathered}
\phi(e^{i\alpha})\boldsymbol{K}\phi(e^{i\alpha})\boldsymbol{A}\phi(c)\boldsymbol{u} = \phi(c)\boldsymbol{K}\boldsymbol{A}\boldsymbol{u}, \\
\phi(e^{-i\alpha})\boldsymbol{K}^T\phi(e^{-i\alpha})\boldsymbol{A}\phi\left(\tfrac{\beta}{c}\right)\boldsymbol{v} = \phi\left(\tfrac{\beta}{c}\right)\boldsymbol{K}^T\boldsymbol{A}\boldsymbol{v}, \\
\rho^*(\boldsymbol{K}\boldsymbol{A}\boldsymbol{u}, \boldsymbol{K}^T\boldsymbol{A}\boldsymbol{v}, 0) = (\boldsymbol{A}\phi(e^{-i\alpha})\boldsymbol{K}^T\boldsymbol{r}_2, \phi(c)\boldsymbol{u}) + (\boldsymbol{A}\phi(e^{i\alpha})\boldsymbol{K}\boldsymbol{r}_1, \phi\left(\tfrac{\beta}{c}\right)\boldsymbol{v}).
\end{gathered} \tag{7.35}$$

We simplify these equations as follows

$$\begin{gathered}
\boldsymbol{K}\boldsymbol{A}\phi(e^{-i\alpha}c)\boldsymbol{u} = \phi(e^{-i\alpha}c)\boldsymbol{K}\boldsymbol{A}\boldsymbol{u}, \\
\boldsymbol{K}^T\boldsymbol{A}\phi\left(\tfrac{e^{i\alpha}}{c}\right)\boldsymbol{v} = \phi\left(\tfrac{e^{i\alpha}}{c}\right)\boldsymbol{K}^T\boldsymbol{A}\boldsymbol{v}, \\
\rho^*(\boldsymbol{K}\boldsymbol{A}\boldsymbol{u}, \boldsymbol{K}^T\boldsymbol{A}\boldsymbol{v}, 0) = (\boldsymbol{K}\boldsymbol{A}\phi(e^{-i\alpha}c)\boldsymbol{u}, \boldsymbol{r}_2) + (\boldsymbol{K}^T\boldsymbol{A}\phi\left(\tfrac{e^{i\alpha}\beta}{c}\right)\boldsymbol{v}, \boldsymbol{r}_2)
\end{gathered}$$

We conclude that $\phi(e^{-i\alpha}c)$ commutes with every matrix. Therefore, $c = \delta e^{i\alpha}$ for some $\delta \in \mathbb{R}$. Substituting this into the equation for ρ^*, we obtain

$$\rho^*(\boldsymbol{u}, \boldsymbol{v}, 0) = \delta(\boldsymbol{u}, \boldsymbol{r}_2) + \frac{\beta}{\delta}(\boldsymbol{v}, \boldsymbol{r}_1).$$

If we assume that $\boldsymbol{u}^*$ and $\boldsymbol{v}^*$ are given by (7.34) we will get a contradiction via the same argument as above. Putting everything together, it is now easy to verify that the map $\boldsymbol{\Psi}$ that we have computed is indeed given by $\boldsymbol{\Psi}(\boldsymbol{P}) = \boldsymbol{X}\boldsymbol{P}\boldsymbol{Y}$ or $\boldsymbol{\Psi}(\boldsymbol{P}) = \boldsymbol{X}\boldsymbol{P}^T\boldsymbol{Y}$, provided we make the identifications $\tau = \beta/\delta$, $\boldsymbol{a} = \boldsymbol{r}_2$ and $\boldsymbol{b} = \boldsymbol{r}_1$. □

Obviously, if Π is a Jordan algebra then so is $\boldsymbol{\Psi}(\Pi)$. Now for each Jordan algebra Π we will try to find a global Jordan automorphism $\boldsymbol{\Psi}$ such that $\Pi_0 = \boldsymbol{\Psi}(\Pi)$ is as simple as possible.

We start by simplifying $\Pi_{2d} = \pi(\Pi)$. The restriction of the global Jordan automorphism $\boldsymbol{\Psi}$ to Π_{2d} (i.e. $\pi \circ \boldsymbol{\Psi} \circ \varpi_0$) is given by the maps $\boldsymbol{K} \mapsto \phi(e^{i\theta})\boldsymbol{K}\phi(e^{i\theta})$ and $\boldsymbol{K} \mapsto \phi(e^{i\theta})\boldsymbol{K}^T\phi(e^{i\theta})$. These symmetries have been found and used in [9]. We will use only the first of these symmetries to simplify the list of Π_{2d}'s. The reason for this is that we need to use the transpose symmetry $\boldsymbol{P} \mapsto \boldsymbol{P}^T$ on $\mathrm{End}(\mathbb{R}^3)$ to simplify the form of Jordan algebras Π. It is easy to verify that each Jordan algebra Π_{2d} on the list in Theorem 7.1 is isomorphic to one of the Π_{2d}'s on the "simplified" list:

SIMPLIFIED LIST OF Π^0_{2d}:

1. $\{\mathbf{0}\}$
2. $\Pi_{\boldsymbol{a},\boldsymbol{a}} = \mathbb{R}(\boldsymbol{a} \otimes \boldsymbol{a})$, $|\boldsymbol{a}| = 1$
3. $\Pi_{\boldsymbol{a}} = \{\boldsymbol{u} \otimes \boldsymbol{a} : \boldsymbol{u} \in \mathbb{R}^2\}$, $|\boldsymbol{a}| = 1$
4. $\Pi^T_{\boldsymbol{a}}$
5. $\mathcal{A}_{2d} = \{\boldsymbol{A} \in \mathrm{Sym}(\mathbb{R}^2) : \mathrm{Tr}\,\boldsymbol{A} = 0\}$
6. $\mathrm{Sym}(\mathbb{R}^2)$
7. $\mathrm{End}(\mathbb{R}^2)$

We note that all the subspaces $\Pi_{\boldsymbol{a}}$ and $\Pi^T_{\boldsymbol{a}}$ are actually isomorphic. However, it will be convenient to keep the parameter $\boldsymbol{a}$ for more esthetically pleasing formulas, since there is no canonical choice of the unit vector $\boldsymbol{a} \in \mathbb{R}^2$.

It is easy to verify that $\varpi(\boldsymbol{K})$ given by (7.23) in Case II, and by (7.8) in Case I can be reduced to $\varpi_0(\boldsymbol{K})$, given by (7.28), by an application of the global Jordan automorphism to Π. We also observe that the subspace $\ker\pi_\Pi$ (determined by (7.21)) is independent of specific form of the map ϖ. Therefore, the analysis of Sections 7.2 and 7.1 yields all possible subspaces $\ker\pi_\Pi$ for each choice of $\Pi_{2d} = \pi(\Pi)$. Applying those global Jordan automorphisms $\boldsymbol{\Psi}$ that map ϖ into ϖ_0, we may also simplify the subspaces $\ker\pi_\Pi$. The required work is voluminous but completely straightforward. We omit the details of the calculation and only list the resulting simplified Jordan algebras Π.

THEOREM **7.6.** *Any Jordan algebra Π is globally isomorphic to one of the following "standard" solutions.*

1. $\left\{\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^2,\ \rho \in \mathbb{R}\right\}$
2. $\left\{\begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \boldsymbol{v} & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ \boldsymbol{v} \in \mathbb{R}^2,\ \rho \in \mathbb{R}\right\}$
3. $\left\{\begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \boldsymbol{v} & 0 \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ \boldsymbol{v} \in \mathbb{R}^2\right\}$
4. $\left\{\begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \mathbf{0} & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ \rho \in \mathbb{R}\right\}$
5. $\left\{\begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d}\right\}$

6. $\left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{u} & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ \boldsymbol{u} \in \mathbb{R}^2,\ \rho \in \mathbb{R} \right\}$, $\Pi_{2d} = \mathrm{Sym}(\mathbb{R}^2)$, $\mathcal{A}_{2d}$, $\Pi_{\boldsymbol{a},\boldsymbol{a}}$, $\{\boldsymbol{0}\}$.

7. $\left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a} & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ t \in \mathbb{R},\ \rho \in \mathbb{R} \right\}$, $\Pi_{2d} = \Pi_{\boldsymbol{a}}$,$\Pi_{\boldsymbol{a},\boldsymbol{a}}$, $\{\boldsymbol{0}\}$

8. $\left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ \boldsymbol{u} \in \mathbb{R}^2,\ t \in \mathbb{R}, \rho \in \mathbb{R} \right\}$, $\Pi_{2d} = \Pi_{\boldsymbol{a}}$, $\Pi_{\boldsymbol{a},\boldsymbol{a}}$, $\{\boldsymbol{0}\}$

9. $\left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a} & 0 \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ t \in \mathbb{R} \right\}$, $\Pi_{2d} = \Pi_{\boldsymbol{a}}$, $\Pi_{\boldsymbol{a},\boldsymbol{a}}$, $\{\boldsymbol{0}\}$

10. $\left\{ \begin{bmatrix} \boldsymbol{K} & t\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ t \in \mathbb{R},\ \rho \in \mathbb{R} \right\}$, $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{a}}$, $\{\boldsymbol{0}\}$

11. $\left\{ \begin{bmatrix} \boldsymbol{K} & s\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ t \in \mathbb{R},\ \rho \in \mathbb{R} \right\}$, $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{a}}$, $\{\boldsymbol{0}\}$

12. $\left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{u} + t\boldsymbol{a} & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d},\ \boldsymbol{u} \in \mathbb{R}^2,\ t \in \mathbb{R},\ \rho \in \mathbb{R} \right\}$, $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{a}}$, $\{\boldsymbol{0}\}$

13. $\left\{ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{u} \\ \boldsymbol{M}\boldsymbol{u} & \rho \end{bmatrix} : \boldsymbol{u} \in \mathbb{R}^2,\ \rho \in \mathbb{R} \right\}$.

There is a common pattern in the structure of all Jordan algebras Π.

THEOREM **7.7.** *Every Jordan algebra is related by a global Jordan automorphism to a subspace* Π *of* $\mathrm{End}(\mathbb{R}^3)$ *with the structure*

$$\Pi = (\varpi_0 \circ \pi)(\Pi) \oplus \ker \pi_\Pi.$$

Moreover, if $\boldsymbol{U}_0 \notin \ker \pi_\Pi$ *then*

$$\ker \pi_\Pi \subset \left\{ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{v} & 0 \end{bmatrix} : \boldsymbol{v} \in \mathbb{R}^2 \right\}.$$

In less abstract terms, all Jordan algebras in Theorem 7.6 have one of the two forms corresponding to either Case I or to Case II. For Case I,

$$\Pi = \left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix} : \boldsymbol{K} \in \Pi_{2d}^0,\ [\boldsymbol{u}, \boldsymbol{v}] \in \mathcal{L},\ \rho \in \mathbb{R} \right\},$$

for some choice of Π^0_{2d} from the simplified list of 2D Jordan algebras and a subspace $\mathcal{L}$ of $\mathbb{R}^4$. For Case II,

$$\Pi = \left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & 0 \end{bmatrix} : \boldsymbol{K} \in \Pi^0_{2d},\ \boldsymbol{v} \in \mathcal{L} \right\},$$

for some choice of Π^0_{2d} from the simplified list of 2D Jordan algebras and a subspace $\mathcal{L}$ of $\mathbb{R}^2$.

8 Derived ideals and volume fraction information

The computation of Jordan squares Π^{*2} is completely routine. Below we summarize the results of the calculation.

THEOREM **8.1** (Squares). $\Pi^{*2} = \Pi$, *except in the following cases.*

1. $\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix}$, *with no restrictions on* $\pi(\Pi)$.

2. $\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & 0 \end{bmatrix}$, *where* $\pi(\Pi)$ *is either* $\mathrm{End}(\mathbb{R}^2)$, $\mathrm{Sym}(\mathbb{R}^2)$, $\mathcal{A}_2$ *or* $\Pi^T_{\boldsymbol{a}}$.

3. $\pi(\Pi) = \Pi_{\boldsymbol{a}}$.

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix}, \qquad \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a} & 0 \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & 0 \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a} & 0 \end{bmatrix}, \qquad \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a} & 0 \end{bmatrix}$$

4. $\pi(\Pi) = \Pi_{\boldsymbol{a},\boldsymbol{a}}$.

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & s\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix}, \qquad \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a} & 0 \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & 0 \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a} & 0 \end{bmatrix}, \qquad \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a} & 0 \end{bmatrix}.$$

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{v} \\ \boldsymbol{v} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & t\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix}, \qquad \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & s\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{u} + t\boldsymbol{a} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & s\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix}.$$

5. $\pi(\Pi) = \{\mathbf{0}\}$. *Any such* Π *has the form*

$$\Pi = \left\{ \begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix} : (\boldsymbol{u}, \boldsymbol{v}, \rho) \in \mathcal{M} \right\},$$

where $\mathcal{M}$ *is a subspace in* $\mathbb{R}^5$. *We have* $\Pi^{*2} = \mathbb{R}\boldsymbol{U}_0$, *if there exists* $\boldsymbol{P} \in \Pi$ *such that* $\boldsymbol{u} \neq \mathbf{0}$ and $\boldsymbol{v} \neq \mathbf{0}$. *Otherwise,* $\Pi^{*2} = \{\mathbf{0}\}$.

We may shorten the above list significantly, if we eliminate those relations that are consequences of some others. Observe that if $\Pi_1 \subset \Pi_2$ then necessarily $\Pi_1^{*2} \subset \Pi_2^{*2} \cap \Pi_1$. If we have equality: $\Pi_1^{*2} = \Pi_2^{*2} \cap \Pi_1$, then the relation corresponding to Π_1^2 can be eliminated. Indeed, recall the inversion formula [12, formula (6.7)]:

$$\mathcal{P}_{(\Pi^{*2})^\perp} \langle W(\boldsymbol{L}(\boldsymbol{x})) \rangle = \mathcal{P}_{(\Pi^{*2})^\perp} W(\boldsymbol{L}^*), \tag{8.1}$$

where $\mathcal{P}_{(\Pi^{*2})^\perp}$ is the orthogonal projection onto the orthogonal complement of Π^{*2} in Π. Then the relation

$$\mathcal{P}_{(\Pi_1^{*2})^\perp} \langle W(\boldsymbol{L}(\boldsymbol{x})) \rangle = \mathcal{P}_{(\Pi_1^{*2})^\perp} W(\boldsymbol{L}^*) \tag{8.2}$$

follows from the relation

$$\mathcal{P}_{(\Pi_2^{*2})^\perp} \langle W(\boldsymbol{L}(\boldsymbol{x})) \rangle = \mathcal{P}_{(\Pi_2^{*2})^\perp} W(\boldsymbol{L}^*). \tag{8.3}$$

Indeed, (8.3) can be written as

$$W(\boldsymbol{L}^*) - \langle W(\boldsymbol{L}(\boldsymbol{x})) \rangle \in \Pi_2^{*2}.$$

But, if $\boldsymbol{L}(\boldsymbol{x}) \in \mathbb{M}_1$—the exact relation surface corresponding to Π_1, then $\boldsymbol{L}^* \in \mathbb{M}_1$ and

$$W(\boldsymbol{L}^*) - \langle W(\boldsymbol{L}(\boldsymbol{x})) \rangle \in \Pi_2^{*2} \cap \Pi_1 = \Pi_1^{*2}.$$

Relation (8.2) then holds.

There is another way to eliminate redundant volume fraction relations. Suppose that $\Pi = \Pi_1 \cap \Pi_2$. Then necessarily $\Pi^{*2} \subset \Pi_1^{*2} \cap \Pi_2^{*2}$. If we have equality: $\Pi^{*2} = \Pi_1^{*2} \cap \Pi_2^{*2}$, then (8.1) follows from (8.2) and (8.3) by the same argument as above. Eliminating redundant volume fraction relations gives us the following reduced list:

1. $\begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \boldsymbol{v} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \boldsymbol{v} & 0 \end{bmatrix}$, $\boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2)$, $\boldsymbol{v} \in \mathbb{R}^2$, $\rho \in \mathbb{R}$.

2. $\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}$, $\{\boldsymbol{u}, \boldsymbol{v}\} \subset \mathbb{R}^2$, $\{t, \rho\} \in \mathbb{R}$.

3. $\begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}^{*2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \rho \end{bmatrix}$, $\{\boldsymbol{u}, \boldsymbol{v}\} \subset \mathbb{R}^2$, $\rho \in \mathbb{R}$.

A simple computation (using formulas (6.11) and (5.6)) gives the exact relations that involve volume fractions.

1. If $\boldsymbol{p}(\boldsymbol{x}) = \mathbf{0}$, then $\alpha^* = \langle \alpha \rangle$.
2. If $\boldsymbol{\Lambda}(\boldsymbol{x})\boldsymbol{e}_0 = \boldsymbol{j}_0$, then $(\boldsymbol{q}^*, \boldsymbol{e}_0) = (\langle \boldsymbol{q} \rangle, \boldsymbol{e}_0)$.
3. If $\boldsymbol{\Lambda}(\boldsymbol{x}) = \boldsymbol{\Lambda}_0$, then $\boldsymbol{p}^* = \langle \boldsymbol{p} \rangle$ and $\boldsymbol{q}^* = \langle \boldsymbol{q} \rangle$.

9 Jordan ideals

According to the general theory described in Section 5, the links between uncoupled problems correspond to pairs of isomorphic factor-algebras

$$\Pi_1/\mathcal{I}_1 \cong \Pi_2/\mathcal{I}_2 \tag{9.1}$$

together with their Jordan isomorphisms, where $\mathcal{I}_i$ is an ideal in Π_i, $i = 1, 2$.

It is sufficient to study factor-algebras in the simplified list of Jordan algebras. Indeed, Suppose that Φ is a Jordan isomorphism between $\Pi_1/\mathcal{I}_1$ and $\Pi_2/\mathcal{I}_2$. Let $\boldsymbol{\Psi}_1 : \Pi_1 \to \Pi_1^0$ and $\boldsymbol{\Psi}_2 : \Pi_2 \to \Pi_2^0$ be global Jordan automorphisms mapping Jordan algebras Π_1 and Π_2 to the Jordan algebras Π_1^0 and Π_2^0 on the simplified list. Then $\mathcal{I}_1^0 = \boldsymbol{\Psi}_1(\mathcal{I}_1)$ and $\mathcal{I}_2^0 = \boldsymbol{\Psi}_2(\mathcal{I}_2)$ are ideals in Π_1^0 and Π_2^0 respectively and $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ induce the isomorphisms between the factors: $\overline{\boldsymbol{\Psi}}_i : \Pi_i/\mathcal{I}_i \to \Pi_i^0/\mathcal{I}_i^0$, $i = 1, 2$. Then, the mapping $\bar{\Phi} = \overline{\boldsymbol{\Psi}}_2 \circ \Phi \circ \overline{\boldsymbol{\Psi}}_1^{-1}$ is the isomorphism between factor-algebras $\Pi_1^0/\mathcal{I}_1^0$ and $\Pi_2^0/\mathcal{I}_2^0$ on the simplified list. Conversely, if $\bar{\Phi}$ is the isomorphism between $\Pi_1^0/\mathcal{I}_1^0$ and $\Pi_2^0/\mathcal{I}_2^0$, then $\Phi = \overline{\boldsymbol{\Psi}}_2^{-1} \circ \bar{\Phi} \circ \overline{\boldsymbol{\Psi}}_1$ is the isomorphism between $\Pi_1/\mathcal{I}_1$ and $\Pi_2/\mathcal{I}_2$. Hence, the link generated by the map Φ can be obtained from the link generated by $\bar{\Phi}$ and two applications of the links (2.2) or (2.3) corresponding to the global Jordan automorphism.

We distinguish between four types of ideals.

- **Type I ideals.** Any subspace $\mathcal{I}$ of Π such that $\Pi^{*2} \subset \mathcal{I}$.
- **Type II ideals.** $\mathcal{I} = \ker \pi_\Pi$ and $\mathcal{I}$ is not of type I.
- **Type III ideals.** $\mathcal{I} = \{\mathbf{0}\}$ and $\mathcal{I}$ is not of type I or II.
- **Type IV ideals.** An ideal which is not of type I–III.

It is easy to recognize ideals of types I–III. Let us describe ideals of type IV.

Lemma **9.1.** *An ideal $\mathcal{I} \subset \Pi$ of type IV is a proper subspace of* $\ker \pi_\Pi$.

Proof. Recall that the projection $\pi : \Pi \to \Pi_{2d}$ is a Jordan homomorphism. Then $\pi(\mathcal{I})$ is an ideal in Π_{2d}. But in [9] we showed that all 2D Jordan algebras are simple (have no non-trivial ideals). Hence, either $\pi(\mathcal{I}) = \{\mathbf{0}\}$ or $\pi(\mathcal{I}) = \Pi_{2d}$. The first possibility is equivalent to $\mathcal{I} \subset \ker \pi_\Pi$. Let us show that the second possibility leads to a contradiction.

Suppose $\pi(\mathcal{I}) = \Pi_{2d} = \pi(\Pi)$. If $\Pi_{2d} = \{\mathbf{0}\}$ there is nothing to prove because $\Pi = \ker \pi_\Pi$. Let us assume that $\Pi_{2d} \neq \{\mathbf{0}\}$. The theorem will follow if we show that $\Pi^{*2} \subset \mathcal{I}$, contradicting the assumption that $\mathcal{I}$ is not an ideal of type I. Suppose there exists $\boldsymbol{P} \in \Pi$ such that $\boldsymbol{P}^{*2} \notin \mathcal{I}$. Then there exists $\boldsymbol{P}_0 \in \mathcal{I}$ such that $\pi(\boldsymbol{P}) = \pi(\boldsymbol{P}_0)$, since by assumption the map $\pi : \mathcal{I} \to \Pi_{2d}$ is surjective. Then $\boldsymbol{P} = \boldsymbol{P}_0 + \boldsymbol{N}$, where $\boldsymbol{N} \in \ker \pi_\Pi$. It follows that $\boldsymbol{P}^{*2} = \boldsymbol{N}^{*2} \mod \mathcal{I}$. Hence, $\boldsymbol{N}^{*2} \notin \mathcal{I}$. Let

$$\boldsymbol{N} = \begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}.$$

Then both $\boldsymbol{u} \neq \mathbf{0}$ and $\boldsymbol{v} \neq \mathbf{0}$, for otherwise $\boldsymbol{N}^{*2} = \mathbf{0} \in \mathcal{I}$. Calculating $\boldsymbol{N}^{*2}$ explicitly, we see that $\boldsymbol{N}^{*2}$ is a non-zero multiple of $\boldsymbol{U}_0$. It follows that $\boldsymbol{U}_0 \in \Pi \setminus \mathcal{I}$.

By our assumptions there exists $\boldsymbol{J} \in \mathcal{I}$ such that $\pi(\boldsymbol{J}) \neq \mathbf{0}$. Let $\boldsymbol{J}' = \boldsymbol{J} *_{\boldsymbol{A}} \boldsymbol{N} \in \mathcal{I} \cap \ker \pi_\Pi$. Then an explicit calculation of $\boldsymbol{J}'$ shows that

$$\boldsymbol{J}' = \begin{bmatrix} \mathbf{0} & \boldsymbol{u}_0 \\ \boldsymbol{v}_0 & \rho_0 \end{bmatrix},$$

where both $\boldsymbol{u}_0 \neq \mathbf{0}$ and $\boldsymbol{v}_0 \neq \mathbf{0}$. But then $\boldsymbol{N} *_{\boldsymbol{A}} \boldsymbol{J}' \in \mathcal{I}$ must be a *non-zero* multiple of $\boldsymbol{U}_0$, giving a contradiction. The lemma is proved. □

A much shorter proof is possible, if we observe that the table of squares in Section 8 implies that $\Pi^{*3} = \Pi^{*2}$ for all Jordan algebras Π with $\pi(\Pi) \neq \{\mathbf{0}\}$. The argument at the beginning of the proof of Lemma 9.1 shows that if $\pi(\mathcal{I}) = \Pi_{2d}$ then for any $\boldsymbol{P} \in \Pi$ we have $\boldsymbol{P}^{*2} = \rho \boldsymbol{U}_0 \mod \mathcal{I}$. In particular, we see that $\Pi^{*3} \subset \mathcal{I}$. Our observation then implies the lemma. Conversely, the lemma implies that $\Pi^{*3} = \Pi^{*2}$, if $\pi(\Pi) \neq \{\mathbf{0}\}$. Indeed, if $\Pi^{*3} \neq \Pi^{*2}$ then $\mathcal{I} = \Pi^{*3}$ is an ideal in Π with $\pi(\mathcal{I}) = \pi(\Pi)$ and such that $\Pi^{*2} \not\subset \mathcal{I}$.

Corollary 9.2. *Let Π and Π' be Jordan algebras. Let $\Phi : \Pi \to \Pi'$ be a Jordan isomorphism. Then $\Phi(\ker \pi_\Pi) = \ker \pi_{\Pi'}$.*

Proof. Assume first that $\Pi_{2d} = \pi(\Pi) \neq \{\mathbf{0}\}$. Let $\mathcal{I}' = \Phi(\ker \pi_\Pi)$. Then $\mathcal{I}'$ is an ideal in Π'. According to Lemma 9.1, either $(\Pi')^{*2} \subset \mathcal{I}' \subset \Pi'$ or $\mathcal{I}' \subset \ker \pi_{\Pi'}$. If the former is the case then the factor family $\Pi'/\mathcal{I}'$ has the trivial multiplicative structure (i.e. $(\Pi'/\mathcal{I}')^{*2} = \{\mathbf{0}\}$). But $\Pi'/\mathcal{I}'$ is isomorphic to $\Pi/\ker \pi_\Pi$ which is isomorphic to Π_{2d}, that by our assumption has a non-trivial multiplicative structure. This is a contradiction. We conclude, therefore, that $\mathcal{I}' \subset \ker \pi_{\Pi'}$. Now, assume that $\Pi_{2d} = \{\mathbf{0}\}$ and that $\Pi'_{2d} = \pi(\Pi') \neq \{\mathbf{0}\}$. Then consider the Jordan isomorphism $\Phi^{-1} : \Pi' \to \Pi$. Let $\mathcal{I} = \Phi^{-1}(\ker \pi_{\Pi'})$. Clearly, Π'_{2d} is isomorphic to $\Pi/\mathcal{I}$. But $(\Pi'_{2d})^{*3} = \Pi'_{2d}$, while $\Pi^{*3} = \{\mathbf{0}\}$, implying that $(\Pi/\mathcal{I})^{*3}$ has a trivial multiplicative structure. The obtained contradiction proves that $\Pi'_{2d} = \{\mathbf{0}\}$ and in all cases $\mathcal{I}' \subset \ker \pi_{\Pi'}$. Applying the same argument to Φ^{-1} we obtain the reverse inclusion:

$$\Phi^{-1}(\ker \pi_{\Pi'}) \subset \ker \pi_\Pi = \Phi^{-1}(\mathcal{I}') \subset \Phi^{-1}(\ker \pi_{\Pi'}).$$

The Corollary is proved now. □

Corollary 9.3. *Let* Π *and* Π' *be Jordan algebras. Let* $\Phi : \Pi \to \Pi'$ *be a Jordan isomorphism. Then* $\pi(\Pi)$ *is isomorphic to* $\pi(\Pi')$*.*

Proof. By Corollary 9.2, $\ker \pi_{\Pi'} = \Phi(\ker \pi_\Pi)$. Then

$$\pi(\Pi') \cong \Pi'/\ker \pi_{\Pi'} = \Phi(\Pi)/\Phi(\ker \pi_\Pi) \cong \Pi/\ker \pi_\Pi \cong \pi(\Pi).$$

□

We can now give an effective characterization of all type IV ideals.

THEOREM **9.4.** *Suppose* $\mathcal{I} \subset \ker \pi_\Pi$ *is a subspace.*

(a) Suppose $\boldsymbol{U}_0 \in \mathcal{I}$*. Then* $\mathcal{I}$ *is an ideal in* Π *if and only if* $\Pi' = \mathcal{I} \oplus (\varpi_0 \circ \pi)(\Pi)$ *is a Jordan algebra.*

(b) Suppose $\boldsymbol{U}_0 \notin \mathcal{I}$*. Then* $\mathcal{I}$ *is an ideal in* Π *if and only if* $\Pi' = \mathcal{I} \oplus (\varpi_0 \circ \pi)(\Pi)$ *is a Jordan algebra and*

$$\ker \pi_\Pi \subset \left\{ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{v} & \rho \end{bmatrix} : \boldsymbol{v} \in \mathbb{R}^2,\ \rho \in \mathbb{R} \right\}.$$

Proof. In order to prove that $\mathcal{I}$ is an ideal we need to show that $\boldsymbol{P} *_{\boldsymbol{A}} \boldsymbol{J}_0 \in \mathcal{I}$ for all $\boldsymbol{P} \in \Pi$ and all $\boldsymbol{J}_0 \in \mathcal{I} \subset \ker \pi_\Pi$. By Theorem 7.7 we have

$$\boldsymbol{P} = \varpi_0(\boldsymbol{K}) + \boldsymbol{N},$$

where $\boldsymbol{K} \in \pi(\Pi)$ and $\boldsymbol{N} \in \ker \pi_\Pi$. We have

$$\varpi_0(\boldsymbol{K}) *_{\boldsymbol{A}} \boldsymbol{J}_0 \in \mathcal{I}$$

because $\Pi' = \mathcal{I} \oplus \varpi_0(\pi(\Pi))$ is a Jordan algebra and therefore

$$\varpi_0(\boldsymbol{K}) *_{\boldsymbol{A}} \boldsymbol{J}_0 \in \ker \pi_{\Pi'} = \mathcal{I}.$$

Also, $\boldsymbol{N} *_{\boldsymbol{A}} \boldsymbol{J}_0 \subset \mathbb{R}\boldsymbol{U}_0$, where $\boldsymbol{U}_0$ is given by (7.6). Thus, if $\boldsymbol{U}_0 \in \mathcal{I}$ then $\mathcal{I}$ is an ideal.

Now assume that $\boldsymbol{U}_0 \notin \mathcal{I}$. Then the previous analysis shows that we must have $\boldsymbol{N} *_{\boldsymbol{A}} \boldsymbol{J}_0 = \boldsymbol{0}$ for any $\boldsymbol{N} \in \ker \pi_\Pi$ and any $\boldsymbol{J}_0 \in \mathcal{I} \subset \ker \pi_\Pi$, in order for $\mathcal{I}$ to be an ideal. It is easy to see that this is possible if and only if

$$\ker \pi_\Pi \subset \left\{ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{v} & \rho \end{bmatrix} : \boldsymbol{v} \in \mathbb{R}^2,\ \rho \in \mathbb{R} \right\}.$$

The converse is proved by reversing the arguments. □

10 Factor algebras and their isomorphisms

In Section 9 we have understood Jordan ideals of Jordan algebras from the list in Theorem 7.6. The problem is that the number of all *pairs* of factor algebras is overwhelming. A closer look at the situation allows us to avoid most comparisons, however. The most basic observation is that in order to compute *all* Jordan isomorphisms between $\Pi_1/\mathcal{I}_1$ and $\Pi_2/\mathcal{I}_2$ it is only necessary to identify a single Jordan isomorphism, if we are able to characterize all Jordan automorphisms of $\Pi_1/\mathcal{I}_1$ or $\Pi_2/\mathcal{I}_2$. We will show that the Jordan algebra structure of $\Pi/\mathcal{I}$ can be of only four types: Π, $\Pi/\mathbb{R}\boldsymbol{U}_0$ and two exceptional ones, where Π is one of the Jordan algebras from the list in Theorem 7.6. The purpose of this section, therefore, will be to identify the canonical type of each factor-algebra $\Pi/\mathcal{I}$ by constructing a single Jordan isomorphism to one of the four canonical types and to characterize all the Jordan automorphisms of each canonical type. The latter task is quite involved and we only state the result here, placing the proof in Appendix C.

THEOREM **10.1.** *Suppose* $\Pi^{*2} \neq \{\boldsymbol{0}\}$. *Then any Jordan automorphism of a Jordan algebra* Π *extends to a global Jordan automorphism of* $\mathrm{End}(\mathbb{R}^3)$ *given by Theorem 7.5.*

Observe that any global Jordan automorphism of $\mathrm{End}(\mathbb{R}^3)$ maps $\mathbb{R}\boldsymbol{U}_0$ onto itself and hence any such automorphism factors through $\mathbb{R}\boldsymbol{U}_0$ to an automorphism of $\mathrm{End}(\mathbb{R}^3)/\mathbb{R}\boldsymbol{U}_0$. The converse also holds, as shown in Appendix C. The Jordan automorphisms of $\Pi/\mathbb{R}\boldsymbol{U}_0$ are charactrized in the following theorem.

THEOREM **10.2.** *Suppose* $\Pi_{2d} \neq \{\boldsymbol{0}\}$. *Then any Jordan automorphism of the factor-algebra* $\Pi/\mathbb{R}\boldsymbol{U}_0$ *extends to a global Jordan automorphism of* $\mathrm{End}(\mathbb{R}^3)/\mathbb{R}\boldsymbol{U}_0$, *except when* $\Pi_{2d} \subset \mathrm{Sym}(\mathbb{R}^2)$, *in which case there is an additional Jordan automorphism*

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & * \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u}+\boldsymbol{v} \\ \boldsymbol{v} & * \end{bmatrix}. \tag{10.1}$$

If $\Pi_{2d} = \mathcal{A}_{2d}$ *then there is yet another family of Jordan automorphisms*

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & * \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \phi(e^{i\theta})\boldsymbol{u} \\ \boldsymbol{v} & * \end{bmatrix}, \qquad \theta \in [0, 2\pi). \tag{10.2}$$

The restrictions of the automorphisms of $\mathrm{End}(\mathbb{R}^3)/\mathbb{R}\boldsymbol{U}_0$ *to* $\Pi/\mathbb{R}\boldsymbol{U}_0$ *and the exceptional automorphisms (10.1) and (10.2), if applicable, generate the group of all automorphisms of* $\Pi/\mathbb{R}\boldsymbol{U}_0$.

The proof is given in Appendix C. We proceed with the discussion of more general cases of factor algebras $\Pi/\mathcal{I}$.

Let us start with the simplest case of the ideal $\mathcal{I}$ of type I in Π. In that case the factor algebra $\Pi/\mathcal{I}$ has a trivial Jordan structure. It follows that if $\mathcal{I}_1$ is an ideal of type I in Π_1

and $\Pi_1/\mathcal{I}_1 \cong \Pi_2/\mathcal{I}_2$ then $\mathcal{I}_2$ must be an ideal of type I in Π_2 and the factor algebras are isomorphic if and only if

$$\dim \Pi_1/\mathcal{I}_1 = \dim \Pi_2/\mathcal{I}_2.$$

Moreover, any invertible linear map Φ between linear spaces $\Pi_1/\mathcal{I}_1$ and $\Pi_2/\mathcal{I}_2$ is a Jordan isomorphism of factor algebras. This link is a consequence of the volume fraction relations, because taking volume averages in (5.13) and using the volume fraction relations (5.6), we obtain that $\widehat{\mathbb{M}}$ given by (5.13) is indeed a link.

The next simplest case is when $\Pi/\mathcal{I}$ is isomorphic to some $\widetilde{\Pi}$ in the list of Theorem 7.6. Observe that ideals of type II and III necessarily belong to that category. In fact, as we will see shortly, most ideals of type IV also belong to that category. The task of identifying all factors $\Pi/\mathcal{I}$ that are isomorphic to one of the Jordan algebras $\widetilde{\Pi}$ is aided by the following observation.

Remark 10.3. *If Π_1' is such that $\mathcal{I}_1 \subset \Pi_1' \subset \Pi_1$. Then $\mathcal{I}_1$ is an ideal in Π_1' as well. If $\Phi : \Pi_1/\mathcal{I}_1 \to \widetilde{\Pi}$ is a Jordan isomorphism, then $\widetilde{\Pi}' = \Phi(\Pi_1'/\mathcal{I}_1)$ is also a Jordan algebra and the restriction of Φ from $\Pi_1/\mathcal{I}_1$ to $\Pi_1'/\mathcal{I}_1$ is an isomorphism between $\Pi_1'/\mathcal{I}_1$ and $\widetilde{\Pi}'$. Hence, for each ideal $\mathcal{I}$ of type IV we need to find maximal Π^* in which $\mathcal{I}$ is an ideal and such that $\Pi^*/\mathcal{I} \cong \widetilde{\Pi}$ for some $\widetilde{\Pi}$. Here "maximal" means that if $\mathcal{I}$ is an ideal in Π then $\Pi \subset \Pi^*$ and $\mathcal{I}$ is still an ideal in Π^*.*

Finally, if $\Pi/\mathcal{I}$ is not isomorphic to any of the previously computed factor algebras, then we add this factor algebra to our list of canonical types and compute the group of its automorphisms. The following lemma describes how such automorphisms act on $\boldsymbol{K}$ (or $\varpi_0(\boldsymbol{K})$, to be precise).

LEMMA **10.4.** *Modulo a global Jordan automorphism of Theorem 7.5 the Jordan isomorphism Φ between $\Pi_1/\mathcal{I}_1$ and $\Pi_2/\mathcal{I}_2$ has the property that*

$$\Phi(\varpi_0(\boldsymbol{K})) = \varpi_0(\boldsymbol{K}), \qquad \boldsymbol{K} \in \pi(\Pi). \tag{10.3}$$

Proof. Let $\varpi(\boldsymbol{K}) = \Phi(\varpi_0(\boldsymbol{K}))$, where we identify the equivalence classes in $\Pi_2/\mathcal{I}_2$ with the orthogonal complement of $\mathcal{I}_2$ in Π_2. Let $T(\boldsymbol{K}) = \pi(\varpi(\boldsymbol{K}))$. Then $T(\boldsymbol{K})$ is a Jordan homomorphism of Π_{2d} into Π_{2d}. The same argument as in the proof of Theorem 7.5 shows that T is a Jordan isomorphism. By applying a global Jordan automorphism $\boldsymbol{\Psi}$, if necessary, we may reduce the map T to identity. In this case, the map $\varpi(\boldsymbol{K})$ satisfies all conditions of Theorem 7.3 for $\Pi_0 = \varpi(\Pi_{2d}) \oplus \mathcal{I}$. Thus, the map ϖ may be reduced to ϖ_0 by a global Jordan automorphism $\boldsymbol{\Psi}$. $\square$

We are now in a position to classify all Jordan factor algebras $\Pi/\mathcal{I}$, for ideals $\mathcal{I}$ of type IV. According to Theorem 9.4, we simply need to go over all possible $\ker \pi_\Pi$ (there are 13 of them) and identify the Jordan structure of algebras $\Pi/\mathcal{I}$ "by hand".

If $\Pi_{2d} = \{\boldsymbol{0}\}$ then either $\Pi^{*2} = \{\boldsymbol{0}\}$, in which case every ideal $\mathcal{I}$ is an ideal of type I, or $\Pi^{*2} = \mathbb{R}\boldsymbol{U}_0$. Then $\mathcal{I} = \{\boldsymbol{0}\}$ is the only ideal that is not of type I. Theorem 10.1 then tells us that all the Jordan automorphisms of Π are restrictions of the global Jordan automorphism. From now on we will assume that $\Pi_{2d} \neq \{\mathbf{0}\}$. The elements of $\Pi/\mathcal{I}$ will be labeled by the elements of the orthogonal complement of $\mathcal{I}$ in Π.

1. $\mathcal{I} = \{\mathbf{0}\}$. The maximal Jordan algebra Π^* in the sense of Remark 10.3 is $\mathrm{End}(\mathbb{R}^3)$. $\Pi^*/\mathcal{I} \cong \mathrm{End}(\mathbb{R}^3)$.

2. $\mathcal{I} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{v} & 0 \end{bmatrix}$. If $\mathcal{I}$ is an ideal in Π then $\Pi \subset \Pi^* = \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \boldsymbol{v} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2)$, and $\mathcal{I}$ is still an ideal in Π^*. Thus, Π^* is maximal in the sense of Remark 10.3. Obviously, $\Pi^*/\mathcal{I} \cong \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \mathbf{0} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2)$.

3. $\mathcal{I} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ t\boldsymbol{a} & 0 \end{bmatrix}$. The maximal Jordan algebra Π^* is $\begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \boldsymbol{v} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}$. However, $\Pi/\mathcal{I}$ is not isomorphic to any other Π because

$$\Pi/\mathcal{I} \cong \Pi_{\boldsymbol{a}} \oplus \mathcal{N}, \qquad \dim \mathcal{N} = 2, \tag{10.4}$$

   where $\mathcal{N}$ annihilates $\Pi/\mathcal{I}$, i.e. $\boldsymbol{P} *_{\boldsymbol{A}} \boldsymbol{N} = \mathbf{0}$ for all $\boldsymbol{P} \in \Pi/\mathcal{I}$ and all $\boldsymbol{N} \in \mathcal{N}$. Therefore, Remark 10.3 does not apply. Therefore, in this case we have to keep going over Jordan algebras in which $\mathcal{I}$ is an ideal.

   If $\Pi = \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \boldsymbol{v} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$, the situation is similar

$$\Pi/\mathcal{I} \cong \Pi_{\boldsymbol{a},\boldsymbol{a}} \oplus \mathcal{N}, \qquad \dim \mathcal{N} = 2. \tag{10.5}$$

   In both cases we have to compute the group of automorphisms of $\Pi/\mathcal{I}$. According to Lemma 10.4, we only need to look for the Jordan automorphisms satisfying (10.3). From the structure of $\Pi/\mathcal{I}$ describe above, any invertible linear map on $(\ker \pi_\Pi)/\mathcal{I}$ will produce a Jordan automorphism. Thus,

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ t\boldsymbol{a}^\perp & \rho \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ (\mu_{11}t + \mu_{12}\rho)\boldsymbol{a}^\perp & \mu_{21}t + \mu_{22}\rho \end{bmatrix}$$

   for both $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}$ and $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$. The two factor algebras (10.4) and (10.5) are the two exceptional cases mentioned at the beginning of this section. Using global Jordan automorphisms of Theorem 7.5 it is possible to reduce the map Φ to the form

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ t\boldsymbol{a}^\perp & \rho \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \rho\boldsymbol{a}^\perp & t \end{bmatrix}.$$

The subspace $\mathcal{I}$ is also an ideal in $\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$ or $\Pi_{\boldsymbol{a}}$ and $\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & 0 \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$ or $\Pi_{\boldsymbol{a}}$. Let us show that Remark 10.3 applies to $\Pi^* = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}$ and $\Pi^* = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & 0 \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}$. In the former case, it is obvious that $\Pi^*/\mathcal{I} \cong \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \rho \end{bmatrix}$. In the latter case, the same is true. Indeed, $\Pi^*/\mathcal{I}$ is a direct sum of an algebra isomorphic to $\Pi_{\boldsymbol{a}}$ and a 1D space spanned by the element $\boldsymbol{V}_0$ such that $\boldsymbol{V}_0 *_{\boldsymbol{A}} \boldsymbol{P} = \boldsymbol{0}$ for all $\boldsymbol{P} \in \Pi^*/\mathcal{I}$. This is exactly the structure of $\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \rho \end{bmatrix}$. The Jordan isomorphism Φ can be given as

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a}^{\perp} & 0 \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & t \end{bmatrix}.$$

4. $\mathcal{I} = \mathbb{R}\boldsymbol{U}_0$. This case is covered by Theorem 10.2 and a simple observation that if $\Pi_1/\mathbb{R}\boldsymbol{U}_0 \cong \Pi_2/\mathbb{R}\boldsymbol{U}_0$ then $\Pi_1 \cong \Pi_2$. A simple inspection shows that any Jordan isomorphism between Π_1 and Π_2 must be a restriction of the global Jordan automorphism to Π_1.

5. $\mathcal{I} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{v} & \rho \end{bmatrix}$, $\Pi^* = \mathrm{End}(\mathbb{R}^3)$ and $\Pi^*/\mathcal{I} \cong \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & 0 \end{bmatrix}$ with the isomorphism

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{0} & 0 \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K}^T & \boldsymbol{0} \\ \boldsymbol{u} & 0 \end{bmatrix}.$$

6. $\mathcal{I} = \ker \pi = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}$, $\Pi^* = \mathrm{End}(\mathbb{R}^3)$ and $\Pi^*/\mathcal{I} \cong \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix}$.

7. $\mathcal{I} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{u} \\ \boldsymbol{u} & \rho \end{bmatrix}$, $\Pi^* = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \mathrm{Sym}(\mathbb{R}^2)$ and $\Pi^*/\mathcal{I} \cong \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & 0 \end{bmatrix}$ with the isomorphism

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & -\boldsymbol{v} \\ \boldsymbol{v} & 0 \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & 0 \end{bmatrix}.$$

8. $\mathcal{I} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ t\boldsymbol{a} & \rho \end{bmatrix}$, $\Pi^* = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}$ and $\Pi^*/\mathcal{I} \cong \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{0} & \rho \end{bmatrix}$, with the isomorphism

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ t\boldsymbol{a}^{\perp} & 0 \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{0} & t \end{bmatrix}.$$

9. $\mathcal{I} = \begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix}$, $\Pi^* = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}$ and $\Pi^*/\mathcal{I} \cong \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \mathbf{0} & \rho \end{bmatrix}$, with the isomorphism

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ t\boldsymbol{a}^{\perp} & 0 \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \mathbf{0} & t \end{bmatrix}.$$

10. $\mathcal{I} = \begin{bmatrix} \mathbf{0} & t\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix}$, $\Pi^* = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$ and $\Pi^*/\mathcal{I} \cong \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \boldsymbol{v} & \rho \end{bmatrix}$, with the isomorphism

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & -\boldsymbol{v} + t\boldsymbol{a}^{\perp} \\ \boldsymbol{v} + t\boldsymbol{a}^{\perp} & 0 \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \boldsymbol{v} & t \end{bmatrix}.$$

11. $\mathcal{I} = \begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{u} + t\boldsymbol{a} & \rho \end{bmatrix}$, $\Pi^* = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$ and $\Pi^*/\mathcal{I} \cong \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \mathbf{0} & \rho \end{bmatrix}$, with the isomorphism

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & t\boldsymbol{a}^{\perp} \\ -t\boldsymbol{a}^{\perp} & 0 \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \mathbf{0} & t \end{bmatrix}.$$

12. $\mathcal{I} = \begin{bmatrix} \mathbf{0} & s\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix}$, $\Pi = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$. The algebraic structure of $\Pi/\mathcal{I}$ is identical to (10.5). In fact, $\Pi/\mathcal{I}$ is isomorphic to $\Pi/\mathcal{I}$ from item 3 with the isomorphism

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & t\boldsymbol{a}^{\perp} \\ s\boldsymbol{a}^{\perp} & 0 \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ s\boldsymbol{a}^{\perp} & t \end{bmatrix}, \quad \boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}.$$

There are two more other Jordan algebras, where $\mathcal{I}$ is an ideal: $\Pi_1 = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$ and $\Pi_2 = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{u} + t\boldsymbol{a} & \rho \end{bmatrix}$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$. We have

$$\Pi_1/\mathcal{I} \cong \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \mathbf{0} & \rho \end{bmatrix}, \qquad \Pi_2/\mathcal{I} \cong \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \mathbf{0} & \rho \end{bmatrix}$$

via the isomorphisms

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & t\boldsymbol{a}^{\perp} \\ \mathbf{0} & 0 \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \mathbf{0} & t \end{bmatrix}, \qquad \Phi\left(\begin{bmatrix} \boldsymbol{K} & t\boldsymbol{a}^{\perp} \\ t\boldsymbol{a}^{\perp} & 0 \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \mathbf{0} & t \end{bmatrix},$$

respectively.

Applying formula (5.12) we construct the corresponding list of subspaces $\widehat{\Pi}$ of $V = \mathrm{End}(\mathbb{R}^3) \oplus \mathrm{End}(\mathbb{R}^3)$ satisfying (5.8). We order this list by decreasing dimension of Π_{2d}.

1. $(\boldsymbol{P}, \boldsymbol{XPY})$, $\boldsymbol{P} \in \mathrm{End}(\mathbb{R}^3)$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are given by (7.27)
2. $(\boldsymbol{P}, \boldsymbol{P}^T)$, $\boldsymbol{P} \in \mathrm{End}(\mathbb{R}^3)$
3. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix} \right)$, $\boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2)$
4. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}, \begin{bmatrix} \boldsymbol{K}^T & \boldsymbol{0} \\ \boldsymbol{u} & 0 \end{bmatrix} \right)$, $\boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2)$
5. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & \rho \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \rho \end{bmatrix} \right)$, $\boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2)$
6. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho_1 \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho_2 \end{bmatrix} \right)$, $\boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2)$.
7. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho_1 \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u}+\boldsymbol{v} \\ \boldsymbol{v} & \rho_2 \end{bmatrix} \right)$, $\boldsymbol{K} \in \mathrm{Sym}(\mathbb{R}^2)$
8. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v}-\boldsymbol{u} & 0 \end{bmatrix} \right)$, $\boldsymbol{K} \in \mathrm{Sym}(\mathbb{R}^2)$
9. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho_1 \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \phi(m)\boldsymbol{u} \\ \boldsymbol{v} & \rho_2 \end{bmatrix} \right)$, $\boldsymbol{K} \in \mathcal{A}_{2d}$, $|m| = 1$
10. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v}_1 & (\boldsymbol{v}_2, \boldsymbol{a}^\perp) \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v}_2 & (\boldsymbol{v}_1, \boldsymbol{a}^\perp) \end{bmatrix} \right)$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}$
11. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ t\boldsymbol{a} & \rho \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \rho \end{bmatrix} \right)$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}$.
12. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & 0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{v}, \boldsymbol{a}^\perp) \end{bmatrix} \right)$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}$.
13. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{0} & (\boldsymbol{v}, \boldsymbol{a}^\perp) \end{bmatrix} \right)$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}$.

14. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{v}, \boldsymbol{a}^\perp) \end{bmatrix}\right)$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}$.

15. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} - \boldsymbol{u} & (\boldsymbol{v} + \boldsymbol{u}, \boldsymbol{a}^\perp) \end{bmatrix}\right)$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$.

16. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{v} - \boldsymbol{u}, \boldsymbol{a}^\perp) \end{bmatrix}\right)$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$.

17. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v}_1 & \rho \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v}_2 & (\boldsymbol{u}, \boldsymbol{a}^\perp) \end{bmatrix}\right)$, $(\boldsymbol{v}_1, \boldsymbol{a}^\perp) = (\boldsymbol{v}_2, \boldsymbol{a}^\perp)$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$.

18. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{u}, \boldsymbol{a}^\perp) \end{bmatrix}\right)$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$.

19. $\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{u} + t\boldsymbol{a} & \rho \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{u}, \boldsymbol{a}^\perp) \end{bmatrix}\right)$, $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{a}}$.

The link corresponding to item 1 is given by (2.2) and (2.3). The derivation of the formulas (2.2) and (2.3) is in the Appendix D. The links corresponding to the remaining subspaces, except item 9 can be easily computed using the simplified formulas (6.11). We therefore omit the calculations. The computation of the link corresponding to item 9 and proof of its redundancy is in the Appendix E. The link corresponding to item 2 is given by (2.4). The link corresponding to items 3 and 4 is given by (2.5). Item 7 corresponds to (2.7). The remaining items correspond to links that upon inspection are seen to be redundant.

A Proof of Theorem 7.2

The proof will be based on (7.1), applying which to (7.7) yields

$$(\mathrm{Tr}\,(\boldsymbol{K}\boldsymbol{A})\boldsymbol{I} - \boldsymbol{K}\boldsymbol{A})w(\boldsymbol{K}) = \det(\boldsymbol{K})w(\mathrm{cof}(\boldsymbol{A})). \tag{A.1}$$

For 2×2 matrices $\boldsymbol{X}$ we have the formula

$$\mathrm{Tr}\,(\boldsymbol{X})\boldsymbol{I} - \boldsymbol{X} = \mathrm{cof}(\boldsymbol{X})^T.$$

Thus, we get

$$\mathrm{cof}(\boldsymbol{K})^T w(\boldsymbol{K}) = \det(\boldsymbol{K})\boldsymbol{A}w(\boldsymbol{A}^{-1}).$$

Consider first the case $\mathcal{A}_{2d} \subset \Pi_{2d}$. If $\boldsymbol{K} \in \Pi_{2d}$ is invertible, then

$$w(\boldsymbol{K}) = \boldsymbol{K}\boldsymbol{A}w(\boldsymbol{A}^{-1}). \tag{A.2}$$

Observe that the set of invertible matrices in those Π_{2d} that contain $\mathcal{A}_{2d}$ is dense. Therefore (A.2) holds for all $\boldsymbol{K} \in \Pi_{2d}$. Notice that the left-hand side in (A.2) does not depend on $\boldsymbol{A}$, we conclude that there exists $\boldsymbol{r} \in \mathbb{R}^2$ such that for all $\boldsymbol{A} \in \mathcal{A}_{2d}$ we have $\boldsymbol{A}w(\boldsymbol{A}^{-1}) = \boldsymbol{r}$, which implies that $w(\boldsymbol{A}) = \boldsymbol{A}\boldsymbol{r}$ for all $\boldsymbol{A} \in \mathcal{A}_{2d}$ and thus, $w(\boldsymbol{K}) = \boldsymbol{K}\boldsymbol{r}$.

Now consider the case $\det(\boldsymbol{K}) = 0$ for all $\boldsymbol{K} \in \Pi_{2d}$. Then (A.1) becomes

$$\mathrm{Tr}\,(\boldsymbol{K}\boldsymbol{A})w(\boldsymbol{K}) = \boldsymbol{K}\boldsymbol{A}w(\boldsymbol{K}). \tag{A.3}$$

By assumption, the matrix $\boldsymbol{K}$ is singular. Therefore, the 2×2 matrix $\boldsymbol{K}$ must have rank one. If $\boldsymbol{K} = \boldsymbol{a} \otimes \boldsymbol{b}$ then (A.3) implies that $w(\boldsymbol{a} \otimes \boldsymbol{b})$ is a multiple of $\boldsymbol{a}$. With this information in hand, let us show that the theorem holds for $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{b}}, \Pi_{\boldsymbol{a}}, \Pi_{\boldsymbol{a}}^T$.

1. $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{b}}$. As we have proved there exist a scalar λ such that $w(\boldsymbol{a} \otimes \boldsymbol{b}) = \lambda\boldsymbol{a}$. Then, letting $\boldsymbol{r} = \lambda\boldsymbol{b}/|\boldsymbol{b}|^2$, we get that $w(\boldsymbol{K}) = \boldsymbol{K}\boldsymbol{r}$ for all $\boldsymbol{K} \in \Pi_{\boldsymbol{a},\boldsymbol{b}}$.

2. $\Pi_{2d} = \Pi_{\boldsymbol{a}}$. By linearity of w, there exists a matrix $\boldsymbol{M}$ such that $w(\boldsymbol{u} \otimes \boldsymbol{a}) = \boldsymbol{M}\boldsymbol{u}$ for all $\boldsymbol{u} \in \mathbb{R}^2$. But we also know that $w(\boldsymbol{u} \otimes \boldsymbol{a})$ must be a scalar multiple of $\boldsymbol{u}$. Thus, every vector $\boldsymbol{u} \in \mathbb{R}^2$ is an eigenvector of $\boldsymbol{M}$. It follows that $\boldsymbol{M} = m\boldsymbol{I}$. So, $w(\boldsymbol{u} \otimes \boldsymbol{a}) = m\boldsymbol{u}$. Now choosing $\boldsymbol{r} = m\boldsymbol{a}/|\boldsymbol{a}|^2$, we get that $w(\boldsymbol{K}) = \boldsymbol{K}\boldsymbol{r}$ for all $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}$.

3. $\Pi_{2d} = \Pi_{\boldsymbol{a}}^T$. We argue as in the preceding item. There exists a 2×2 matrix $\boldsymbol{M}$, such that $w(\boldsymbol{a} \otimes \boldsymbol{v}) = \boldsymbol{M}\boldsymbol{v}$. But then $\boldsymbol{M}\boldsymbol{v}$ must be a multiple of $\boldsymbol{a}$. Thus, $\boldsymbol{M} = \boldsymbol{a} \otimes \boldsymbol{r}$ for some vector $\boldsymbol{r} \in \mathbb{R}^2$. Thus, $w(\boldsymbol{K}) = \boldsymbol{K}\boldsymbol{r}$ for all $\boldsymbol{K} \in \Pi_{\boldsymbol{a}}^T$.

B Solution of equation (5.1) in Case II

We go over the cases in order of decreasing dimension of $\mathcal{L}$.

1. $\dim \mathcal{L} = 3$, i.e. $(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{L}$ if they satisfy $(\boldsymbol{u}, \boldsymbol{a}) + (\boldsymbol{v}, \boldsymbol{b}) = 0$ for some $\{\boldsymbol{a}, \boldsymbol{b}\} \subset \mathbb{R}^2$, provided that $|\boldsymbol{a}| + |\boldsymbol{b}| \neq 0$. We need to consider separately two cases: $\boldsymbol{a} = \boldsymbol{0}$, $\boldsymbol{b} \neq \boldsymbol{0}$ and $\boldsymbol{a} \neq \boldsymbol{0}$, $\boldsymbol{b} \neq \boldsymbol{0}$. The case $\boldsymbol{a} \neq \boldsymbol{0}$, $\boldsymbol{b} = \boldsymbol{0}$ can be reduced to the first case by means of the transpose symmetry.

    (a) Consider first the case $\boldsymbol{a} = \boldsymbol{0}$, $\boldsymbol{b} \neq \boldsymbol{0}$. Then $\mathcal{L} = \{(\boldsymbol{u}, t\boldsymbol{b}^\perp) : \boldsymbol{u} \in \mathbb{R}^2,\ t \in \mathbb{R}\}$. Here $\boldsymbol{b}^\perp = \boldsymbol{S}\boldsymbol{b} = (-b_2, b_1)$. Observe that the projection of $\mathcal{L}$ on the second copy of $\mathbb{R}^2$ is a 1D subspace $\mathbb{R}\boldsymbol{b}^\perp$. Therefore, in this case Π_{2d} maybe either $\Pi_{\boldsymbol{a},\boldsymbol{b}^\perp}$ or $\Pi_{\boldsymbol{b}^\perp}$. Then, since ϖ maps into the orthogonal complement of $\ker \pi_\Pi$, there exists a 2×2 matrix $\boldsymbol{M}$ such that

    $$\varpi(\boldsymbol{K}) = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ (\boldsymbol{K}, \boldsymbol{M})\boldsymbol{b} & 0 \end{bmatrix}.$$

    But then

    $$\varpi(\boldsymbol{K})^{*2} - \varpi(\boldsymbol{K}^{*2}) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ (\boldsymbol{K}, \boldsymbol{M})\boldsymbol{K}^T\boldsymbol{A}\boldsymbol{b} - (\boldsymbol{K}^{*2}, \boldsymbol{M})\boldsymbol{b} & 0 \end{bmatrix} \in \ker \pi_\Pi.$$

For $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{b}^\perp}$ or $\Pi_{\boldsymbol{b}^\perp}$, we have $\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{K}^T\boldsymbol{A}\boldsymbol{b} & 0 \end{bmatrix} \in \ker \pi_\Pi$. It follows that $(\boldsymbol{K}^{*2}, \boldsymbol{M}) = 0$ for all $\boldsymbol{K} \in \Pi_{2d}$ and all $\boldsymbol{A} \in \mathcal{A}_{2d}$. But since for any Π_{2d} we have $\Pi_{2d}^{*2} = \Pi_{2d}$ (as we proved in [9]), we conclude that the map $\boldsymbol{K} \mapsto (\boldsymbol{K}, \boldsymbol{M})$ is identically zero. Thus,

$$\varpi(\boldsymbol{K}) = \begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}. \tag{B.1}$$

(b) Assume that $\boldsymbol{a} \neq \mathbf{0}$ and $\boldsymbol{b} \neq \mathbf{0}$. Then for all $(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{L}$, for all $\boldsymbol{K} \in \Pi_{2d}$ and all $\boldsymbol{A} \in \mathcal{A}_{2d}$ we must have, according to (7.21), $(\boldsymbol{K}\boldsymbol{A}\boldsymbol{u}, \boldsymbol{a}) + (\boldsymbol{K}^T\boldsymbol{A}\boldsymbol{v}, \boldsymbol{b}) = 0$.

Claim **B.1.** *For any* $\{\boldsymbol{p}, \boldsymbol{q}\} \subset \mathbb{R}^2$ *there exists* $(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{L}$ *and* $\boldsymbol{A} \in \mathcal{A}_{2d}$ *such that* $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{p}$ *and* $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{q}$.

If we assume the claim, the equation (7.21) implies that Π_{2d} can only be $\Pi_{\boldsymbol{a}^\perp,\boldsymbol{b}^\perp}$. A similar to the previous case argument shows again that the map ϖ is given by (B.1).

Proof. In order to prove the Claim B.1 we use the same complex variable formalism used to prove Claim 6.2.
We parameterize $\mathcal{A}_{2d}$ by $\boldsymbol{A} = \psi(z)$, $z \in \mathbb{C}$. Then the equations $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{p}$ and $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{q}$ become $z\overline{u} = p$ and $z\overline{v} = q$. Thus, the condition that $(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{L}$ is written as $\mathfrak{Re}(\overline{u}a + \overline{v}b) = 0$. Therefore,

$$\mathfrak{Re}\left\{\frac{pa}{z} + \frac{qb}{z}\right\} = 0, \qquad z \neq 0.$$

If $pa + qb = 0$, then any $z \neq 0$ solves the equation. If $pa + qb \neq 0$, then $z = it(pa + qb)$, $t \in \mathbb{R} \setminus \{0\}$ is the general solution. The claim is proved. ☐

2. $\dim \mathcal{L} = 2$. Here again we have two cases. Either $\mathcal{L} = \mathbb{R}(\boldsymbol{u}_0, \mathbf{0}) \oplus \mathbb{R}(\mathbf{0}, \boldsymbol{v}_0)$ or there exist a 2×2 matrix $\boldsymbol{M}$ such that $\mathcal{L} = \{(\boldsymbol{u}, \boldsymbol{M}\boldsymbol{u}) : \boldsymbol{u} \in \mathbb{R}^2\}$. The case $\mathcal{L} = \{(\boldsymbol{M}\boldsymbol{v}, \boldsymbol{v}) : \boldsymbol{v} \in \mathbb{R}^2\}$ can be reduced to the second case by means of the transpose symmetry.

(a) Assume that $\mathcal{L} = \mathbb{R}(\boldsymbol{u}_0, \mathbf{0}) \oplus \mathbb{R}(\mathbf{0}, \boldsymbol{v}_0)$. The technique used in the proof of Claim B.1 can be used to show that $\boldsymbol{A}\boldsymbol{u}_0$ (and similarly $\boldsymbol{A}\boldsymbol{v}_0$) can be any vector in $\mathbb{R}^2$. Thus $\boldsymbol{K}$ must map all vectors into multiples of $\boldsymbol{u}_0$, while $\boldsymbol{K}^T$ must map all vectors into multiples of $\boldsymbol{v}_0$. Thus, $\Pi_{2d} = \Pi_{\boldsymbol{u}_0,\boldsymbol{v}_0}$ and the map ϖ must have the form

$$\varpi(t(\boldsymbol{u}_0 \otimes \boldsymbol{v}_0)) = \begin{bmatrix} t(\boldsymbol{u}_0 \otimes \boldsymbol{v}_0) & \lambda_1 t\boldsymbol{u}_0^\perp \\ \lambda_2 t\boldsymbol{v}_0^\perp & 0 \end{bmatrix},$$

according to (7.19). Once again, the condition that $\varpi(\boldsymbol{K})^{*2} - \varpi(\boldsymbol{K}^{*2}) \in \ker \pi_\Pi$ results in ϖ having to have the form (B.1).

(b) Assume that $\mathcal{L} = \{(\boldsymbol{u}, \boldsymbol{M}\boldsymbol{u}) : \boldsymbol{u} \in \mathbb{R}^2\}$ for some $\boldsymbol{M} \in \mathrm{End}(\mathbb{R}^2)$. Then (7.21) implies that for all $\boldsymbol{A} \in \mathcal{A}$ and all $\boldsymbol{K} \in \Pi_{2d}$ we must have

$$\boldsymbol{K}^T \boldsymbol{A}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{K}\boldsymbol{A}. \tag{B.2}$$

We just need to go over each choice of Π_{2d} and figure out which matrices $\boldsymbol{M}$ satisfy (B.2). The table below summarizes our findings:

Π_{2d}	$\Pi_{\boldsymbol{a},\boldsymbol{b}}$	$\Pi_{\boldsymbol{a}}$	$\Pi_{\boldsymbol{a}}^T$	$\mathcal{A}_{2d}$	Π_β	$\mathrm{End}(\mathbb{R}^2)$
$\boldsymbol{M}$	$\mathbb{R}\phi(\overline{a}b)$	$\boldsymbol{0}$	$\boldsymbol{0}$	$\phi(\mathbb{C})$	$\mathbb{R}\phi(e^{i\theta_\beta})$	$\boldsymbol{0}$

where $\cot\theta_\beta = \beta$. Now, for each cell in the table we have to identify the map ϖ, such that the equations (7.20) and (7.19) are satisfied. The equations (7.19) imply that

$$w(\boldsymbol{K}) = -\boldsymbol{M}^T q(\boldsymbol{K}), \tag{B.3}$$

since the orthogonal complement of $\mathcal{L}$ is given by $\mathcal{L}^\perp = \{(-\boldsymbol{M}^T\boldsymbol{v}, \boldsymbol{v}) : \boldsymbol{v} \in \mathbb{R}^2\}$. The equation $(7.20)_2$ then results in

$$\boldsymbol{K}^T\boldsymbol{A}q(\boldsymbol{K}) - q(\boldsymbol{K}^{*2}) = \boldsymbol{M}\boldsymbol{K}\boldsymbol{A}w(\boldsymbol{K}) - \boldsymbol{M}w(\boldsymbol{K}^{*2}).$$

Applying (B.3) and (B.2) we obtain

$$\boldsymbol{K}^T\boldsymbol{A}(\boldsymbol{I} + \boldsymbol{M}\boldsymbol{M}^T)q(\boldsymbol{K}) = (\boldsymbol{I} + \boldsymbol{M}\boldsymbol{M}^T)q(\boldsymbol{K}^{*2}).$$

The matrices $\boldsymbol{M}$, given by the table above have a remarkable property that the matrix $\boldsymbol{M}\boldsymbol{M}^T$ is always a multiple of the identity. Thus, we conclude, that $q(\boldsymbol{K})$ satisfies $(7.7)_2$. Let us show that $w(\boldsymbol{K})$ satisfies $(7.7)_2$. Indeed, if we multiply $(7.7)_2$ by $\boldsymbol{M}^T$ on the left and use (B.3) then we get $\boldsymbol{M}^T\boldsymbol{K}^T\boldsymbol{A}q(\boldsymbol{K}) = -w(\boldsymbol{K}^{*2})$. Next, we use the relation $\boldsymbol{M}^T\boldsymbol{K}^T\boldsymbol{A} = \boldsymbol{K}\boldsymbol{A}\boldsymbol{M}^T$ that follows from (B.2) and the fact that $\boldsymbol{M}\boldsymbol{M}^T$ is a multiple of the identity (for in that case, $\boldsymbol{M}$ is either $\boldsymbol{0}$ or invertible). Therefore, according to Theorem 7.2, there exist $\{\boldsymbol{r}_1, \boldsymbol{r}_2\} \subset \mathbb{R}^2$, such that ϖ has the form (7.23). The vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ cannot be chosen arbitrarily. They have to satisfy the relation $\boldsymbol{K}\boldsymbol{r}_1 = -\boldsymbol{M}^T\boldsymbol{K}^T\boldsymbol{r}_2$, induced by (B.3). Using complex variable formalism we may prove that $\boldsymbol{M}^T\boldsymbol{K}^T = \boldsymbol{K}\boldsymbol{M}$. Indeed, the table of values of $\boldsymbol{M}$ on page 44 shows that in each case $\boldsymbol{M} = \phi(m)$ for some complex number m. Then, equation (B.2) can be written as $\boldsymbol{K}^T\psi(c)\phi(m) = \phi(m)\boldsymbol{K}\psi(c)$. Observing that $\psi(c)\phi(m) = \phi(\overline{m})\psi(c)$, we conclude that $\boldsymbol{K}^T\phi(\overline{m}) = \phi(m)\boldsymbol{K}$. Multiplying this identity by $\phi(m)$ on the right and $\phi(\overline{m})$ on the left we obtain the desired equality. Hence, we may always take $\boldsymbol{r}_1 = -\boldsymbol{M}\boldsymbol{r}_2$.

3. $\dim\mathcal{L} = 1$. Then $\mathcal{L} = \mathbb{R}(\boldsymbol{u}_0, \boldsymbol{v}_0)$. Again we have two cases: $\boldsymbol{u}_0 = \boldsymbol{0}$ or both $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ are non-zero vectors.

(a) Assume that $\boldsymbol{u}_0 = 0$. Then, equation $(7.20)_2$ implies that $w(\boldsymbol{K})$ satisfies $(7.7)_1$ and therefore, according to Theorem 7.2, $w(\boldsymbol{K}) = \boldsymbol{K}\boldsymbol{r}$ for some $\boldsymbol{r} \in \mathbb{R}^2$. We also observe that the only spaces Π_{2d} that satisfy (7.21) are $\Pi_{\boldsymbol{a},\boldsymbol{v}_0}$ and $\Pi_{\boldsymbol{v}_0}$.

i. $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{v}_0}$. Then, taking into account that $w(\boldsymbol{K}) = \boldsymbol{K}\boldsymbol{r}$,

$$\varpi(\boldsymbol{a} \otimes \boldsymbol{v}_0) = \begin{bmatrix} \boldsymbol{a} \otimes \boldsymbol{v}_0 & \lambda \boldsymbol{a} \\ \mu \boldsymbol{v}_0^{\perp} & 0 \end{bmatrix}$$

for some $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Equation $(7.20)_2$ is satisfied if and only if $\mu = 0$. So,

$$\varpi(\boldsymbol{a} \otimes \boldsymbol{v}_0) = \begin{bmatrix} \boldsymbol{a} \otimes \boldsymbol{v}_0 & \lambda \boldsymbol{a} \\ \boldsymbol{0} & 0 \end{bmatrix}.$$

ii. $\Pi_{2d} = \Pi_{\boldsymbol{v}_0}$. Then, taking into account that $w(\boldsymbol{K}) = \boldsymbol{K}\boldsymbol{r}$,

$$\varpi(\boldsymbol{u} \otimes \boldsymbol{v}_0) = \begin{bmatrix} \boldsymbol{u} \otimes \boldsymbol{v}_0 & \lambda \boldsymbol{u} \\ (\boldsymbol{\nu}, \boldsymbol{u})\boldsymbol{v}_0^{\perp} & 0 \end{bmatrix}$$

for some $\lambda \in \mathbb{R}$ and $\boldsymbol{\nu} \in \mathbb{R}^2$. Equation $(7.20)_2$ is satisfied if and only if $\boldsymbol{\nu} = \boldsymbol{0}$. So,

$$\varpi(\boldsymbol{u} \otimes \boldsymbol{v}_0) = \begin{bmatrix} \boldsymbol{u} \otimes \boldsymbol{v}_0 & \lambda \boldsymbol{u} \\ \boldsymbol{0} & 0 \end{bmatrix}.$$

(b) Assume that $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ are non-zero vectors. Then, the only admissible Π_{2d} is $\Pi_{\boldsymbol{u}_0,\boldsymbol{v}_0}$. Suppose, $\varpi(\boldsymbol{u}_0 \otimes \boldsymbol{v}_0) = \begin{bmatrix} \boldsymbol{u}_0 \otimes \boldsymbol{v}_0 & \boldsymbol{a} \\ \boldsymbol{b} & 0 \end{bmatrix}$. Then equation (7.19) becomes

$$(\boldsymbol{a}, \boldsymbol{u}_0) + (\boldsymbol{b}, \boldsymbol{v}_0) = 0. \tag{B.4}$$

Equation(7.21) is identically satisfied, while equation $(7.20)_2$ is equivalent to $\boldsymbol{a} = \lambda \boldsymbol{u}_0$, $\boldsymbol{b} = \mu \boldsymbol{v}_0$. Equation (B.4) then relates λ and μ:

$$\lambda |\boldsymbol{u}_0|^2 + \mu |\boldsymbol{v}_0|^2 = 0.$$

So,

$$\varpi(\boldsymbol{u}_0 \otimes \boldsymbol{v}_0) = \begin{bmatrix} \boldsymbol{u}_0 \otimes \boldsymbol{v}_0 & -\dfrac{\mu |\boldsymbol{v}_0|^2 \boldsymbol{u}_0}{|\boldsymbol{u}_0|^2} \\ \mu \boldsymbol{v}_0 & 0 \end{bmatrix}.$$

C Proof of Theorems 10.1 and 10.2

We can conduct the proofs of Theorems 10.1 and 10.2 simultaneously, ignoring everything that pertains to "ρ-component" or $\boldsymbol{U}_0$ for Theorem 10.2. We observe that $\boldsymbol{U}_0$ is the only non-zero element (up to a scalar multiple) in Π that annihilates Π, i.e. $\boldsymbol{U}_0 *_{\boldsymbol{A}} \boldsymbol{P} = \boldsymbol{0}$ for all $\boldsymbol{P} \in \Pi$, provided $\Pi^{*2} \neq \boldsymbol{0}$. Therefore,

$$\Phi(\boldsymbol{U}_0) = \beta \boldsymbol{U}_0 \tag{C.1}$$

for some $\beta \neq 0$. Now we consider the cases $\Pi_{2d} = \{\mathbf{0}\}$ and $\Pi_{2d} \neq \{\mathbf{0}\}$ separately. We start with $\Pi_{2d} = \{\mathbf{0}\}$. This case is explicitly excluded in Theorem 10.2. Therefore, we will only be concerned with Theorem 10.1 in this part of the proof. We have the following cases (satisfying $\Pi^{*2} \neq \{\mathbf{0}\}$)

- $\dim \Pi = 2$, $\Pi = \begin{bmatrix} \mathbf{0} & t\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix}$.

$$\Phi\left(\begin{bmatrix} \mathbf{0} & t\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} & mt\boldsymbol{a} \\ mt\boldsymbol{a} & \mu t + \beta\rho \end{bmatrix}.$$

  Φ is a Jordan automorphism if and only if $\beta = m^2 \neq 0$. It is easy to verify that all such automorphisms are restrictions of the global Jordan automorphism to Π.

- $\dim \Pi = 3$, $\Pi = \begin{bmatrix} \mathbf{0} & s\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix}$ and $\Pi = \begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{M}\boldsymbol{u} & \rho \end{bmatrix}$

$$\Phi\left(\begin{bmatrix} \mathbf{0} & s\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} & (\mu_{11}s + \mu_{12}t)\boldsymbol{a} \\ (\mu_{21}s + \mu_{22}t)\boldsymbol{a} & s\mu_{13} + t\mu_{23} + \beta\rho \end{bmatrix}.$$

  Φ is a Jordan automorphism if and only if

$$(\mu_{11}s + \mu_{12}t)(\mu_{21}s + \mu_{22}t) = \beta st,$$

  as polynomials in (t, s). Therefore, we may have either

$$\Phi\left(\begin{bmatrix} \mathbf{0} & s\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} & \mu_{11}s\boldsymbol{a} \\ \mu_{22}t\boldsymbol{a} & \mu_{11}\mu_{22}\rho \end{bmatrix} \text{ or } \Phi\left(\begin{bmatrix} \mathbf{0} & s\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} & \mu_{12}t\boldsymbol{a} \\ \mu_{21}s\boldsymbol{a} & \mu_{12}\mu_{21}\rho \end{bmatrix}.$$

  It is easy to verify that in either case all such automorphisms are restrictions of the global Jordan automorphism to Π.

$$\Phi\left(\begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{M}\boldsymbol{u} & \rho \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} & \boldsymbol{M}_1\boldsymbol{u} \\ \boldsymbol{M}\boldsymbol{M}_1\boldsymbol{u} & (\boldsymbol{u}, \boldsymbol{r}) + \beta\rho \end{bmatrix}$$

  is the most general linear map satisfying (C.1). Φ is a Jordan automorphism if and only if

$$(\boldsymbol{A}\boldsymbol{M}_1\boldsymbol{u}, \boldsymbol{M}\boldsymbol{M}_1\boldsymbol{u}) = \beta(\boldsymbol{A}\boldsymbol{u}, \boldsymbol{M}\boldsymbol{u}),$$

  for all $\boldsymbol{u} \in \mathbb{R}^2$ and all $\boldsymbol{A} \in \mathcal{A}$. First we observe that using the global Jordan automorphism transformation, we may always reduce $\boldsymbol{M}$ to a trace-free matrix. Then using complex formalism we find that $\boldsymbol{M}_1 = m\boldsymbol{I}$ with $\beta = m^2$ if $\det \boldsymbol{M} = 0$. If $\det \boldsymbol{M} \neq 0$, then there is a second class of solutions $\boldsymbol{M}_1 = \mu\boldsymbol{M}$ with $\beta = -\mu^2 \det \boldsymbol{M}$. By the Cayley-Hamilton theorem, the trace-free 2×2 matrices $\boldsymbol{M}$ satisfy $\boldsymbol{M}^2 = -(\det \boldsymbol{M})\boldsymbol{I}$. Therefore in both cases the map Φ is a restriction of the global Jordan automorphism to Π.

- $\dim \Pi = 4$, $\Pi = \begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix}$ and $\Pi = \begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{u} + t\boldsymbol{a} & \rho \end{bmatrix}$. For the first case we look for the Jordan automorphism Φ in the form

$$\Phi\left(\begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} & \boldsymbol{M}\boldsymbol{u} + t\boldsymbol{m} \\ (\mu t + (\boldsymbol{u}, \boldsymbol{r}_1))\boldsymbol{a} & (\boldsymbol{u}, \boldsymbol{r}_2) + \lambda t + \beta\rho \end{bmatrix}.$$

The map Φ is a Jordan automorphism if and only if $\boldsymbol{m} = \boldsymbol{r}_1 = \mathbf{0}$, $\boldsymbol{M} = \nu \boldsymbol{I}$ and $\beta = \mu\nu$, i.e.

$$\Phi\left(\begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} & \nu\boldsymbol{u} \\ \mu t\boldsymbol{a} & (\boldsymbol{u}, \boldsymbol{r}_2) + \lambda t + \nu\mu\rho \end{bmatrix}.$$

Φ is a restriction of the global Jordan automorphism to Π. For the second case, the most general linear map satisfying (C.1) has the form

$$\Phi\left(\begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{u} + t\boldsymbol{a} & \rho \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} & \boldsymbol{M}\boldsymbol{u} + t\boldsymbol{m} \\ \boldsymbol{M}\boldsymbol{u} + t\boldsymbol{m} + (\mu t + (\boldsymbol{u}, \boldsymbol{r}_1))\boldsymbol{a} & (\boldsymbol{u}, \boldsymbol{r}_2) + \lambda t + \beta\rho \end{bmatrix}.$$

The map Φ is a Jordan automorphism if and only if it is either

$$\Phi\left(\begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{u} + t\boldsymbol{a} & \rho \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} & \mu\boldsymbol{u} \\ \mu\boldsymbol{u} + \mu t\boldsymbol{a} & (\boldsymbol{u}, \boldsymbol{r}_2) + \lambda t + \mu^2\rho \end{bmatrix}$$

or

$$\Phi\left(\begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{u} + t\boldsymbol{a} & \rho \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} & \mu\boldsymbol{u} + \mu t\boldsymbol{a} \\ \mu\boldsymbol{u} & (\boldsymbol{u}, \boldsymbol{r}_2) + \lambda t + \mu^2\rho \end{bmatrix}.$$

In both cases Φ is a restriction of the global Jordan automorphism to Π.

- $\dim \Pi = 5$, $\Pi = \begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}$. We look for the automorphism Φ in the form

$$\Phi\left(\begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} & \boldsymbol{M}_{11}\boldsymbol{u} + \boldsymbol{M}_{12}\boldsymbol{v} \\ \boldsymbol{M}_{21}\boldsymbol{u} + \boldsymbol{M}_{22}\boldsymbol{v} & (\boldsymbol{u}, \boldsymbol{r}_1) + (\boldsymbol{v}, \boldsymbol{r}_2) + \beta\rho \end{bmatrix}.$$

The map Φ is a Jordan automorphism if and only if it has the form

$$\Phi\left(\begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}\right) = \begin{bmatrix} \mathbf{0} & \phi(m)\boldsymbol{u} \\ \phi\left(\dfrac{m}{\beta}\right)\boldsymbol{v} & (\boldsymbol{u}, \boldsymbol{r}_1) + (\boldsymbol{v}, \boldsymbol{r}_2) + \beta\rho \end{bmatrix}.$$

Φ is a restriction of the global Jordan automorphism to Π.

Now, let us assume that $\Pi_{2d} \neq \{\mathbf{0}\}$. Lemma 10.4 describes how an automorphism Φ acts on $\{\varpi_0(\boldsymbol{K}) : \boldsymbol{K} \in \Pi_{2d}\}$. It remains to understand how Φ acts on $\ker \pi_\Pi$. The list in Theorem 7.6 shows that $\ker \pi_\Pi$ is a direct sum of one or more of the following 1D spaces: $\mathbb{R}\boldsymbol{U}_0$, $\mathbb{R}\boldsymbol{P}(\boldsymbol{c})$, $\mathbb{R}\boldsymbol{P}(\boldsymbol{c})^T$ and $\mathbb{R}\boldsymbol{Q}(\boldsymbol{c})$ for some $\boldsymbol{c} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, where

$$\boldsymbol{P}(\boldsymbol{c}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{c} & 0 \end{bmatrix}, \qquad \boldsymbol{Q}(\boldsymbol{c}) = \begin{bmatrix} \mathbf{0} & \boldsymbol{c} \\ \boldsymbol{c} & 0 \end{bmatrix}.$$

Next, consider $\boldsymbol{P}(\boldsymbol{c})$ (the analysis of $\boldsymbol{P}(\boldsymbol{c})^T$ can be reduced to that of $\boldsymbol{P}(\boldsymbol{c})$ by means of the transpose symmetry). The property $\boldsymbol{P}(\boldsymbol{c})^{*2} = \mathbf{0}$ implies that in the context of Theorem 10.1 we have either

$$\Phi(\boldsymbol{P}(\boldsymbol{c})) \subset \left\{ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{v} & \rho \end{bmatrix} : \boldsymbol{v} \in \mathbb{R}^2,\ \rho \in \mathbb{R} \right\} \tag{C.2}$$

or

$$\Phi(\boldsymbol{P}(\boldsymbol{c})) \subset \left\{ \begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \mathbf{0} & \rho \end{bmatrix} : \boldsymbol{u} \in \mathbb{R}^2,\ \rho \in \mathbb{R} \right\}. \tag{C.3}$$

The transpose symmetry permits us to consider only the first possibility for those Π for which $\Pi_{2d} \subset \mathrm{Sym}(\mathbb{R}^2)$. For those Π for which $\Pi_{2d} = \Pi_{\boldsymbol{a}}$, $\Pi_{\boldsymbol{a}}^T$ or $\mathrm{End}(\mathbb{R}^3)$ both possibilities need to be considered. By contrast, in the context of Theorem 10.2 we have that

$$\Phi(\boldsymbol{P}(\boldsymbol{c})) \subset \left\{ \begin{bmatrix} \mathbf{0} & \boldsymbol{u} \\ \boldsymbol{v} & * \end{bmatrix} : \boldsymbol{u},\ \boldsymbol{v} \in \mathbb{R}^2 \right\}.$$

Let us do the analysis for Theorem 10.1 first. Let us start with (C.2). There exists a 2×2 matrix $\boldsymbol{M}$ and a vector $\boldsymbol{r} \in \mathbb{R}^2$ such that

$$\Phi(\boldsymbol{P}(\boldsymbol{c})) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{M}\boldsymbol{c} & (\boldsymbol{c}, \boldsymbol{r}) \end{bmatrix}.$$

Applying the Jordan isomorphism Φ to the formula

$$\begin{bmatrix} \boldsymbol{K} & \mathbf{0} \\ \boldsymbol{c} & 0 \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K}^{*2} & \mathbf{0} \\ \boldsymbol{K}^T\boldsymbol{A}\boldsymbol{c} & 0 \end{bmatrix}, \tag{C.4}$$

we obtain

$$\boldsymbol{M}\boldsymbol{K}^T\boldsymbol{A}\boldsymbol{c} = \boldsymbol{K}^T\boldsymbol{A}\boldsymbol{M}\boldsymbol{c}, \qquad (\boldsymbol{K}^T\boldsymbol{A}\boldsymbol{c}, \boldsymbol{r}) = 0 \tag{C.5}$$

for all $\boldsymbol{A} \in \mathcal{A}_{2d}$ and all $\boldsymbol{K} \in \Pi_{2d}$. The second equation in (C.5) is equivalent to $\boldsymbol{K}\boldsymbol{r} = 0$. Therefore, either $\boldsymbol{r} = \mathbf{0}$ or Π_{2d} contains only matrices of the form $\boldsymbol{w} \otimes \boldsymbol{r}^\perp$. The first equation in (C.5) is equivalent to $\boldsymbol{M}\boldsymbol{X}\boldsymbol{c} = \boldsymbol{X}\boldsymbol{M}\boldsymbol{c}$ for all $\boldsymbol{X} \in \mathrm{Span}\{\boldsymbol{K}^T\boldsymbol{A} : \boldsymbol{A} \in \mathcal{A}_{2d},\ \boldsymbol{K} \in \Pi_{2d}\}$. The table below shows the spaces that $\boldsymbol{X}$ belongs to, depending on Π_{2d}.

Π_{2d}	$\Pi_{\boldsymbol{a},\boldsymbol{a}}$	$\Pi_{\boldsymbol{a}}$	$\Pi_{\boldsymbol{a}}^T$	$\mathcal{A}_{2d}$	$\mathrm{Sym}(\mathbb{R}^2)$	$\mathrm{End}(\mathbb{R}^2)$
$\mathrm{Span}\{\boldsymbol{K}^T\boldsymbol{A}\}$	$\Pi_{\boldsymbol{a}}^T$	$\Pi_{\boldsymbol{a}}^T$	$\mathrm{End}(\mathbb{R}^2)$	$\{\phi(z) : z \in \mathbb{C}\}$	$\mathrm{End}(\mathbb{R}^2)$	$\mathrm{End}(\mathbb{R}^2)$

We see that for all Π_{2d}, except $\Pi_{2d} = \mathcal{A}_{2d}$ we can take $\boldsymbol{X} = \boldsymbol{b} \otimes \boldsymbol{v}$ for some $\boldsymbol{b} \in \mathbb{R}^2 \setminus \{\boldsymbol{0}\}$ ($\boldsymbol{b}$ must be $\boldsymbol{a}$ for $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{a}}$ and $\Pi_{\boldsymbol{a}}$) and all choices of $\boldsymbol{v} \in \mathbb{R}^2$. The first equation in (C.5) then becomes $\boldsymbol{b} \otimes \boldsymbol{M}\boldsymbol{c} = \boldsymbol{M}\boldsymbol{b} \otimes \boldsymbol{c}$. It follows that $\boldsymbol{M}\boldsymbol{c} = m\boldsymbol{c}$ and $\boldsymbol{M}\boldsymbol{X} = m\boldsymbol{X}$ for all $\boldsymbol{X} \in \mathrm{Span}\{\boldsymbol{K}^T\boldsymbol{A} : \boldsymbol{A} \in \mathcal{A}_{2d},\ \boldsymbol{K} \in \Pi_{2d}\}$. For $\Pi_{2d} = \mathcal{A}_{2d}$ we can use complex variable formalism to show that $\boldsymbol{M} = \phi(m)$ for some $m \in \mathbb{C}$. Putting everything together we conclude that $\Phi(\boldsymbol{P}(\boldsymbol{c})) = m\boldsymbol{P}(\boldsymbol{c})$ for all $\boldsymbol{c} \in \mathbb{R}^2$, except in the following cases

1. $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{a}}$ or $\Pi_{\boldsymbol{a}}$ and $\{\boldsymbol{P}(\boldsymbol{v}) : \boldsymbol{v} \in \mathbb{R}^2\} \subset \ker \pi_\Pi$. Then
$$\Phi(\boldsymbol{P}(\boldsymbol{v})) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ m\boldsymbol{v} & \mu(\boldsymbol{v}, \boldsymbol{a}^\perp) \end{bmatrix},\ m \neq 0.$$

2. $\Pi_{2d} = \mathcal{A}_{2d}$. Then
$$\Phi(\boldsymbol{P}(\boldsymbol{c})) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \phi(m)\boldsymbol{c} & 0 \end{bmatrix},\ m \in \mathbb{C} \setminus \{0\}.$$

Now consider those Jordan algebras Π for which
$$\Phi(\boldsymbol{P}(\boldsymbol{c})) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{M}\boldsymbol{c} \\ \boldsymbol{0} & (\boldsymbol{c}, \boldsymbol{r}) \end{bmatrix}. \tag{C.6}$$

We are going to show that (C.6) is impossible. Applying Φ to the formula (C.4) we get $\boldsymbol{K}\boldsymbol{r} = \boldsymbol{0}$, as before, and
$$\boldsymbol{K}\boldsymbol{A}\boldsymbol{M}\boldsymbol{c} = \boldsymbol{M}\boldsymbol{K}^T\boldsymbol{A}\boldsymbol{c}. \tag{C.7}$$
Recall that we only need to consider the cases when $\Pi_{2d} = \Pi_{\boldsymbol{a}}$, $\Pi_{\boldsymbol{a}}^T$, or $\mathrm{End}(\mathbb{R}^2)$. Let $\Pi_{2d} = \Pi_{\boldsymbol{a}}$. Then we conclude from (C.7) that $\boldsymbol{M}\boldsymbol{a} = \boldsymbol{M}\boldsymbol{c} = \boldsymbol{0}$. If $\boldsymbol{c}$ is a multiple of $\boldsymbol{a}$ then $\Phi(\boldsymbol{P}(\boldsymbol{a})) = \boldsymbol{0}$ in contradiction to the invertibility of the Jordan isomorphism Φ. If $\boldsymbol{c}$ is not a multiple of $\boldsymbol{a}$ then $\mathcal{V} = \{\boldsymbol{P}(\boldsymbol{v}) : \boldsymbol{v} \in \mathbb{R}^2\} \subset \Pi$ and Φ maps the two-dimensional subspace $\mathcal{V}$ onto a one-dimensional subspace $\mathbb{R}\boldsymbol{U}_0$, resulting in contradiction. If $\Pi_{2d} = \Pi_{\boldsymbol{a}}^T$ then (C.7) results in $\boldsymbol{M} = 0$ with the same problem: $\Phi(\boldsymbol{P}(\boldsymbol{c})) = \boldsymbol{0}$. Thus, (C.6) is also impossible for $\Pi_{2d} = \mathrm{End}(\mathbb{R}^2)$.

In the context of Theorem 10.2, the arguments for both (C.2) and (C.3) apply, resulting in $\Phi(\boldsymbol{P}(\boldsymbol{c})) = m\boldsymbol{P}(\boldsymbol{c})$, if $\Pi_{2d} = \Pi_{\boldsymbol{a}}$, $\Pi_{\boldsymbol{a}}^T$ or $\mathrm{End}(\mathbb{R}^2)$ and
$$\Phi(\boldsymbol{P}(\boldsymbol{c})) = \begin{bmatrix} \boldsymbol{0} & m_1\boldsymbol{c} \\ m_2\boldsymbol{c} & * \end{bmatrix},$$
if $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{a}}$ or $\mathrm{Sym}(\mathbb{R}^2)$. If $\Pi_{2d} = \mathcal{A}_{2d}$, we have
$$\Phi(\boldsymbol{P}(\boldsymbol{c})) = \begin{bmatrix} \boldsymbol{0} & \phi(m_1)\boldsymbol{c} \\ \phi(m_2)\boldsymbol{c} & * \end{bmatrix},$$

It remains to consider $\boldsymbol{Q}(\boldsymbol{c})$. This is only relevant to Jordan algebras Π for which $\Pi_{2d} = \mathrm{Sym}(\mathbb{R}^2)$, $\mathcal{A}_{2d}$ or $\Pi_{\boldsymbol{a},\boldsymbol{a}}$, i.e. to those Jordan algebras for which $\boldsymbol{K}^T = \boldsymbol{K}$. The analysis proceeds along the same lines as that for $\boldsymbol{P}(\boldsymbol{c})$. We know that $\Phi(\boldsymbol{Q}(\boldsymbol{c})) \in \ker \pi$. Therefore, there exist two 2×2 matrices $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ and a vector $\boldsymbol{r} \in \mathbb{R}^2$ such that

$$\Phi(\boldsymbol{Q}(\boldsymbol{c})) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{M}_1\boldsymbol{c} \\ \boldsymbol{M}_2\boldsymbol{c} & (\boldsymbol{r}, \boldsymbol{c}) \end{bmatrix}. \tag{C.8}$$

Applying Φ to the formula

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{c} \\ \boldsymbol{c} & 0 \end{bmatrix}^{*2} = \begin{bmatrix} \boldsymbol{K}^{*2} & \boldsymbol{K}\boldsymbol{A}\boldsymbol{c} \\ \boldsymbol{K}\boldsymbol{A}\boldsymbol{c} & (\boldsymbol{A}\boldsymbol{c}, \boldsymbol{c}) \end{bmatrix}$$

we obtain

$$\boldsymbol{K}\boldsymbol{A}\boldsymbol{M}_1\boldsymbol{c} = \boldsymbol{M}_1\boldsymbol{K}\boldsymbol{A}\boldsymbol{c}, \quad \boldsymbol{K}\boldsymbol{A}\boldsymbol{M}_2\boldsymbol{c} = \boldsymbol{M}_2\boldsymbol{K}\boldsymbol{A}\boldsymbol{c}, \quad (\boldsymbol{A}\boldsymbol{M}_1\boldsymbol{c}, \boldsymbol{M}_2\boldsymbol{c}) = (\boldsymbol{K}\boldsymbol{A}\boldsymbol{c}, \boldsymbol{r}) + \beta(\boldsymbol{A}\boldsymbol{c}, \boldsymbol{c}).$$

The last equation is equivalent to

$$\boldsymbol{K}\boldsymbol{r} = \boldsymbol{0}, \quad (\boldsymbol{A}\boldsymbol{M}_1\boldsymbol{c}, \boldsymbol{M}_2\boldsymbol{c}) = \beta(\boldsymbol{A}\boldsymbol{c}, \boldsymbol{c}) \tag{C.9}$$

for all $\boldsymbol{K} \in \Pi_{2d}$ and all $\boldsymbol{A} \in \mathcal{A}_{2d}$. In particular, $\boldsymbol{r} = \boldsymbol{0}$ for $\Pi_{2d} = \mathrm{Sym}(\mathbb{R}^2)$ and $\mathcal{A}_{2d}$ and $\boldsymbol{r} = \mu\boldsymbol{a}^\perp$ for $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{a}}$. The remaining two equations are actually one and the same equation that both $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ have to satisfy: $\boldsymbol{X}\boldsymbol{M}\boldsymbol{c} = \boldsymbol{M}\boldsymbol{X}\boldsymbol{c}$ for all $\boldsymbol{X} \in \mathrm{Span}\{\boldsymbol{K}\boldsymbol{A} : \boldsymbol{A} \in \mathcal{A}_{2d},\ \boldsymbol{K} \in \Pi_{2d}\}$. This equation has already been analyzed above. And it follows that $\boldsymbol{M}_i\boldsymbol{c} = m_i\boldsymbol{c}$, $i = 1, 2$ for $\Pi_{2d} = \mathrm{Sym}(\mathbb{R}^2)$ and $\Pi_{\boldsymbol{a},\boldsymbol{a}}$, while $\boldsymbol{M}_i = \phi(m_i)$ for $\Pi_{2d} = \mathcal{A}_{2d}$. The remaining equation

$$(\boldsymbol{A}\boldsymbol{M}_1\boldsymbol{c}, \boldsymbol{M}_2\boldsymbol{c}) = \beta(\boldsymbol{A}\boldsymbol{c}, \boldsymbol{c})$$

is then equivalent to $m_1 m_2 = \beta$ in all cases. Thus, we arrive at the following conclusions regarding $\Phi(\boldsymbol{Q}(\boldsymbol{u}))$.

1. $\Pi_{2d} = \mathrm{Sym}(\mathbb{R}^2)$.

$$\Phi(\boldsymbol{Q}(\boldsymbol{u})) = \begin{bmatrix} \boldsymbol{0} & m_1\boldsymbol{u} \\ \dfrac{\beta}{m_1}\boldsymbol{u} & 0 \end{bmatrix}.$$

   for any $\boldsymbol{u} \in \mathbb{R}^2$.

2. $\Pi_{2d} = \mathcal{A}_{2d}$.

$$\Phi(\boldsymbol{Q}(\boldsymbol{u})) = \begin{bmatrix} \boldsymbol{0} & \phi(m_1)\boldsymbol{u} \\ \phi\left(\dfrac{\beta}{m_1}\right)\boldsymbol{u} & 0 \end{bmatrix}.$$

   for any $\boldsymbol{u} \in \mathbb{R}^2$.

3. $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{a}}$.

$$\Phi(\boldsymbol{Q}(\boldsymbol{u})) = \begin{bmatrix} \boldsymbol{0} & m_1\boldsymbol{u} \\ \dfrac{\beta}{m_1}\boldsymbol{u} & \mu(\boldsymbol{u},\boldsymbol{a}^\perp) \end{bmatrix}.$$

for any $\boldsymbol{u} \in \mathbb{R}^2$.

In the context of Theorem 10.2 the relation between m_1, m_2 and β is absent and therefore we have

$$\Phi(\boldsymbol{Q}(\boldsymbol{u})) = \begin{bmatrix} \boldsymbol{0} & m_1\boldsymbol{u} \\ m_2\boldsymbol{u} & * \end{bmatrix},$$

if Π_{2d} is either $\Pi_{\boldsymbol{a},\boldsymbol{a}}$ or $\mathrm{Sym}(\mathbb{R}^2)$, and

$$\Phi(\boldsymbol{Q}(\boldsymbol{u})) = \begin{bmatrix} \boldsymbol{0} & \phi(m_1)\boldsymbol{u} \\ \phi(m_2)\boldsymbol{u} & * \end{bmatrix},$$

if $\Pi_{2d} = \mathcal{A}_{2d}$.

So far the action of Φ agrees with the action of some global Jordan automorphism $\boldsymbol{\Psi}$. It now remains to go over all the 13 classes of solutions from Theorem 7.6 and apply our formulas to determine all the Jordan automorphisms of each Π. It is only necessary to consider cases where $\ker \pi_\Pi$ combines $\boldsymbol{P}(\boldsymbol{c})$ and $\boldsymbol{P}^T(\boldsymbol{c})$ or $\boldsymbol{P}(\boldsymbol{c})$ and $\boldsymbol{Q}(\boldsymbol{c})$ or $\boldsymbol{P}^T(\boldsymbol{c})$ and $\boldsymbol{Q}(\boldsymbol{c})$.

- $\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}$, with no restrictions on Π_{2d}.

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}\right) = \varpi_0(\boldsymbol{K}) + \beta\rho\boldsymbol{U}_0 + \Phi(\boldsymbol{P}(\boldsymbol{v})) + (\mathbb{T}\circ\Phi^{\mathbb{T}})(\boldsymbol{P}(\boldsymbol{u})),$$

where $\mathbb{T}$ is the map $\boldsymbol{P} \mapsto \boldsymbol{P}^T$ and $\Phi^{\mathbb{T}} = \mathbb{T}\circ\Phi\circ\mathbb{T}$. Observe that $\Phi^{\mathbb{T}}$ is a Jordan automorphism of Π^T satisfying $\Phi(\varpi_0(\boldsymbol{K})) = \varpi_0(\boldsymbol{K})$.

1. $\Pi_{2d} = \mathrm{Sym}(\mathbb{R}^2)$ or $\mathrm{End}(\mathbb{R}^2)$. Then

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & m_1\boldsymbol{u} \\ m_2\boldsymbol{v} & \beta\rho \end{bmatrix}.$$

The map Φ is a Jordan automorphism if and only if $\beta = m_1m_2$, which implies that Φ is a restriction of the global Jordan automorphism. The result is the same for Theorem 10.2 if $\Pi_{2d} = \mathrm{End}(\mathbb{R}^2)$. However, if $\Pi_{2d} = \mathrm{Sym}(\mathbb{R}^2)$ then the maps

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & * \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & m_{11}\boldsymbol{u} + m_{12}\boldsymbol{v} \\ m_{21}\boldsymbol{u} + m_{22}\boldsymbol{v} & * \end{bmatrix}. \tag{C.10}$$

are automorphisms. Moreover, not all of them can be obtained as a restriction of the global Jordan automorphism onto Π. However, if we adjoin the isomorphism

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & * \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u}+\boldsymbol{v} \\ \boldsymbol{v} & * \end{bmatrix}. \tag{C.11}$$

to the global ones, then all the isomorphisms (C.10) can be generated.

2. $\Pi_{2d} = \mathcal{A}_{2d}$. Then

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \phi(m_1)\boldsymbol{u} \\ \phi(m_2)\boldsymbol{v} & \beta\rho \end{bmatrix}.$$

The map Φ is a Jordan automorphism if and only if $\beta = m_1 m_2$, which implies that Φ is a restriction of the global Jordan automorphism. In the case of Theorem 10.2 the result is a bit different

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & * \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \phi(m_{11})\boldsymbol{u} + \phi(m_{12})\boldsymbol{v} \\ \phi(m_{21})\boldsymbol{u} + \phi(m_{22})\boldsymbol{v} & * \end{bmatrix}. \tag{C.12}$$

Not all automorphisms (C.12) are restrictions of the global Jordan automorphism to Π. However, if we adjoint the isomorphisms (C.11) and

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & * \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & \phi(m)\boldsymbol{u} \\ \boldsymbol{v} & * \end{bmatrix}. \tag{C.13}$$

to the global ones, then all the isomorphisms (C.12) can be generated.

3. $\Pi_{2d} = \Pi_{\boldsymbol{a}}$. Then

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & m_1\boldsymbol{u} \\ m_2\boldsymbol{v} & \beta\rho + \mu(\boldsymbol{v}, \boldsymbol{e}_2) \end{bmatrix}.$$

The map Φ is a Jordan automorphism if and only if $\beta = m_1 m_2$, which implies that Φ is a restriction of the global Jordan automorphism. The result is the same for Theorem 10.2 as well, since it is the same for the image of $\boldsymbol{P}(\boldsymbol{c})$.

4. $\Pi_{2d} = \Pi_{\boldsymbol{a}}^T$. Then $\Phi^{\mathbb{T}} : \Pi^T \to \Pi^T$ with $(\Pi^T)_{2d} = \Pi_{\boldsymbol{a}}$. Hence, $\Phi^{\mathbb{T}}$ must be given by the formula for $\Pi_{2d} = \Pi_{\boldsymbol{a}}$. So,

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & m_1\boldsymbol{u} \\ m_2\boldsymbol{v} & m_1 m_2\rho + \mu(\boldsymbol{u}, \boldsymbol{e}_2) \end{bmatrix}.$$

Thus, Φ is a restriction of the global Jordan automorphism. The result is the same for Theorem 10.2 as well, since it is the same for the image of $\boldsymbol{P}(\boldsymbol{c})$.

5. $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{a}}$. Then

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & m_1\boldsymbol{u} \\ m_2\boldsymbol{v} & \beta\rho + (\mu_1\boldsymbol{u} + \mu_2\boldsymbol{v}, \boldsymbol{a}^\perp) \end{bmatrix}.$$

The map Φ is a Jordan automorphism if and only if $\beta = m_1 m_2$, which implies that Φ is a restriction of the global Jordan automorphism. In the case of Theorem 10.2 we have a sightly different result

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & * \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & m_{11}\boldsymbol{u} + m_{12}\boldsymbol{v} \\ m_{21}\boldsymbol{u} + m_{22}\boldsymbol{v} & * \end{bmatrix}.$$

These maps are restrictions of the automorphisms (C.10).

- $\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix}$, where $\Pi_{2d} = \Pi_{\boldsymbol{a}}$ or $\Pi_{\boldsymbol{a},\boldsymbol{a}}$. Then

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix}\right) = \varpi_0(\boldsymbol{K}) + \beta\rho\boldsymbol{U}_0 + m_2 t\boldsymbol{P}(\boldsymbol{a}) + \mathbb{T}\circ\Phi^{\mathbb{T}}(\boldsymbol{P}(\boldsymbol{u})).$$

1. $\Pi_{2d} = \Pi_{\boldsymbol{a}}$. Then
$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & m_1\boldsymbol{u} \\ m_2 t\boldsymbol{a} & \beta\rho \end{bmatrix}.$$

The map Φ is a Jordan automorphism if and only if $\beta = m_1 m_2$, which implies that Φ is a restriction of the global Jordan automorphism. The result is the same for Theorem 10.2 as well, since it is the same for the image of $\boldsymbol{P}(\boldsymbol{c})$.

2. $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{a}}$. Then

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ t\boldsymbol{a} & \rho \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & m_1\boldsymbol{u} \\ m_2 t\boldsymbol{a} & \beta\rho + \mu(\boldsymbol{u}, \boldsymbol{a}^\perp) \end{bmatrix}.$$

The map Φ is a Jordan automorphism if and only if $\beta = m_1 m_2$, which implies that Φ is a restriction of the global Jordan automorphism.

In the case of Theorem 10.2 we have a sightly different result

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ t\boldsymbol{a} & * \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & m_{11}\boldsymbol{u} + m_{12}t\boldsymbol{a} \\ m_{22}t\boldsymbol{a} & * \end{bmatrix}.$$

These automorphisms are just restrictions of (C.10).

- $\begin{bmatrix} \boldsymbol{K} & s\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix}$, $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{a}}$. Then

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & s\boldsymbol{a} \\ t\boldsymbol{a} & \rho \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & m_1 s\boldsymbol{a} \\ m_2 t\boldsymbol{a} & \beta\rho \end{bmatrix}.$$

  The map Φ is a Jordan automorphism if and only if $\beta = m_1 m_2$, which implies that Φ is a restriction of the global Jordan automorphism. In the case of Theorem 10.2 we have a sightly different result

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & s\boldsymbol{a} \\ t\boldsymbol{a} & * \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & (m_{11}s + m_{12}t)\boldsymbol{a} \\ (m_{12}s + m_{22}t)\boldsymbol{a} & * \end{bmatrix}.$$

  These automorphisms are just restrictions of (C.10).

- $\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{u} + t\boldsymbol{a} & \rho \end{bmatrix}$, $\Pi_{2d} = \Pi_{\boldsymbol{a},\boldsymbol{a}}$.

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{u} + t\boldsymbol{a} & \rho \end{bmatrix}\right) = \varpi_0(\boldsymbol{K}) + \beta\rho\boldsymbol{U}_0 + tm_1\boldsymbol{P}(\boldsymbol{a}) + \Phi(\boldsymbol{Q}(\boldsymbol{u})).$$

  We have

$$\Phi(\boldsymbol{Q}(\boldsymbol{u})) = \begin{bmatrix} \boldsymbol{0} & m_2\boldsymbol{u} \\ \dfrac{\beta}{m_2}\boldsymbol{u} & \mu(\boldsymbol{u}, \boldsymbol{a}^\perp) \end{bmatrix}.$$

  It is easy to check that $\Phi(\boldsymbol{Q}(\boldsymbol{u})) \in \Pi$, if and only if $\beta = m_2^2$. Then

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{u} + t\boldsymbol{a} & \rho \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & m_2\boldsymbol{u} \\ m_2\boldsymbol{u} + m_1 t\boldsymbol{a} & m_2^2\rho + \mu(\boldsymbol{u}, \boldsymbol{a}^\perp) \end{bmatrix}.$$

  The map Φ is a Jordan automorphism if and only if $m_1 = m_2$, which implies that Φ is a restriction of the global Jordan automorphism. Once again the answer is a little different in the case of Theorem 10.2.

$$\Phi\left(\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{u} + t\boldsymbol{a} & * \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{K} & m_1\boldsymbol{u} + m_2 t\boldsymbol{a} \\ m_1\boldsymbol{u} + m_3 t\boldsymbol{a} & * \end{bmatrix}.$$

  These automorphisms are restrictions of (C.10).

Theorems 10.1 and 10.2 are now proved.

D Derivation of the links (2.2) and (2.3)

In this section we compute the link $\widehat{\mathbb{M}}$ corresponding to the subspace

$$\widehat{\Pi} = \{[\boldsymbol{P}, \boldsymbol{XPY}] : \boldsymbol{P} \in \mathrm{End}(\mathbb{R}^3)\}, \tag{D.1}$$

where the matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ are given by (7.27). To maintain full generality, we must assume that

$$\widehat{\boldsymbol{L}}_0 = [\boldsymbol{L}_0^{(1)}, \boldsymbol{L}_0^{(2)}] = \left[\begin{bmatrix} \boldsymbol{\sigma}_0 + r_0^{(1)}\boldsymbol{S} & \boldsymbol{p}_0^{(1)} \\ \boldsymbol{q}_0^{(1)} & \alpha_0^{(1)} \end{bmatrix}, \begin{bmatrix} s_0\boldsymbol{\sigma}_0 + r_0^{(2)}\boldsymbol{S} & \boldsymbol{p}_0^{(2)} \\ \boldsymbol{q}_0^{(2)} & \alpha_0^{(2)} \end{bmatrix}\right], \tag{D.2}$$

where $\boldsymbol{\sigma}_0^T = \boldsymbol{\sigma}_0$. The simplifications afforded by formulas (6.10) cannot be used to avoid circular argument, because these simplifications have been obtained by applying the link (D.1) in its most general form. However, the simplification obtained by the transformation (6.1) can be used. In that case the more general inversion formulas (5.10) simplify to

$$\widehat{\boldsymbol{L}} = \widehat{\boldsymbol{L}}_0 - \widehat{\boldsymbol{C}}_1[\widehat{\boldsymbol{I}} + \widehat{\boldsymbol{P}}\widehat{\boldsymbol{M}_0}]^{-1}\widehat{\boldsymbol{P}}\widehat{\boldsymbol{C}}_2\widehat{\boldsymbol{L}}_0, \tag{D.3}$$

where $\widehat{\boldsymbol{P}} \in \widehat{\Pi}$, $\widehat{\boldsymbol{M}_0} = [\boldsymbol{e}\otimes\boldsymbol{e}, \boldsymbol{e}\otimes\boldsymbol{e}]$, $\widehat{\boldsymbol{C}}_j = [\boldsymbol{C}_j^{(1)}, \boldsymbol{C}_j^{(2)}]$, $j = 1, 2$ and where $\boldsymbol{C}_1^{(i)}$ and $\boldsymbol{C}_2^{(i)}$ are given by (6.3) for each $i = 1, 2$. Writing (D.3) in block-components we obtain for each block-component (suppressing superscripts (1) and (2) to simplify notation)

$$\begin{aligned} &\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_0 - \sigma_0^{1/2}\overline{\boldsymbol{K}}\sigma_0^{1/2}, \quad \boldsymbol{p} = \boldsymbol{\Lambda}\widehat{\boldsymbol{p}}_0 - \widehat{\alpha}_0\sigma_0^{1/2}\overline{\boldsymbol{u}}, \\ &\boldsymbol{q} = \boldsymbol{q}_0 - \sigma_0^{1/2}\overline{\boldsymbol{v}}, \quad \alpha = \widehat{\alpha}_0(1-\overline{\rho}) + (\boldsymbol{q}, \widehat{\boldsymbol{p}}_0), \\ &\overline{\boldsymbol{K}} = [\boldsymbol{I} + \boldsymbol{Ke}\otimes\boldsymbol{e}]^{-1}\boldsymbol{K}, \quad \overline{\boldsymbol{u}} = \boldsymbol{u} - (\boldsymbol{u}, \boldsymbol{e})\overline{\boldsymbol{K}}\boldsymbol{e}, \\ &\overline{\boldsymbol{v}} = \boldsymbol{v} - (\boldsymbol{v}, \boldsymbol{e})\overline{\boldsymbol{K}}^T\boldsymbol{e}, \quad \overline{\rho} = \rho - \frac{(\boldsymbol{u}, \boldsymbol{e})(\boldsymbol{v}, \boldsymbol{e})}{1 + (\boldsymbol{Ke}, \boldsymbol{e})}, \end{aligned} \tag{D.4}$$

where $\widehat{\boldsymbol{p}}_0 = \boldsymbol{\Lambda}_0^{-1}\boldsymbol{p}_0$ and $\widehat{\alpha}_0 = \alpha_0 - (\widehat{\boldsymbol{p}}_0, \boldsymbol{q}_0)$. We remark that $\widehat{\alpha}_0 = \det\boldsymbol{L}_0/\det\boldsymbol{\Lambda}_0 > 0$, if $\boldsymbol{L}_0$ is positive definite. Formulas (D.4) apply to both components of $\widehat{\boldsymbol{L}}$, except the constants (quantities with zero subscript) for $\boldsymbol{L}_1$ come from $\boldsymbol{L}_0^{(1)}$, while the constants for $\boldsymbol{L}_2$ come from $\boldsymbol{L}_0^{(2)}$.

To simplify notation we denote $\phi(e^{i\theta})$ by $\boldsymbol{R}$, since it is a rotation matrix. Let $\boldsymbol{P}_2 = \boldsymbol{XPY}$. Then we have

$$\boldsymbol{K}_2 = \boldsymbol{RKR}, \quad \boldsymbol{u}_2 = \boldsymbol{RKb} + \delta\boldsymbol{Ru}, \quad \boldsymbol{v}_2 = \boldsymbol{R}^T\boldsymbol{K}^T\boldsymbol{a} + \tau\boldsymbol{R}^T\boldsymbol{v}, \tag{D.5}$$

$$\rho_2 = (\boldsymbol{a}, \boldsymbol{Kb}) + \delta(\boldsymbol{u}, \boldsymbol{a}) + \tau(\boldsymbol{v}, \boldsymbol{b}) + \delta\tau\rho. \tag{D.6}$$

The goal is to express $\boldsymbol{L}_2$ in terms of $\boldsymbol{L}$.

D.1 Computation for the Λ-block

We have, according to (D.4) and (D.5)

$$\overline{\boldsymbol{K}}_2 = [\boldsymbol{I} + \boldsymbol{R}\boldsymbol{K}\boldsymbol{R}\boldsymbol{e}\otimes\boldsymbol{e}]^{-1}\boldsymbol{R}\boldsymbol{K}\boldsymbol{R} = [\boldsymbol{R}^T + \boldsymbol{K}\boldsymbol{R}\boldsymbol{e}\otimes\boldsymbol{e}]^{-1}\boldsymbol{K}\boldsymbol{R}.$$

$$\boldsymbol{K}\boldsymbol{R}\boldsymbol{e} = [\boldsymbol{I} - \overline{\boldsymbol{K}}\boldsymbol{e}\otimes\boldsymbol{e}]^{-1}\overline{\boldsymbol{K}}\boldsymbol{R}\boldsymbol{e}.$$

Therefore,

$$\overline{\boldsymbol{K}}_2 = [(\boldsymbol{I} - \overline{\boldsymbol{K}}\boldsymbol{e}\otimes\boldsymbol{e})\boldsymbol{R}^T + \overline{\boldsymbol{K}}\boldsymbol{R}\boldsymbol{e}\otimes\boldsymbol{e}]^{-1}(\boldsymbol{I} - \overline{\boldsymbol{K}}\boldsymbol{e}\otimes\boldsymbol{e})\boldsymbol{K}\boldsymbol{R}.$$

$$\overline{\boldsymbol{K}}_2 = [\boldsymbol{R}^T - \overline{\boldsymbol{K}}\boldsymbol{e}\otimes\boldsymbol{R}\boldsymbol{e} + \overline{\boldsymbol{K}}\boldsymbol{R}\boldsymbol{e}\otimes\boldsymbol{e}]^{-1}\overline{\boldsymbol{K}}\boldsymbol{R} = [\boldsymbol{R}^T + \overline{\boldsymbol{K}}(\boldsymbol{R}\boldsymbol{e}\otimes\boldsymbol{e} - \boldsymbol{e}\otimes\boldsymbol{R}\boldsymbol{e})]^{-1}\overline{\boldsymbol{K}}\boldsymbol{R}.$$

It is easy to verify (using complex variable formalism (6.4) for example) that

$$\boldsymbol{R}\boldsymbol{e}\otimes\boldsymbol{e} - \boldsymbol{e}\otimes\boldsymbol{R}\boldsymbol{e} = \boldsymbol{S}\sin\theta.$$

Therefore, we obtain

$$\overline{\boldsymbol{K}}_2 = [\boldsymbol{R}^T + \overline{\boldsymbol{K}}\boldsymbol{S}\sin\theta]^{-1}\overline{\boldsymbol{K}}\boldsymbol{R}.$$

Expressing $\overline{\boldsymbol{K}}$ and $\overline{\boldsymbol{K}}_2$ in terms of $\boldsymbol{\Lambda}^{(1)} = \boldsymbol{\sigma} + r\boldsymbol{S}$ and $\boldsymbol{\Lambda}^{(2)} = \boldsymbol{\sigma}_2 + r_2\boldsymbol{S}$, respectively, we obtain

$$\overline{\boldsymbol{K}}_2 = \boldsymbol{I} - \boldsymbol{\sigma}_2' + r_2'\boldsymbol{S}, \qquad \overline{\boldsymbol{K}} = \boldsymbol{I} - \boldsymbol{\sigma}' + r'\boldsymbol{S},$$

where

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma}_0^{-1/2}\boldsymbol{\sigma}\boldsymbol{\sigma}_0^{-1/2}; \quad \boldsymbol{\sigma}_2' = \frac{1}{s_0}\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{\sigma}_2\boldsymbol{\sigma}_0^{-1/2}; \quad r_2' = \frac{r_0^{(2)} - r_2}{s_0\sqrt{\det\boldsymbol{\sigma}_0}}, \quad r' = \frac{r_0^{(1)} - r}{\sqrt{\det\boldsymbol{\sigma}_0}}.$$

We have

$$\boldsymbol{R}^T + \overline{\boldsymbol{K}}\boldsymbol{S}\sin\theta = \boldsymbol{R}^T + \boldsymbol{S}\sin\theta - r'\boldsymbol{I}\sin\theta - \boldsymbol{\sigma}'\boldsymbol{S}\sin\theta = ((\cos\theta - r'\sin\theta)\boldsymbol{S} + \boldsymbol{\sigma}'\sin\theta)\boldsymbol{S}^T,$$

Because $\boldsymbol{R} = \boldsymbol{I}\cos\theta + \boldsymbol{S}\sin\theta$. Therefore,

$$\boldsymbol{\sigma}_2' = \boldsymbol{I} + r_2'\boldsymbol{S} - \boldsymbol{S}(\boldsymbol{\sigma}'\sin\theta + (\cos\theta - r'\sin\theta)\boldsymbol{S})^{-1}(\boldsymbol{I} + r'\boldsymbol{S} - \boldsymbol{\sigma}')\boldsymbol{R}.$$

$$\boldsymbol{\sigma}_2' - r_2'\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{\sigma}'\sin\theta + (\cos\theta - r'\sin\theta)\boldsymbol{S})^{-1}\left\{\boldsymbol{\sigma}'\cos\theta - (r'\cos\theta + \sin\theta)\boldsymbol{S}\right\}.$$

$$\boldsymbol{\sigma}'\cos\theta - (r'\cos\theta + \sin\theta)\boldsymbol{S} = (\boldsymbol{\sigma}'\sin\theta + (\cos\theta - r'\sin\theta)\boldsymbol{S})\cot\theta - \csc\theta\boldsymbol{S}.$$

Thus

$$\boldsymbol{\sigma}_2' - r_2'\boldsymbol{S} = \boldsymbol{S}\cot\theta + \boldsymbol{S}(\boldsymbol{\sigma}'\sin\theta + (\cos\theta - r'\sin\theta)\boldsymbol{S})^{-1}\boldsymbol{S}^T\csc\theta.$$

Now we use the fact that for any invertible 2×2 matrix $\boldsymbol{A}$ we have

$$\boldsymbol{A}^{-1} = \frac{\boldsymbol{S}\boldsymbol{A}^T\boldsymbol{S}^T}{\det\boldsymbol{A}}.$$

We obtain

$$\boldsymbol{\sigma}_2' - r_2'\boldsymbol{S} = \boldsymbol{S}\cot\theta + \csc^2\theta\frac{\boldsymbol{\sigma}' + (r' - \cot\theta)\boldsymbol{S}}{\det\boldsymbol{\sigma}' + (r' - \cot\theta)^2}.$$

Hence,

$$\boldsymbol{\sigma}_2 + (r_2 - r_0^{(2)})\boldsymbol{S} = s_0\beta_0\boldsymbol{S} + s_0 \det \boldsymbol{\sigma}_0 \csc^2\theta \frac{\boldsymbol{\sigma} + (r_0^{(1)} - r - \beta_0)\boldsymbol{S}}{\det \boldsymbol{\sigma} + (r_0^{(1)} - r - \beta_0)^2},$$

where $\beta_0 = \sqrt{\det \boldsymbol{\sigma}_0} \cot\theta$. The last formula can be written as

$$\boldsymbol{\Lambda}_2 - r_0'\boldsymbol{S} = \tau_0 \frac{(\boldsymbol{\Lambda} - r_0\boldsymbol{S})^T}{\det(\boldsymbol{\Lambda} - r_0\boldsymbol{S})}. \tag{D.7}$$

where

$$r_0' = s_0\beta_0 + r_0^{(2)}, \quad \tau_0 = s_0 \det \boldsymbol{\sigma}_0 \csc^2\theta, \quad r_0 = r_0^{(1)} - \beta_0. \tag{D.8}$$

If $\theta = 0$ or π then we obtain $\boldsymbol{\sigma}_2' - r_2'\boldsymbol{S} = \boldsymbol{\sigma}' - r'\boldsymbol{S}$ and

$$\boldsymbol{\Lambda}_2 - r_0^{(2)}\boldsymbol{S} = s_0(\boldsymbol{\Lambda} - r_0^{(1)}\boldsymbol{S}). \tag{D.9}$$

D.2 Computation for the $\boldsymbol{p}$ and $\boldsymbol{q}$-blocks

Let $\boldsymbol{b}' = \boldsymbol{R}^T\boldsymbol{b}$ and $\boldsymbol{a}' = \boldsymbol{R}\boldsymbol{a}$. Then (D.5) can be written as

$$\boldsymbol{u}_2 = \boldsymbol{K}_2\boldsymbol{b}' + \delta\boldsymbol{R}\boldsymbol{u}, \qquad \boldsymbol{v}_2 = \boldsymbol{K}_2^T\boldsymbol{a}' + \tau\boldsymbol{R}^T\boldsymbol{v},$$

By (D.4) we get

$$\overline{\boldsymbol{u}}_2 = \boldsymbol{u}_2 - (\boldsymbol{u}_2, \boldsymbol{e})\overline{\boldsymbol{K}}_2\boldsymbol{e} = \boldsymbol{K}_2\boldsymbol{b}' + \delta\boldsymbol{R}\boldsymbol{u} - \delta(\boldsymbol{R}\boldsymbol{u}, \boldsymbol{e})\overline{\boldsymbol{K}}_2\boldsymbol{e} - \frac{(\boldsymbol{K}_2\boldsymbol{b}', \boldsymbol{e})\boldsymbol{K}_2\boldsymbol{e}}{1 + (\boldsymbol{K}_2\boldsymbol{e}, \boldsymbol{e})},$$

Since

$$\overline{\boldsymbol{K}} = \boldsymbol{K} - \frac{\boldsymbol{K}\boldsymbol{e} \otimes \boldsymbol{K}^T\boldsymbol{e}}{1 + (\boldsymbol{K}\boldsymbol{e}, \boldsymbol{e})}, \qquad \overline{\boldsymbol{K}}\boldsymbol{e} = \frac{\boldsymbol{K}\boldsymbol{e}}{1 + (\boldsymbol{K}\boldsymbol{e}, \boldsymbol{e})}. \tag{D.10}$$

Therefore,

$$\overline{\boldsymbol{u}}_2 = \overline{\boldsymbol{K}}_2\boldsymbol{b}' + \delta\boldsymbol{R}\boldsymbol{u} - \delta(\boldsymbol{R}\boldsymbol{u}, \boldsymbol{e})\overline{\boldsymbol{K}}_2\boldsymbol{e}.$$

We also have

$$\boldsymbol{R}\boldsymbol{u} = \boldsymbol{R}\overline{\boldsymbol{u}} + (\overline{\boldsymbol{u}}, \boldsymbol{e})\boldsymbol{K}_2\boldsymbol{R}^T\boldsymbol{e}.$$

Thus,

$$\overline{\boldsymbol{u}}_2 = \overline{\boldsymbol{K}}_2\boldsymbol{b}' + \delta\boldsymbol{R}\overline{\boldsymbol{u}} + \delta(\overline{\boldsymbol{u}}, \boldsymbol{e})\boldsymbol{K}_2\boldsymbol{R}^T\boldsymbol{e} - \delta(\boldsymbol{R}\overline{\boldsymbol{u}}, \boldsymbol{e})\overline{\boldsymbol{K}}_2\boldsymbol{e} - \delta(\overline{\boldsymbol{u}}, \boldsymbol{e})\frac{(\boldsymbol{K}_2\boldsymbol{R}^T\boldsymbol{e}, \boldsymbol{e})\boldsymbol{K}_2\boldsymbol{e}}{1 + (\boldsymbol{K}_2\boldsymbol{e}, \boldsymbol{e})}.$$

Combining the third and the last term, using (D.10), we obtain

$$\overline{\boldsymbol{u}}_2 = \overline{\boldsymbol{K}}_2\boldsymbol{b}' + \delta\boldsymbol{R}\overline{\boldsymbol{u}} + \delta(\overline{\boldsymbol{u}}, \boldsymbol{e})\overline{\boldsymbol{K}}_2\boldsymbol{R}^T\boldsymbol{e} - \delta(\boldsymbol{R}\overline{\boldsymbol{u}}, \boldsymbol{e})\overline{\boldsymbol{K}}_2\boldsymbol{e} = \overline{\boldsymbol{K}}_2\boldsymbol{b}' + \delta\boldsymbol{R}\overline{\boldsymbol{u}} - \delta\sin\theta\overline{\boldsymbol{K}}_2\boldsymbol{S}\overline{\boldsymbol{u}}.$$

Similarly, $\overline{\boldsymbol{v}}_2 = \overline{\boldsymbol{K}}_2^T\boldsymbol{a}' + \tau\boldsymbol{R}^T\overline{\boldsymbol{v}} + \tau\sin\theta\overline{\boldsymbol{K}}_2^T\boldsymbol{S}\overline{\boldsymbol{v}}$. Let us start with the $\boldsymbol{q}$-block. It will be convenient to rewrite $\overline{\boldsymbol{v}}_2$ in the following form.

$$\overline{\boldsymbol{v}}_2 = \tau(\sin\theta\overline{\boldsymbol{K}}_2^T + \boldsymbol{R}^T\boldsymbol{S}^T)\boldsymbol{S}\left(\overline{\boldsymbol{v}} - \frac{\boldsymbol{S}\boldsymbol{a}'}{\tau\sin\theta}\right) + \frac{\boldsymbol{R}^T\boldsymbol{S}\boldsymbol{a}'}{\sin\theta}.$$

Therefore,

$$\boldsymbol{\sigma}_0^{1/2}\overline{\boldsymbol{v}}_2 = \frac{\tau}{\sqrt{\det\boldsymbol{\sigma}_0}}(\sin\theta\boldsymbol{\sigma}_0^{1/2}\overline{\boldsymbol{K}}_2^T\boldsymbol{\sigma}_0^{1/2} + \boldsymbol{\sigma}_0^{1/2}\boldsymbol{R}^T\boldsymbol{S}^T\boldsymbol{\sigma}_0^{1/2})\boldsymbol{S}\left(\boldsymbol{\sigma}_0^{1/2}\overline{\boldsymbol{v}} - \frac{\boldsymbol{\sigma}_0^{1/2}\boldsymbol{S}\boldsymbol{a}'}{\tau\sin\theta}\right) + \frac{\boldsymbol{\sigma}_0^{1/2}\boldsymbol{R}^T\boldsymbol{S}\boldsymbol{a}'}{\sin\theta}.$$

Using (D.4) we obtain

$$\boldsymbol{q}_2 - \boldsymbol{q}_0^{(2)} = \frac{\tau\sqrt{s_0}}{\sqrt{\det\boldsymbol{\sigma}_0}}\left(\frac{\sin\theta}{s_0}(\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_0^{(2)})^T - \boldsymbol{\sigma}_0^{1/2}\boldsymbol{R}^T\boldsymbol{S}^T\boldsymbol{\sigma}_0^{1/2}\right)\boldsymbol{S}\left(\boldsymbol{q}_0^{(1)} - \boldsymbol{q} - \frac{\boldsymbol{\sigma}_0^{1/2}\boldsymbol{S}\boldsymbol{a}'}{\tau\sin\theta}\right) - \frac{\sqrt{s_0}\boldsymbol{\sigma}_0^{1/2}\boldsymbol{R}^T\boldsymbol{S}\boldsymbol{a}'}{\sin\theta}$$

Observe that

$$\frac{\sin\theta}{s_0}(\boldsymbol{\Lambda}_0^{(2)})^T + \boldsymbol{\sigma}_0^{1/2}\boldsymbol{R}^T\boldsymbol{S}^T\boldsymbol{\sigma}_0^{1/2} = (\boldsymbol{\sigma}_0 - \frac{r_0^{(2)}}{s_0}\boldsymbol{S})\sin\theta - \sqrt{\det\boldsymbol{\sigma}_0}\cos\theta\boldsymbol{S} - \boldsymbol{\sigma}_0\sin\theta = -\frac{\sin\theta}{s_0}r_0'\boldsymbol{S},$$

where r_0' is defined in (D.8). Let us define new constants

$$\boldsymbol{q}_0 = \boldsymbol{q}_0^{(1)} - \frac{\boldsymbol{\sigma}_0^{1/2}\boldsymbol{S}\boldsymbol{a}'}{\tau\sin\theta}, \quad \boldsymbol{q}_0' = \boldsymbol{q}_0^{(2)} - \frac{\sqrt{s_0}\boldsymbol{\sigma}_0^{1/2}\boldsymbol{R}^T\boldsymbol{S}\boldsymbol{a}'}{\sin\theta}, \quad \nu_0 = -\frac{\tau\sqrt{s_0\det\boldsymbol{\sigma}_0}}{\sin\theta}. \tag{D.11}$$

Then, we obtain

$$\boldsymbol{q}_2 = \boldsymbol{q}_0' + \nu_0\boldsymbol{\Psi}(\boldsymbol{\Lambda})^T(\boldsymbol{q} - \boldsymbol{q}_0)^\perp,$$

where $\boldsymbol{\Psi}(\boldsymbol{\Lambda})$ is given by (2.1). Here we have chosen to define the constant ν_0 with the "$-$" sign, so as to produce the same expression for the $\boldsymbol{q}$-block as in (2.2).

Now let us turn to the computation of the $\boldsymbol{p}$-block. Our first step is the same as in the computation of the $\boldsymbol{q}$-block. We rewrite $\overline{\boldsymbol{u}}_2$ as

$$\overline{\boldsymbol{u}}_2 = -\delta(\boldsymbol{R}\boldsymbol{S} + \sin\theta\overline{\boldsymbol{K}}_2)\boldsymbol{S}(\overline{\boldsymbol{u}} + \frac{\boldsymbol{S}\boldsymbol{b}'}{\delta\sin\theta}) - \frac{\boldsymbol{R}\boldsymbol{S}\boldsymbol{b}'}{\sin\theta},$$

multiply the equation by $\boldsymbol{\sigma}_0^{1/2}$ and use (D.4):

$$\frac{\boldsymbol{p}_2 - \boldsymbol{\Lambda}_2\widehat{\boldsymbol{p}}_0^{(2)}}{\sqrt{s_0}\widehat{\alpha}_0^{(2)}} = \frac{\tau_0\delta\sin\theta}{s_0\sqrt{\det\boldsymbol{\sigma}_0}}\boldsymbol{\Psi}(\boldsymbol{\Lambda})\boldsymbol{S}\left(\frac{\boldsymbol{p} - \boldsymbol{\Lambda}\widehat{\boldsymbol{p}}_0^{(1)}}{\widehat{\alpha}_0^{(1)}} - \frac{\boldsymbol{\sigma}_0^{1/2}\boldsymbol{S}\boldsymbol{b}'}{\delta\sin\theta}\right) + \frac{\boldsymbol{\sigma}_0^{1/2}\boldsymbol{R}\boldsymbol{S}\boldsymbol{b}'}{\sin\theta},$$

where we had used the same calculation as for the $\boldsymbol{q}$-block:

$$\boldsymbol{\sigma}_0^{1/2}\boldsymbol{R}\boldsymbol{S}\boldsymbol{\sigma}_0^{1/2}+\frac{\sin\theta}{s_0}\boldsymbol{\Lambda}_0^{(2)}=\frac{r_0'\sin\theta}{s_0}\boldsymbol{S},\qquad \boldsymbol{\Lambda}_2-r_0'\boldsymbol{S}=\tau_0\boldsymbol{\Psi}(\boldsymbol{\Lambda}).$$

To simplify the expression for $\boldsymbol{p}_2$ we observe that

$$\boldsymbol{\Psi}(\boldsymbol{\Lambda})\boldsymbol{S}(\boldsymbol{\Lambda}-r_0\boldsymbol{S})=\boldsymbol{S},$$

Therefore,

$$\frac{\boldsymbol{p}_2-\boldsymbol{\Lambda}_2\widehat{\boldsymbol{p}}_0^{(2)}}{\sqrt{s_0}\widehat{\alpha}_0^{(2)}}=\frac{\tau_0\delta\sin\theta}{s_0\widehat{\alpha}_0^{(1)}\sqrt{\det\boldsymbol{\sigma}_0}}\boldsymbol{\Psi}(\boldsymbol{\Lambda})\boldsymbol{S}\left(\boldsymbol{p}-r_0\boldsymbol{S}\widehat{\boldsymbol{p}}_0^{(1)}-\frac{\widehat{\alpha}_0^{(1)}\boldsymbol{\sigma}_0^{1/2}\boldsymbol{S}\boldsymbol{b}'}{\delta\sin\theta}\right)+\frac{\boldsymbol{\sigma}_0^{1/2}\boldsymbol{R}\boldsymbol{S}\boldsymbol{b}'}{\sin\theta}-\frac{\tau_0\delta\sin\theta}{s_0\widehat{\alpha}_0^{(1)}\sqrt{\det\boldsymbol{\sigma}_0}}\boldsymbol{S}\widehat{\boldsymbol{p}}_0^{(1)}.$$

Finally, we replace $\boldsymbol{\Lambda}_2$ on the left-hand side with $r_0'\boldsymbol{S}+\tau_0\boldsymbol{\Psi}(\boldsymbol{\Lambda})$. Combining the terms with $\boldsymbol{\Psi}(\boldsymbol{\Lambda})$ we obtain

$$\boldsymbol{p}_2=\boldsymbol{p}_0'+\mu_0\boldsymbol{\Psi}(\boldsymbol{\Lambda})(\boldsymbol{p}-\boldsymbol{p}_0)^{\perp},$$

where

$$\mu_0=\frac{\widehat{\alpha}_0^{(2)}\tau_0\delta\sin\theta}{\widehat{\alpha}_0^{(1)}\sqrt{s_0\det\boldsymbol{\sigma}_0}},\quad \boldsymbol{p}_0=r_0\boldsymbol{S}\widehat{\boldsymbol{p}}_0^{(1)}+\frac{\widehat{\alpha}_0^{(1)}\boldsymbol{\sigma}_0^{1/2}\boldsymbol{S}\boldsymbol{b}'}{\delta\sin\theta}+\frac{\widehat{\alpha}_0^{(1)}\sqrt{s_0\det\boldsymbol{\sigma}_0}}{\widehat{\alpha}_0^{(2)}\delta\sin\theta}\boldsymbol{S}\widehat{\boldsymbol{p}}_0^{(2)} \tag{D.12}$$

$$\boldsymbol{p}_0'=r_0'\boldsymbol{S}\widehat{\boldsymbol{p}}_0^{(2)}+\frac{\sqrt{s_0}\widehat{\alpha}_0^{(2)}}{\sin\theta}\boldsymbol{\sigma}_0^{1/2}\boldsymbol{R}\boldsymbol{S}\boldsymbol{b}'-\frac{\widehat{\alpha}_0^{(2)}\tau_0\delta\sin\theta}{\widehat{\alpha}_0^{(1)}\sqrt{s_0\det\boldsymbol{\sigma}_0}}\boldsymbol{S}\widehat{\boldsymbol{p}}_0^{(1)}, \tag{D.13}$$

The case $\theta=0$ or π needs to be considered separately. In this case

$$\overline{\boldsymbol{u}}_2=\overline{\boldsymbol{K}}_2\boldsymbol{b}+\delta\overline{\boldsymbol{u}},\qquad \overline{\boldsymbol{v}}_2=\overline{\boldsymbol{K}}_2^T\boldsymbol{a}+\tau\overline{\boldsymbol{v}}.$$

Applying (D.4) and (D.9) we obtain

$$\boldsymbol{p}_2=\boldsymbol{\Lambda}\boldsymbol{p}_0+\mu_0\boldsymbol{p}+\boldsymbol{p}_0',\qquad \boldsymbol{q}_2=\boldsymbol{\Lambda}^T\boldsymbol{q}_0+\nu_0\boldsymbol{q}+\boldsymbol{q}_0' \tag{D.14}$$

where

$$\mu_0=\frac{\widehat{\alpha}_0^{(2)}\delta\sqrt{s_0}}{\widehat{\alpha}_0^{(1)}},\quad \boldsymbol{p}_0=s_0\widehat{\boldsymbol{p}}_0^{(2)}-\mu_0\widehat{\boldsymbol{p}}_0^{(1)}+\sqrt{s_0}\widehat{\alpha}_0^{(2)}\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{b}, \tag{D.15}$$

$$\boldsymbol{p}_0'=(r_0^{(2)}-s_0r_0^{(1)})\boldsymbol{S}\widehat{\boldsymbol{p}}_0^{(2)}-\sqrt{s_0}\widehat{\alpha}_0^{(2)}\boldsymbol{\Lambda}_0^{(1)}\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{b}. \tag{D.16}$$

$$\nu_0=\sqrt{s_0}\tau,\quad \boldsymbol{q}_0=\sqrt{s_0}\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{a},\quad \boldsymbol{q}_0'=\boldsymbol{q}_0^{(2)}-\nu_0\boldsymbol{q}_0^{(1)}-\sqrt{s_0}(\boldsymbol{\Lambda}_0^{(1)})^T\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{a}. \tag{D.17}$$

D.3 Computation for the α-block

The relation (D.6) can be written as

$$\rho_2 = (\boldsymbol{u}_2, \boldsymbol{a}') + \tau(\boldsymbol{v}, \boldsymbol{b}) + \delta\tau\rho. \tag{D.18}$$

Lemma **D.1.**

$$\overline{\rho}_2 = (\overline{\boldsymbol{u}}_2, \boldsymbol{a}') + \tau\sin\theta(\boldsymbol{S}\overline{\boldsymbol{v}}, \overline{\boldsymbol{u}}_2) + \tau(\overline{\boldsymbol{v}}, \boldsymbol{b}) + \tau\delta\overline{\rho}. \tag{D.19}$$

Proof. We "solve" (D.4) to express $\boldsymbol{u}_2$, $\boldsymbol{v}$, ρ and $\rho_2$in terms of $\overline{\boldsymbol{u}}_2$, $\overline{\boldsymbol{v}}$, $\overline{\rho}$ and $\overline{\rho}_2$:

$$\boldsymbol{u}_2 = \overline{\boldsymbol{u}}_2 + (\overline{\boldsymbol{u}}_2, \boldsymbol{e})\boldsymbol{K}_2\boldsymbol{e}, \quad \boldsymbol{v} = \overline{\boldsymbol{v}} + (\overline{\boldsymbol{v}}, \boldsymbol{e})\boldsymbol{K}^T\boldsymbol{e}, \quad \rho = \overline{\rho} + (\overline{\boldsymbol{u}}, \boldsymbol{e})(\overline{\boldsymbol{v}}, \boldsymbol{e})(1 + (\boldsymbol{K}\boldsymbol{e}, \boldsymbol{e})). \tag{D.20}$$

Substituting these into (D.18) we obtain

$$\overline{\rho}_2 = (\overline{\boldsymbol{u}}_2, \boldsymbol{a}') + \tau(\overline{\boldsymbol{v}}, \boldsymbol{b}) + \tau\delta\overline{\rho} + \Xi_1 + \Xi_2,$$

where

$$\Xi_1 = \tau(\overline{\boldsymbol{v}}, \boldsymbol{e})(\boldsymbol{K}\boldsymbol{b}, \boldsymbol{e}) + \delta\tau(\overline{\boldsymbol{u}}, \boldsymbol{e})(\overline{\boldsymbol{v}}, \boldsymbol{e})(1 + (\boldsymbol{K}\boldsymbol{e}, \boldsymbol{e}))$$

and

$$\Xi_2 = (\overline{\boldsymbol{u}}_2, \boldsymbol{e})(\boldsymbol{K}_2^T\boldsymbol{a}', \boldsymbol{e}) - (\overline{\boldsymbol{u}}_2, \boldsymbol{e})(\overline{\boldsymbol{v}}_2, \boldsymbol{e})(1 + (\boldsymbol{K}_2\boldsymbol{e}, \boldsymbol{e})).$$

We have

$$\Xi_1 = \tau(\overline{\boldsymbol{v}}, \boldsymbol{e})(\boldsymbol{K}\boldsymbol{b} + \delta\boldsymbol{u}, \boldsymbol{e}), \quad \Xi_2 = (\overline{\boldsymbol{u}}_2, \boldsymbol{e})(\boldsymbol{K}_2^T\boldsymbol{a}' - \boldsymbol{v}_2, \boldsymbol{e}).$$

Recalling (D.5), we have

$$\Xi_1 = \tau(\overline{\boldsymbol{v}}, \boldsymbol{e})(\boldsymbol{u}_2, \boldsymbol{R}\boldsymbol{e}), \qquad \Xi_2 = -\tau(\overline{\boldsymbol{u}}_2, \boldsymbol{e})(\boldsymbol{v}, \boldsymbol{R}\boldsymbol{e}).$$

Using (D.20) again together with $\boldsymbol{K}_2 = \boldsymbol{R}\boldsymbol{K}\boldsymbol{R}$, we obtain We also have

$$\Xi_1 + \Xi_2 = \tau(\overline{\boldsymbol{v}}, \boldsymbol{e})(\overline{\boldsymbol{u}}_2, \boldsymbol{R}\boldsymbol{e}) - \tau(\overline{\boldsymbol{v}}, \boldsymbol{R}\boldsymbol{e})(\overline{\boldsymbol{u}}_2, \boldsymbol{e}) = \tau\sin\theta(\boldsymbol{S}\overline{\boldsymbol{v}}, \overline{\boldsymbol{u}}_2).$$

The lemma is proved. □

Now we rewrite the relation (D.19) using formulas (D.4)

$$\alpha_2 = \widehat{\alpha}_0^{(2)}(1-\tau\delta) + \frac{\widehat{\alpha}_0^{(2)}\tau\delta}{\widehat{\alpha}_0^{(1)}}\alpha + (\boldsymbol{q}_2, \widehat{\boldsymbol{p}}_0^{(2)}) + \left(\frac{\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{a}'}{\sqrt{s_0}}, \boldsymbol{p}_2 - \boldsymbol{\Lambda}_2\widehat{\boldsymbol{p}}_0^{(2)}\right) - \frac{\widehat{\alpha}_0^{(2)}\tau\delta}{\widehat{\alpha}_0^{(1)}}(\boldsymbol{q}, \boldsymbol{p}_0^{(1)}) + \frac{\tau\sin\theta}{\sqrt{s_0\det\boldsymbol{\sigma}_0}}(\boldsymbol{S}(\boldsymbol{q}_0^{(1)} - \boldsymbol{q}), \boldsymbol{p}_2 - \boldsymbol{\Lambda}_2\widehat{\boldsymbol{p}}_0^{(2)}) + \widehat{\alpha}_0^{(2)}\tau(\boldsymbol{q} - \boldsymbol{q}_0^{(1)}, \boldsymbol{\sigma}_0^{-1/2}\boldsymbol{b}).$$

Now, we replace $\boldsymbol{q}_2$, $\boldsymbol{p}_2$ and $\boldsymbol{\Lambda}_2$ on the right-hand side with their expressions given by (2.2), while observing that

$$\frac{\widehat{\alpha}_0^{(2)}\tau\delta}{\widehat{\alpha}_0^{(1)}} = -\frac{\mu_0\nu_0}{\tau_0}, \qquad \frac{\tau\sin\theta}{\sqrt{s_0\det\boldsymbol{\sigma}_0}} = -\frac{\nu_0}{\tau_0}.$$

We obtain

$$\alpha_2 = \alpha_0 - \frac{\mu_0\nu_0}{\tau_0}\alpha + \nu_0(\boldsymbol{\Psi}(\boldsymbol{\Lambda})^T(\boldsymbol{q}-\boldsymbol{q}_0)^\perp, \widehat{\boldsymbol{p}}_0^{(2)}) + \frac{\mu_0}{\sqrt{s_0}}(\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{a}', \boldsymbol{\Psi}(\boldsymbol{\Lambda})(\boldsymbol{p}-\boldsymbol{p}_0)^\perp) - \frac{\tau_0}{\sqrt{s_0}}(\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{a}', \boldsymbol{\Psi}(\boldsymbol{\Lambda})\widehat{\boldsymbol{p}}_0^{(2)}) + \widehat{\alpha}_0^{(2)}\tau\left(\boldsymbol{q}, \boldsymbol{\sigma}_0^{-1/2}\boldsymbol{b} - \frac{\delta\boldsymbol{p}_0^{(1)}}{\widehat{\alpha}_0^{(1)}}\right) - \frac{\nu_0}{\tau_0}(\boldsymbol{q}, \boldsymbol{S}\boldsymbol{p}_0' + r_0'\widehat{\boldsymbol{p}}_0^{(2)}) + \frac{\nu_0}{\tau_0}((\boldsymbol{q}-\boldsymbol{q}_0^{(1)})^\perp, \mu_0\boldsymbol{\Psi}(\boldsymbol{\Lambda})(\boldsymbol{p}-\boldsymbol{p}_0)^\perp - \tau_0\boldsymbol{\Psi}(\boldsymbol{\Lambda})\widehat{\boldsymbol{p}}_0^{(2)}).$$

where

$$\alpha_0 = \widehat{\alpha}_0^{(2)}(1-\tau\delta) + (\boldsymbol{q}_0', \widehat{\boldsymbol{p}}_0^{(2)}) + \left(\frac{\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{a}'}{\sqrt{s_0}}, \boldsymbol{p}_0' - r_0'\boldsymbol{S}\widehat{\boldsymbol{p}}_0^{(2)}\right) - \widehat{\alpha}_0^{(2)}\tau(\boldsymbol{q}_0^{(1)}, \boldsymbol{\sigma}_0^{-1/2}\boldsymbol{b}) + \frac{\nu_0}{\tau_0}(\boldsymbol{q}_0^{(1)}, \boldsymbol{S}\boldsymbol{p}_0' + r_0'\widehat{\boldsymbol{p}}_0^{(2)}).$$

Hence,

$$\alpha_2 = \alpha_0 + \frac{\mu_0\nu_0}{\tau_0}((\boldsymbol{\Psi}(\boldsymbol{\Lambda})(\boldsymbol{p}-\boldsymbol{p}_0)^\perp, (\boldsymbol{q}-\boldsymbol{q}_0)^\perp) - \alpha) + (\boldsymbol{q}, \widehat{\boldsymbol{q}}_0) + (\boldsymbol{\Psi}(\boldsymbol{\Lambda})(\boldsymbol{p}-\boldsymbol{p}_0)^\perp, \boldsymbol{A}_0) - \frac{\tau_0}{\mu_0}(\boldsymbol{A}_0, \boldsymbol{\Psi}(\boldsymbol{\Lambda})\widehat{\boldsymbol{p}}_0^{(2)}),$$

where

$$\widehat{\boldsymbol{q}}_0 = \widehat{\alpha}_0^{(2)}\tau\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{b} - \frac{\widehat{\alpha}_0^{(2)}\tau\delta}{\widehat{\alpha}_0^{(1)}}\boldsymbol{p}_0^{(1)} - \frac{\nu_0}{\tau_0}(\boldsymbol{S}\boldsymbol{p}_0' + r_0'\widehat{\boldsymbol{p}}_0^{(2)}), \quad \boldsymbol{A}_0 = \frac{\mu_0}{\sqrt{s_0}}\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{a}' + \frac{\mu_0\nu_0}{\tau_0}(\boldsymbol{q}_0 - \boldsymbol{q}_0^{(1)})^\perp.$$

Substituting the expressions for $\boldsymbol{p}_0'$ from (D.13) and for $\boldsymbol{q}_0$ from (D.11) into the formulas for $\widehat{\boldsymbol{q}}_0$ and $\boldsymbol{A}_0$, respectively, and using the identity

$$\frac{\boldsymbol{S}^T\boldsymbol{\sigma}_0^{1/2}\boldsymbol{S}}{\sqrt{\det\boldsymbol{\sigma}_0}} = \boldsymbol{\sigma}_0^{-1/2},$$

we easily verify that $\widehat{\boldsymbol{q}}_0 = \boldsymbol{A}_0 = \boldsymbol{0}$. Hence,

$$\alpha_2 = \alpha_0 + \frac{\mu_0\nu_0}{\tau_0}((\boldsymbol{\Psi}(\boldsymbol{\Lambda})(\boldsymbol{p}-\boldsymbol{p}_0)^\perp, (\boldsymbol{q}-\boldsymbol{q}_0)^\perp) - \alpha).$$

Once again, the case $\sin\theta = 0$ need to be considered separately. In this case we have

$$\overline{\rho}_2 = (\overline{\boldsymbol{u}}_2, \boldsymbol{a}) + \tau(\overline{\boldsymbol{v}}, \boldsymbol{b}) + \tau\delta\overline{\rho}.$$

Applying formulas (D.4), we obtain

$$\alpha_2 = \widehat{\alpha}_0^{(2)}(1-\tau\delta) + (\boldsymbol{p}_2 - \boldsymbol{\Lambda}_2\widehat{\boldsymbol{p}}_0^{(2)}, (s_0\boldsymbol{\sigma}_0)^{-1/2}\boldsymbol{a}) + \widehat{\alpha}_0^{(2)}\tau(\boldsymbol{q}-\boldsymbol{q}_0^{(1)}, \boldsymbol{\sigma}_0^{-1/2}\boldsymbol{b}) + \frac{\widehat{\alpha}_0^{(2)}\tau\delta}{\widehat{\alpha}_0^{(1)}}(\alpha - (\boldsymbol{q}, \widehat{\boldsymbol{p}}_0^{(1)})) + (\boldsymbol{q}_2, \widehat{\boldsymbol{p}}_0^{(2)})$$

Using formulas (D.14) we obtain

$$\alpha_2 = \alpha_0 + \frac{\widehat{\alpha}_0^{(2)}\tau\delta}{\widehat{\alpha}_0^{(1)}}\alpha + (\boldsymbol{\Lambda p}_0, (s_0\boldsymbol{\sigma}_0)^{-1/2}\boldsymbol{a}) + (\boldsymbol{\Lambda}\widehat{\boldsymbol{p}}_0^{(2)}, \boldsymbol{q}_0 - \sqrt{s_0}\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{a}) +$$
$$\mu_0(\boldsymbol{p}, (s_0\boldsymbol{\sigma}_0)^{-1/2}\boldsymbol{a}) + (\boldsymbol{q}, \widehat{\alpha}_0^{(2)}\tau\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{b} - \frac{\widehat{\alpha}_0^{(2)}\tau\delta}{\widehat{\alpha}_0^{(1)}}\widehat{\boldsymbol{p}}_0^{(1)} + \nu_0\widehat{\boldsymbol{p}}_0^{(2)})$$

where

$$\alpha_0 = \widehat{\alpha}_0^{(2)}(1-\tau\delta) - \widehat{\alpha}_0^{(2)}(\boldsymbol{b}, \boldsymbol{a}) - \frac{\widehat{\alpha}_0^{(2)} r_0^{(1)}}{\sqrt{\det\boldsymbol{\sigma}_0}}(\boldsymbol{b}^{\perp}, \boldsymbol{a}) - \widehat{\alpha}_0^{(2)}\tau(\boldsymbol{q}_0^{(1)}, \boldsymbol{\sigma}_0^{-1/2}\boldsymbol{b}) + (\boldsymbol{q}_0', \widehat{\boldsymbol{p}}_0^{(2)}).$$

Now, using the formulas (D.15)-(D.17) we obtain

$$\alpha_2 = \alpha_0 + \frac{\mu_0\nu_0}{s_0}\alpha + \frac{1}{s_0}(\boldsymbol{\Lambda p}_0, \boldsymbol{q}_0) + \frac{\mu_0}{s_0}(\boldsymbol{p}, \boldsymbol{q}_0) + \frac{\nu_0}{s_0}(\boldsymbol{q}, \boldsymbol{p}_0). \tag{D.21}$$

E Redundancy of the link corresponding to item 9 at the end of Section 10

Now consider the link corresponding to item 9. In order to compute the link corresponding to item 9 we have to use $\widehat{\boldsymbol{M}} = [\boldsymbol{e}\otimes\boldsymbol{e}, \boldsymbol{e}\otimes\boldsymbol{e}]$ and formulas (6.10). For ease of notation it is convenient to set

$$\boldsymbol{\Lambda}' = \boldsymbol{\sigma}_0^{-1/2}\boldsymbol{\Lambda}\boldsymbol{\sigma}_0^{-1/2}, \qquad \boldsymbol{p}' = -\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{p}, \qquad \boldsymbol{q}' = -\boldsymbol{\sigma}_0^{-1/2}\boldsymbol{q}, \qquad \alpha' = 1-\alpha. \tag{E.1}$$

Then the formulas (6.10) can be written as

$$\boldsymbol{\Lambda}' = \boldsymbol{I} - [\boldsymbol{I} + \boldsymbol{K}\boldsymbol{e}\otimes\boldsymbol{e}]^{-1}\boldsymbol{K}, \quad \boldsymbol{p}' = [\boldsymbol{I} + \boldsymbol{K}\boldsymbol{e}\otimes\boldsymbol{e}]^{-1}\boldsymbol{u}, \quad \boldsymbol{q}' = [\boldsymbol{I} + \boldsymbol{K}\boldsymbol{e}\otimes\boldsymbol{e}]^{-1}\boldsymbol{v}, \quad \alpha' = \rho.$$

We have $\boldsymbol{u}_2 = \phi(m)\boldsymbol{u}_1$, $|m| = 1$ and $\boldsymbol{v}_2 = \boldsymbol{v}_1$. Then we have

$$\boldsymbol{u}_1 = (\boldsymbol{I} + \boldsymbol{K}\boldsymbol{e}\otimes\boldsymbol{e})\boldsymbol{p}_1', \qquad \boldsymbol{p}_2' = [\boldsymbol{I} + \boldsymbol{K}\boldsymbol{e}\otimes\boldsymbol{e}]^{-1}\phi(m)\boldsymbol{u}_1.$$

Then

$$\boldsymbol{p}_2' = [\boldsymbol{I} + \boldsymbol{K}\boldsymbol{e}\otimes\boldsymbol{e}]^{-1}(\phi(m) + \phi(m)\boldsymbol{K}\boldsymbol{e}\otimes\boldsymbol{e})\boldsymbol{p}_1'.$$

Now let us use the fact that $\boldsymbol{K}\in\mathcal{A}_{2d}$. In that case $\phi(m)\boldsymbol{K}\phi(m) = \boldsymbol{K}$. Thus, we get

$$\boldsymbol{p}_2' = [\boldsymbol{I} + \boldsymbol{K}\boldsymbol{e}\otimes\boldsymbol{e}]^{-1}(\boldsymbol{I} + \boldsymbol{K}\phi(\overline{m})\boldsymbol{e}\otimes\phi(m)\boldsymbol{e})\phi(m)\boldsymbol{p}_1'. \tag{E.2}$$

Let us now use the identity

$$\boldsymbol{e}\otimes\boldsymbol{e} - \phi(\overline{m})\boldsymbol{e}\otimes\phi(m)\boldsymbol{e} = \phi\left(\frac{1-\overline{m}^2}{2}\right)$$

to eliminate $\phi(\overline{m})\boldsymbol{e} \otimes \phi(m)\boldsymbol{e}$ from (E.2). We have

$$\boldsymbol{p}_2' = \left(\boldsymbol{I} - [\boldsymbol{I} + \boldsymbol{K}\boldsymbol{e} \otimes \boldsymbol{e}]^{-1}\boldsymbol{K}\phi\left(\frac{1-\overline{m}^2}{2}\right)\right)\phi(m)\boldsymbol{p}_1'.$$

Hence,

$$\boldsymbol{p}_2' = \left(\phi\left(\frac{m+\overline{m}}{2}\right) + \boldsymbol{\Lambda}'\phi\left(\frac{m-\overline{m}}{2}\right)\right)\boldsymbol{p}_1'.$$

Thus, $\boldsymbol{p}_2' = \mu_1\boldsymbol{p}_1' + \mu_2\boldsymbol{\Lambda}_1'(\boldsymbol{p}_1')^\perp$, where μ_1 and μ_2 are the real and imaginary parts of m, respectively. Therefore, we have the following link

$$\boldsymbol{\Lambda}_2 = \boldsymbol{\Lambda}_1, \qquad \boldsymbol{p}_2 = \mu_1\boldsymbol{p}_1 + \frac{\mu_2}{\sqrt{\det \boldsymbol{\sigma}_0}}\boldsymbol{\Lambda}_1\boldsymbol{p}_1^\perp, \qquad \boldsymbol{q}_2 = \boldsymbol{q}_1. \tag{E.3}$$

We first observe that $\boldsymbol{K} \in \mathcal{A}_{2d}$ means that $\boldsymbol{\Lambda}$ is symmetric and $\det \boldsymbol{\Lambda} = d_0$, constant. In that case it is not difficult to use (2.2) to show that

$$\boldsymbol{\Lambda}_2 = \boldsymbol{\Lambda}_1 = \boldsymbol{\Lambda}, \qquad \boldsymbol{p}_2 = \mu_1\boldsymbol{p}_1 + \mu_2\boldsymbol{\Lambda}\boldsymbol{p}_1^\perp, \qquad \boldsymbol{q}_2 = \nu_1\boldsymbol{q}_1 + \frac{\nu_1\mu_2}{\mu_1}\boldsymbol{\Lambda}\boldsymbol{q}_1^\perp.$$

In that case $\boldsymbol{p}_2^* = \mu_1\boldsymbol{p}_1^* + \mu_2\boldsymbol{\Lambda}^*(\boldsymbol{p}_1^*)^\perp$. However, we know that $\boldsymbol{p}_2^*$ depends only on $\boldsymbol{\Lambda}$ and $\boldsymbol{p}_2$, while $\boldsymbol{q}_2^*$ depends only on $\boldsymbol{\Lambda}$ and $\boldsymbol{q}_2$. Thus, the link (E.3) is a consequence of (2.2) and (2.5).

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