Algebra, geometry and computations of exact relations for effective moduli of composites.

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Yury Grabovsky
Department of Mathematics
Temple University
Philadelphia, PA 19122-6094

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Abstract

In this paper we will review and extend the results of [21], which covered the case of 3D thermo-piezoelectric polycrystals. In that context the settings of conductivity, elasticity, pyroelectricity, piezoelectricity, thermo-electricity and thermo-elasticity can be viewed as particular cases. We will consider a class of composites more general than polycrystals, where the set of allowable materials is not constrained in any way. In addition, the tensors of material properties are not assumed to be symmetric—an assumption we made in [21]. For example, the Hall effect for conduction in a weak magnetic field is described by a non-symmetric conductivity tensor. We explain the step-by-step process of finding all exact relations for the simple example of 2D Hall effect. The paper concludes with a discussion of new algebraic and geometric questions posed by the theory of exact relations.

1 Introduction

Composite materials play an increasingly important role in our everyday life and technology. They are used everywhere from skis and golf clubs to sensors and actuators in high tech components. By "composite" we mean a perfectly bonded mixture of two or more materials, where the mixing occurs on length scales much smaller than the human size but much larger than the inter-atomic distances. The physical properties of a composite (thermal, electric, elastic, etc.) are described by a tensor—the *effective tensor* of a composite. In order to create a composite with desired properties two basic problems become important: prediction of the effective properties of composite materials and determination of the effective tensor of a given composite by as few measurements as possible.

The problem of analytical prediction of effective properties of composite materials is important for both practical and theoretical points of view. Understanding the effective behavior mathematically may help save many costly and time consuming measurements, or may prevent spending time and money on many trial-and-error approaches to material design. Recent years have seen a lot of success in that direction. I would like to describe in this chapter a theory of exact relations for effective moduli of composites. The theory combines algebra, geometry, analysis and mechanics in a beautiful symbiosis. It also achieves a high level of generality encompassing conductivity, elasticity, piezo-electricity and many other coupled problem contexts. The theory and many of the physical results of this paper are also discussed in the new book by Milton [39].

The principal difficulty in the prediction of effective properties is the universally recognized fact that the effective tensors of composites depend on the microstructure (spatial arrangement of component materials), in general. Therefore, the object of importance is the set of all possible effective properties of a composite made with given materials taken in prescribed volume fractions (the so called G-closure set). Unfortunately, aside from a few particular cases the G-closures are extremely difficult to compute analytically.

Exact relations are dependencies between various material properties that "survive" homogenization. For example, if we mix two isotropic elastic materials the resulting composite does not have to be isotropic. So, the relations defining isotropy are not preserved under homogenization. Yet isotropy survives if both isotropic materials have the same shear modulus. In fact the composite will also have that same shear modulus, *no matter how the materials are mixed*. Statements about effective tensors that do not depend on the geometric arrangement of the constituent materials are called exact relations. The example described above is the exact relation due to Hill [22, 23] in 1963.

Exact relations and, more generally, G-closures provide the information about all composites regardless of their origin, details and complexity of microstructure. If we want to move beyond the G-closure and obtain a more detailed information about the effective tensors of composites we need to distinguish composites according to the type of microstructure they have.

One class of composites is where the (possibly infinitely many) scales of inhomogeneity are well separated. For example, imagine a material that looks homogeneous to the naked eye. Yet, when we examine this material under a microscope we may observe that it is in fact composed of several seemingly homogeneous materials, each of which, upon examination in a more powerful microscope, reveals to be composed of other seemingly homogeneous materials, and so on for several, or possible infinite number of steps. We also assume that on each step of our microscopic examination of an apparently homogeneous component, we find a rather simple geometric arrangement of phases, each occupying regions of approximately the same size. This broad class of composites is adequately described by the homogenization theory, (see e.g. [5, 24]) which we briefly review in this paper.

On the other end of the spectrum are disordered media where the microstructure at one point is almost uncorrelated to the microstructure at a somewhat distant point. In this situation it is usually assumed that the composite is random, and the stochastic approach [18, 30, 42] works well here.

The third and a very important type of composite has the power law distribution of sizes and properties of the constituents. The power law is the tell-tail sign of the self-organized criticality—a theory proposed by Bak, Tang and Weisenfeld [3, 4]. According to this theory many open systems with constant influx of energy self-organize into a poised state on the border between order and chaos. Examples include earthquakes and sandpiles, extinction of species and traffic jams [2]. I can speculate that the structure of real geological media is self-organized critical. The upper crust of our home planet can be considered an open system with constant energy flow from inner layers of Earth in the forms of seismic and volcanic activity. As a result the Earth's crust is a complicated multi-scale structure with inhomogeneities on a vast range of scales. Under the microscope we can see tiny particles making up clay, with a naked eye we can see small individual grains of sand, we can also see boulders—the heterogeneities on the scale of meters. The scale staircase goes on and on. The Earth's crust is a heterogeneous mixture of materials on a continuous spectrum of length scales from microscopic to global (continental plates, oceans). I would like to call such composites *critical*. The appropriate mathematical tools for a rigorous discussion of effective properties of such composites are only beginning to emerge. On the one hand there is a non-rigorous renormalization group approach [27, 44], on the other there is a more rigorous micro-mechanics based approach leading to non-local constitutive laws [10, 11, 41]. Yet, hydrologists, for example, have to deal with the effective hydrolic permittivity of such media as a rule [8].

As we mentioned earlier, we will focus on the microstructure-independent aspect of the theory of composite materials, which applies to all composites equally well. The discussion of the different types of composites was needed in order to place the subject of this paper in a broader context.

2 G-closures

We start this discussion with the example of conductivity before we go on to the abstract framework encompassing a variety of physical contexts.

The conduction of the electric current is described by the two fields, the electric field $\boldsymbol{e}$ and the current field $\boldsymbol{j}$. The two fields satisfy differential constraints: the electric field is curl-free, the current field is divergence-free. The two fields are related via a tensorial Ohm's law. Thus, we have the following equations:

$$\nabla \times \boldsymbol{e} = 0, \quad \nabla \cdot \boldsymbol{j} = 0, \quad \boldsymbol{j} = \boldsymbol{\sigma e}, \tag{2.1}$$

where the 3×3 matrix $\boldsymbol{\sigma}$ is symmetric and positive definite.

An n phase composite made with n materials $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n$ is a Borel measurable matrix field $\boldsymbol{\sigma}(\boldsymbol{x})$, such that $\boldsymbol{\sigma}(\boldsymbol{x}) \in D = \{\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n\}$. An n phase polycrystal is a Borel measurable matrix field $\boldsymbol{\sigma}(\boldsymbol{x})$, such that $\boldsymbol{\sigma}(\boldsymbol{x}) \in D = \bigcup SO(3) \cdot \boldsymbol{\sigma}_j$. In general, we fix a set D (finite, or infinite) of materials and consider the the set

$$\mathcal{D} = \{\boldsymbol{\sigma}(\boldsymbol{x}) \in L^\infty(\Omega) \mid \boldsymbol{\sigma}(\boldsymbol{x}) \in D \ a.e. \ \boldsymbol{x} \in \Omega\} \tag{2.2}$$

From the applied point of view one needs to solve the elliptic boundary value problem in a domain Ω occupied by the composite.

$$\begin{cases} \nabla \cdot (\boldsymbol{\sigma}(\boldsymbol{x})\nabla\phi) = f, & \boldsymbol{x} \in \Omega \\ \phi = 0, & \boldsymbol{x} \in \partial\Omega. \end{cases} \tag{2.3}$$

However, if $\boldsymbol{\sigma}(\boldsymbol{x})$ has a very complicated geometry, the numerical solution of (2.3) is not feasible. The fruitful idea is to represent the local conductivity tensor $\boldsymbol{\sigma}(\boldsymbol{x})$ as a member in a sequence $\boldsymbol{\sigma}^\epsilon(\boldsymbol{x})$ for small ϵ. As $\epsilon \to 0$ the length scales in the microstructure become more and more separated. In the limit, as $\epsilon \to 0$ the material properties are described by a *homogenized* tensor $\boldsymbol{\sigma}^*(\boldsymbol{x})$, which may be much simpler than the local conductivity tensor. This is especially true in the case of composites with inhomogeneities on the well-separated length scales. If $\boldsymbol{\sigma}^*(\boldsymbol{x})$ is particularly simple then (2.3) may become solvable numerically.

Putting the practical questions aside and turning to mathematical justification of the homogenization procedure, we need to answer the following question. Given a sequence of bounded measurable functions $\boldsymbol{\sigma}^\epsilon(\boldsymbol{x})$ is there a sense in which we can say that $\boldsymbol{\sigma}^\epsilon \to \boldsymbol{\sigma}^*$, so that the solutions ϕ^ϵ of (2.3) with $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\epsilon$ converges to the solution ϕ^* of the homogenized equation? Such a notion was proposed by Spagnolo [45, 46] and further developed by De Giorgi and Spagnolo [7]. Murat and Tartar [40](English translation of the French original) extended G-convergence to the case of non-symmetric tensors $\boldsymbol{\sigma}$ and proved that G-convergence of symmetric elliptic operators in (2.3) implies the convergence of fluxes

$$\boldsymbol{\sigma}^\epsilon\nabla\phi^\epsilon \rightharpoonup \boldsymbol{\sigma}^*\nabla\phi^* \tag{2.4}$$

weakly in $L^2(\Omega)$.

The non-symmetric tensors of material properties may arise in a variety of contexts. For example the Hall effect in the electric current conduction in a very weak magnetic field is governed by the same basic equations of conductivity (2.1), except the tensor $\boldsymbol{\sigma}$ is no longer symmetric

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_s + \pi(\boldsymbol{r}),$$

where $\boldsymbol{\sigma}_s$ is the symmetric and positive definite conductivity tensor, $\boldsymbol{r}$ is the Hall vector and π is the "cross-product" mapping between vectors in $\mathbb{R}^3$ and 3×3 skew symmetric matrices such that for any $\{\boldsymbol{u}, \boldsymbol{x}\} \subset \mathbb{R}^3$ we have $\pi(\boldsymbol{u})\boldsymbol{x} = \boldsymbol{u} \times \boldsymbol{x}$.

Following Murat and Tartar, the name H-convergence is attached to the kind of G-convergence, which is appropriate for non-symmetric material tensors.

Definition 1 *A sequence of not necessarily symmetric tensors $\boldsymbol{\sigma}^\epsilon$ H-converges to $\boldsymbol{\sigma}^*$ if for any f in (2.3)*
(i) $\phi^\epsilon \rightharpoonup \phi^*$ weakly in $H_0^1(\Omega)$.
(ii) $\boldsymbol{\sigma}^\epsilon\nabla\phi^\epsilon \rightharpoonup \boldsymbol{\sigma}^*\nabla\phi^*$ weakly in $L^2(\Omega)$.

Murat and Tartar [40] noticed that for symmetric matrices $\boldsymbol{\sigma}$ the second condition is redundant. It follows from the first.

The key result here is the Compactness theorem proved by De Giorgi and Spagnolo for the case of symmetric $\boldsymbol{\sigma}$ and by Murat and Tartar in general. The theorem states that

any sequence of uniformly positive definite and bounded matrix fields $\boldsymbol{\sigma}^\epsilon(\boldsymbol{x})$ contains an H-convergent subsequence. In this connection the problem of the closure of the set $\mathcal{D}$, defined in (2.2) with respect to the H-convergence becomes important. In order to formulate a fundamental result, we must first give the basic non-trivial example of an H-converging sequence $\boldsymbol{\sigma}^\epsilon$. Suppose that the matrix valued function $\boldsymbol{\sigma}(\boldsymbol{y})$ is periodic with a parallelepiped of periods $Q = [0,1]^3$. We further assume that $\boldsymbol{\sigma}(\boldsymbol{y})$ is uniformly positive definite and uniformly bounded. Then $\boldsymbol{\sigma}^\epsilon(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x}/\epsilon)$ H-converges to a constant positive definite matrix $\boldsymbol{\sigma}^*$, defined via a solution of the so called periodic cell problem. We will write the cell problem in the form that will be useful later.

$$\nabla \cdot \boldsymbol{j} = 0, \quad \nabla \times \boldsymbol{e} = 0, \quad \boldsymbol{j}(\boldsymbol{y}) = \boldsymbol{\sigma}(\boldsymbol{y})\boldsymbol{e}(\boldsymbol{y}), \tag{2.5}$$

where all functions and differential operators are Q-periodic. The equations (2.5) have a unique solution, if we fix the mean value of the electric field $\boldsymbol{e}$ over the period cell. Suppose, $\langle \boldsymbol{e} \rangle = \boldsymbol{e}^*$, where $\langle \cdot \rangle$ denotes the average over the period cell Q. Then the unique field $\boldsymbol{j}$ satisfying (2.5) depends linearly on $\boldsymbol{e}^*$, and therefore, there exists a 3×3 matrix $\boldsymbol{\sigma}^*$ such that

$$\boldsymbol{\sigma}^* \boldsymbol{e}^* = \langle \boldsymbol{j} \rangle. \tag{2.6}$$

Definition 2 *Let D be the set of matrices representing conductivities of the materials constituting the composite. Then the* ***G-closure*** *[31, 49] $G(D)$ of the set of materials D is the set of all effective tensors $\boldsymbol{\sigma}^*$ of* periodic *composites made with materials from the set D.*

Kohn and Dal Maso realized that the set $G(D)$ is sufficient to describe the H-closure (closure with respect to the H-convergence topology) of the set $\mathcal{D}$ defined in (2.2). The result has been recently rigorously proved in a very general context by Raitums [43]. The theorem states that the matrix field $\boldsymbol{\sigma}^*(\boldsymbol{x})$ is in the H-closure of $\mathcal{D}$ if and only if $\boldsymbol{\sigma}^*(\boldsymbol{x}) \in G(D)$ for a.e. $\boldsymbol{x} \in \Omega$.

Our primary interest is the G-closure sets of materials in a variety of physical contexts. We must mention that the explicit form of the G-closure is known only in a very few cases [15, 19, 20, 29, 32, 33, 34, 35, 37, 49]. Only the paper [15] characterizes G-closure sets for *arbitrary* sets of materials D for 2D conducting polycrystals.

At this point we would like to make several remarks. Observe that the G-closure problem is the problem of computing a subset in a *finite dimensional* space, given another subset of the same space. Yet, it involves solving a periodic PDE problem (2.5). It would be nice, if we can have a completely geometric description of how to construct $G(D)$ knowing D, as was done in [15] for 2D conducting polycrystals. The first step towards such a geometric description in the general case was made in [16] for sets closed under lamination (it will be the cornerstone of this paper), but a major new geometric breakthrough is still needed for the method to be truly useful.

Another remark, is that G-closure lives in a rather high dimensional space (6D for conductivity, 9D for Hall effect). Therefore, even if we have exact formulas for the boundary of $G(D)$, they are not immediately useful in applications without further (often very complicated) algebra. Instead, a more useful goal would be an efficient numerical algorithm for

computing quantities like

$$f_D = \min_{\boldsymbol{\sigma} \in G(D)} f(\boldsymbol{\sigma})$$

for a class of functions f. For each f the answer is a *number* f_D—something an engineer can relate to.

Getting closer to the subject of the paper, there is a fundamental dichotomy in the shapes of G-closure sets. Most G-closure sets (aside from single points) have a non-empty interior. In exceptional cases the G-closure sets lie on surfaces (of various codimensions). It is these exceptional cases that we call *exact relations* and it is these cases that we are after. The trick here is that we are no longer in the loosing game of computing the boundaries of the G-closure sets. Instead, we just want to identify the surfaces that contain G-closure sets. To give a trivial example, consider the Hall effect and assume that we are mixing materials whose Hall tensors are all zero. From the physical point of view, it is obvious that Hall effect will not arise in a composite if every component does not have it. Geometrically, a six dimensional surface (all 3×3 symmetric matrices) in a nine dimensional space (all 3×3 matrices) is an exact relation. In identifying this exact relation we do not need to compute any G-closure sets, yet the result is not entirely devoid of useful information.

3 Hilbert Space formalism

Milton [38] has observed that the periodic cell problems in various physical contexts follow the same abstract pattern, which is best described using the language of Hilbert spaces (see also [9, 18, 25, 28, 42, 50] for similar formal approaches). Milton observed that regardless of the particular physics of the problem there are two fields $\boldsymbol{E}$—the intensity field and $\boldsymbol{J}$—the flux field. They take their values in a certain finite dimensional tensor space $\mathcal{T}$. For the Hall effect example, the electric field $\boldsymbol{e}$ is the intensity field and the current field $\boldsymbol{j}$ is the flux field (in fact the names are purely formal, since the theory is symmetric with respect to the swapping of the fields). The fields are vectors, so $\mathcal{T} = \mathbb{R}^3$ for the Hall effect. All intensity fields and fluxes will be assumed to belong to the ambient Hilbert space $\mathcal{H} = L^2(Q) \otimes \mathcal{T}$. The most important observation of Milton concerns the general structure of the differential constraints satisfied by the fields $\boldsymbol{E}$ and $\boldsymbol{J}$. If we denote $\mathcal{U} = \mathbb{R} \otimes \mathcal{T}$ the subspace of constant fields in $\mathcal{H}$ then the Hilbert space $\mathcal{H}$ is split into the orthogonal sum $\mathcal{H} = \mathcal{E} \oplus \mathcal{J} \oplus \mathcal{U}$, where $\mathcal{E}$ and $\mathcal{J}$ are the subspaces of mean zero intensity fields and fluxes respectively. The subspaces $\mathcal{E}$ and $\mathcal{J}$ correspond to the differential constraints on $\boldsymbol{E}$ and $\boldsymbol{J}$ respectively. For that reason Milton required that the orthogonal projection Γ onto $\mathcal{E}$ be local in Fourier space. In other words there is a degree zero function $\Gamma(\boldsymbol{k})$, such that for each $\boldsymbol{k} \in \mathbb{S}^2$ the matrix $\Gamma(\boldsymbol{k})$ is an orthogonal projection onto a subspace $\mathcal{E}_{\boldsymbol{k}}$ of $\mathcal{T}$ and such that

$$\widehat{\Gamma \boldsymbol{f}}(\boldsymbol{k}) = \begin{cases} \Gamma(\boldsymbol{k}) \widehat{\boldsymbol{f}}(\boldsymbol{k}), & \text{if } \boldsymbol{k} \neq \boldsymbol{0}, \\ \boldsymbol{0}, & \text{if } \boldsymbol{k} = \boldsymbol{0}. \end{cases}$$

The subspaces $\mathcal{E}_{\boldsymbol{k}}$ cannot be arbitrary. For example, they have to be of the same dimension. The basic $O(3)$ symmetry of our space implies that the subspaces $\mathcal{E}_{\boldsymbol{k}}$ are permuted by

rotations. In other words, the vector space $\mathcal{T}$ is not just a linear space, it is a representation space for the group $O(3)$. For $\boldsymbol{\tau} \in \mathcal{T}$ we denote the action of $\boldsymbol{R} \in O(3)$ by $\boldsymbol{R} \cdot \boldsymbol{\tau}$. With this notation we can write the basic property of the subspaces $\mathcal{E}_{\boldsymbol{k}}$: $\boldsymbol{R} \cdot \mathcal{E}_{\boldsymbol{k}} = \mathcal{E}_{\boldsymbol{Rk}}$. We use the same notation for the action of rotations on $\mathrm{End}(\mathcal{T})$, the space of linear operators on $\mathcal{T}$. For any $\boldsymbol{A} \in \mathrm{End}(\mathcal{T})$ and any $\boldsymbol{\tau} \in \mathcal{T}$ we define $\boldsymbol{R} \cdot \boldsymbol{A}$ by $(\boldsymbol{R} \cdot \boldsymbol{A})\boldsymbol{\tau} = \boldsymbol{R} \cdot (\boldsymbol{A}(\boldsymbol{R}^{-1} \cdot \boldsymbol{\tau}))$. With this notation the basic property of the subspaces $\mathcal{E}_{\boldsymbol{k}}$ can also be written as $\boldsymbol{R} \cdot \Gamma(\boldsymbol{k}) = \Gamma(\boldsymbol{Rk})$. This last formula tells us that the function $\Gamma(\boldsymbol{k})$ is uniquely determined by a single matrix $\Gamma_0 = \Gamma((1,0,0))$, which in turn is uniquely determined by a subspace $\mathcal{E}_0 = \mathcal{E}_{(1,0,0)}$. The only restriction on this subspace is that the subgroup $O(2)$ of $SO(3)$ that leaves $\boldsymbol{e}_1 = (1,0,0)$ direction invariant should leave the subspace $\mathcal{E}_0$ invariant. Thus, a standard representation theory of $SO(3)$ can give us a list of all possible subspaces $\mathcal{E}_0$ satisfying all of our constraints. For example for $\mathcal{T} = \mathbb{R}^3$, there are just two proper subspaces $\mathcal{E}_0$ satisfying our constraints. One is $\mathbb{R}\boldsymbol{e}_1$, while the other is its orthogonal complement. Thus, for $\boldsymbol{k} \in \mathbb{S}^2$ there are two choices for the function $\Gamma(\boldsymbol{k})$. Either $\Gamma(\boldsymbol{k}) = \boldsymbol{k} \otimes \boldsymbol{k}$ or $\Gamma(\boldsymbol{k}) = \boldsymbol{I} - \boldsymbol{k} \otimes \boldsymbol{k}$, where $\boldsymbol{I}$ is a 3×3 identity matrix. The first choice corresponds to the equations (2.5), and so does the second, with $\boldsymbol{e}$ and $\boldsymbol{j}$ interchanged. Thus, in the context of conductivity, the equations (2.5) are the only possibility consistent with a general Hilbert space framework described here.

The composite microstructure is given by an L^∞ mapping $\boldsymbol{L}(\boldsymbol{x})$ of Q into $\mathrm{End}(\mathcal{T})$. The function $\boldsymbol{L}(\boldsymbol{x})$ can also be viewed as an operator L mapping $\mathcal{H} = L^2(Q) \otimes \mathcal{T}$ into $\mathcal{H}$: for any $\boldsymbol{f} \in \mathcal{H}$ we have $(\mathsf{L}\boldsymbol{f})(\boldsymbol{x}) = \boldsymbol{L}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})$.

The effective tensor $\boldsymbol{L}^*$ is defined by analogy with (2.6):

$$\boldsymbol{J}^* = \boldsymbol{L}^* \boldsymbol{E}^*, \tag{3.1}$$

where $\boldsymbol{E}^* = \langle \boldsymbol{E} \rangle$ and $\boldsymbol{J}^* = \langle \boldsymbol{J} \rangle$.

4 Lamination formula

In this section we derive a lamination formula that has the form of convex combination. This formula was first derived by Milton [38] and independently by Zhikov [51]. Other linear lamination formulas were derived by Backus [1] and Tartar [48] for elasticity. Their idea was to rewrite the constitutive relation so that the continuous and discontinuous components of the elastic fields are separated. The laminate of two materials $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ with layer normal $\boldsymbol{n}$ and volume fractions θ_1, θ_2 ($\theta_1 + \theta_2 = 1$) is a periodic stricture with the period cell shown in Figure 1. More generally, a laminate is a function $\boldsymbol{L}(\boldsymbol{y})$ that depends only on $\boldsymbol{y} \cdot \boldsymbol{n}$. In order to formulate the theorem we introduce the W-transformation of Milton [38]. Let $\boldsymbol{L}_0$ be a reference medium which is assumed to be positive definite, but which otherwise is completely arbitrary. Let $\Gamma'(\boldsymbol{n})$ denote the non-orthogonal projection onto the subspace $\boldsymbol{L}_0\mathcal{E}_{\boldsymbol{n}}$ along the subspace $\mathcal{J}_{\boldsymbol{n}}$. In fact, the projection $\Gamma'(\boldsymbol{n})$ is well defined as long as $\boldsymbol{L}_0$ is positive definite on $\mathcal{E}_{\boldsymbol{n}}$ (to ensure that $\boldsymbol{L}_0\mathcal{E}_{\boldsymbol{n}}$ and $\mathcal{J}_{\boldsymbol{n}}$ have trivial intersection). One can check that

$$\Gamma'(\boldsymbol{n}) = \boldsymbol{L}_0(\boldsymbol{I} - \Gamma(\boldsymbol{n}) + \Gamma(\boldsymbol{n})\boldsymbol{L}_0)^{-1}\Gamma(\boldsymbol{n}).$$

First define the S-transformation

$$S(\boldsymbol{L}) = (\boldsymbol{I} - \boldsymbol{L}\boldsymbol{L}_0^{-1})^{-1}. \tag{4.1}$$

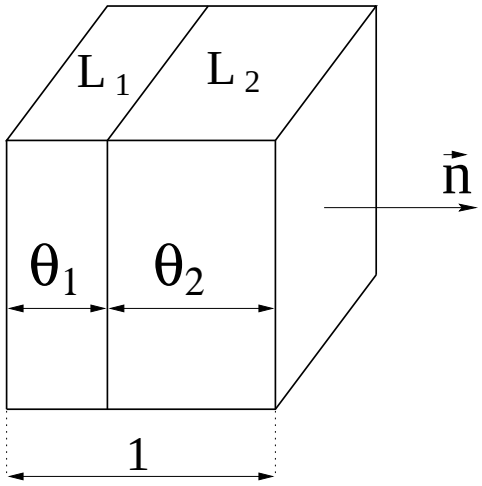

Figure 1: The period cell of the laminate.

Now, define

$$W_{\boldsymbol{n}}(\boldsymbol{L}) = [S(\boldsymbol{L}) - \Gamma'(\boldsymbol{n})]^{-1}\,. \tag{4.2}$$

THEOREM **4.1** *Let $\boldsymbol{L}(\boldsymbol{y})$ be the laminate with the normal $\boldsymbol{n}$ and let $\boldsymbol{L}_0$ be an arbitrary positive definite reference medium (positive definite on $\mathcal{E}_{\boldsymbol{n}}$ is enough). Then*

$$\langle W_{\boldsymbol{n}}(\boldsymbol{L}(\boldsymbol{y}))\rangle = W_{\boldsymbol{n}}(\boldsymbol{L}^*).$$

The proof is analogous to the proof of the corresponding theorem in [21, Theorem 3.1] for symmetric matrices.

PROOF: Taking the average of the polarization field $\boldsymbol{P} = (\boldsymbol{L} - \boldsymbol{L}_0)\boldsymbol{E}$ we obtain $\boldsymbol{P}^* = (\boldsymbol{L}^* - \boldsymbol{L}_0)\boldsymbol{E}^*$. These relations can also be written using the S-transformation defined in(4.1):

$$\boldsymbol{L}_0\boldsymbol{E} = -S(\boldsymbol{L})\boldsymbol{P}, \quad \boldsymbol{L}_0\boldsymbol{E}^* = -S(\boldsymbol{L}^*)\boldsymbol{P}^*. \tag{4.3}$$

Applying the projection operator Γ' to $\boldsymbol{P}$ we obtain

$$\Gamma'(\boldsymbol{n})(\boldsymbol{P} - \boldsymbol{P}^*) = -\boldsymbol{L}_0(\boldsymbol{E} - \boldsymbol{E}^*).$$

Substituting the values for $\boldsymbol{L}_0\boldsymbol{E}$ and $\boldsymbol{L}_0\boldsymbol{E}^*$ from (4.3) we obtain

$$\Gamma'(\boldsymbol{n})(\boldsymbol{P} - \boldsymbol{P}^*) = S(\boldsymbol{L})\boldsymbol{P} - S(\boldsymbol{L}^*)\boldsymbol{P}^*.$$

Solving for $\boldsymbol{P}$ and using the definition of the W-transformation we obtain

$$\boldsymbol{P}(\boldsymbol{y}) = W_{\boldsymbol{n}}(\boldsymbol{L}(\boldsymbol{y}))(W_{\boldsymbol{n}}(\boldsymbol{L}^*))^{-1}\boldsymbol{P}^*.$$

Taking averages we get

$$\boldsymbol{P}^* = \langle W_{\boldsymbol{n}}(\boldsymbol{L}(\boldsymbol{y}))\rangle(W_{\boldsymbol{n}}(\boldsymbol{L}^*))^{-1}\boldsymbol{P}^*.$$

The Theorem now follows because the constant field $\boldsymbol{P}^*$ can be arbitrary. ∎

For the Hall effect $\mathcal{E}_{\boldsymbol{n}} = \mathbb{R}\boldsymbol{n}$ and the requirement that $\boldsymbol{L}_0$ is positive definite on $\mathcal{E}_{\boldsymbol{n}} = \mathbb{R}\boldsymbol{n}$ for all $\boldsymbol{n}$ is equivalent to $\boldsymbol{L}_0$ being positive definite. (For elasticity the corresponding condition is equivalent to the Legendre-Hadamard condition for $\boldsymbol{L}_0$.) In this case

$$\Gamma'(\boldsymbol{n}) = \frac{\boldsymbol{L}_0\boldsymbol{n} \otimes \boldsymbol{n}}{\boldsymbol{L}_0\boldsymbol{n} \cdot \boldsymbol{n}}. \tag{4.4}$$

The W-transformation maps the classical lamination formula [17, 49] into a convex combination. Namely, if $\boldsymbol{L}^*$ is an effective tensor of a laminate made with materials $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ taken in volume fractions θ_1 and θ_2 with lamination normal $\boldsymbol{n}$ then

$$W_{\boldsymbol{n}}(\boldsymbol{L}^*) = \theta_1 W_{\boldsymbol{n}}(\boldsymbol{L}_1) + \theta_2 W_{\boldsymbol{n}}(\boldsymbol{L}_2). \tag{4.5}$$

A corollary is that for any direction $\boldsymbol{n}$ a $W_{\boldsymbol{n}}$-image of any set stable under lamination must be a convex set. The idea to use this property to study geometry of sets stable under lamination is due to Francfort and Milton [16].

5 The main ideas

Theorem 4.1 implies that if a set G is G-closed then $W_{\boldsymbol{n}}(G)$ is a convex set for all $\boldsymbol{n} \in \mathbb{S}^2$. If, on the other hand $W_{\boldsymbol{n}}(G)$ is a convex set for all $\boldsymbol{n} \in \mathbb{S}^2$ then the set G is L-closed (or stable under lamination). In almost all cases the G-closure and L-closure coincide, however there is an example of Milton of a set that is L-closed but not G-closed [39]. In any case, from the practical point of view L-closure would provide a very good approximation to the G-closure. Thus, the *geometric* problem of finding the smallest set $G(D)$ containing D and such that $W_{\boldsymbol{n}}(G(D))$ is convex replaces the problem of solving the cell problem. In the important polycrystalline case, the geometric problem admits a very attractive formulation. (Is there an attractive answer?)

Assume that if a material $\boldsymbol{L}$ is in D then all of its rotations $\{\boldsymbol{R} \cdot \boldsymbol{L} \mid \boldsymbol{R} \in SO(3)\}$ are also in D. If we choose an isotropic reference medium then we can easily verify that

$$\boldsymbol{R} \cdot W_{\boldsymbol{n}}(\boldsymbol{L}) = W_{\boldsymbol{R}\boldsymbol{n}}(\boldsymbol{R} \cdot \boldsymbol{L}). \tag{5.1}$$

It follows then, that if a set $G \subset \mathrm{End}(\mathcal{T})$ is rotationally invariant and $W_{\boldsymbol{n}}(G)$ is convex for *a single choice of* $\boldsymbol{n}$ then $W_{\boldsymbol{n}}(G)$ is a convex set for all $\boldsymbol{n} \in \mathbb{S}^2$. Thus, we can fix one unit vector $\boldsymbol{n}$, for example $\boldsymbol{n} = (1, 0, 0)$ and consider just one map $W(\boldsymbol{L}) = W_{\boldsymbol{n}}(\boldsymbol{L})$. The question is this:

> Find the smallest rotationally invariant set $G(D)$ containing a given set D such that $W(G(D))$ is convex.

We can ask a different question.

> Can we characterize convex functions $g(\boldsymbol{K})$ such that the functions $f(\boldsymbol{L}) = g(W(\boldsymbol{L}))$ are rotationally invariant, $f(\boldsymbol{R} \cdot \boldsymbol{L}) = f(\boldsymbol{L})$ for all $\boldsymbol{R} \in SO(3)$?

This question sounds very similar to the one answered by Chandler Davis [6]. Davis proved that a rotationally invariant function defined on hermitean matrices is convex if and only if its restriction to the diagonal matrices is convex. The method of Davis does not apply to our problem because the action of $SO(3)$ on the W variables is no longer linear. Yet, a similar result is hoped for. We conjecture that there is a subspace in $\mathrm{End}(\mathcal{T})$ transversal to the action of the rotation group $SO(3)$, such that the convexity of the restriction of g to this transversal is equivalent to the convexity of g. (Of course, the restriction of g to the transversal has to be invariant under the action of the finite Weil subgroup S_3 of $SO(3)$.) It is our guess that the geometric results of [15] can be reformulated in terms of convexity in the transversal.

6 Exact relations

We now turn to another implication of convexity discussed in Section 5 above. If we are searching for submanifolds $\mathbb{M}$ of $\mathrm{End}(\mathcal{T})$ corresponding to exact relations, then convexity of the submanifolds $\Pi_{\boldsymbol{n}} = W_{\boldsymbol{n}}(\mathbb{M})$ is equivalent to saying that $\Pi_{\boldsymbol{n}}$ are affine subspaces of $\mathrm{End}(\mathcal{T})$. Observe now that $W_{\boldsymbol{n}}(\boldsymbol{L}_0) = \boldsymbol{0}$. Thus, choosing $\boldsymbol{L}_0$ to lie on $\mathbb{M}$ we make sure that $\Pi_{\boldsymbol{n}}$ is a subspace for each $\boldsymbol{n}$. The following theorem gives necessary conditions for $\mathbb{M}$ to be an exact relation.

THEOREM **6.1** *Suppose the submanifold $\mathbb{M}$ is an exact relation. Then the subspaces $\Pi_{\boldsymbol{n}} = W_{\boldsymbol{n}}(\mathbb{M})$ do not depend on $\boldsymbol{n}$. Moreover, this single subspace Π is closed with respect to the family of Jordan multiplications, defined by*

$$\boldsymbol{K}_1 *_{\boldsymbol{A}} \boldsymbol{K}_2 = \frac{1}{2}(\boldsymbol{K}_1\boldsymbol{A}\boldsymbol{K}_2 + \boldsymbol{K}_2\boldsymbol{A}\boldsymbol{K}_1), \tag{6.1}$$

where $\boldsymbol{A}$ can be any matrix from the subspace

$$\mathcal{A} = \mathrm{Span}\{\Gamma'(\boldsymbol{n}) - \Gamma'(\boldsymbol{e}_1) \mid \boldsymbol{n} \in \mathbb{S}^2\}. \tag{6.2}$$

The proof of this theorem follows word for word the proof of the corresponding theorem in [21, Theorem 3.5] for the symmetric case. From now on we will say that a theorem is proved in [21] if the proof for the general case is the same as for the symmetric case with obvious modifications.

Another important question is about stability under homogenization. To this end, in [21], we have derived a formula for the effective tensor $\boldsymbol{L}^*$.

THEOREM **6.2** *Let $\boldsymbol{L}^*$ be the effective tensor for the composite with the local tensor $\boldsymbol{L}(\boldsymbol{x})$. Then for any $\boldsymbol{n} \in \mathbb{S}^2$*

$$W_{\boldsymbol{n}}(\boldsymbol{L}^*) = \langle (\mathsf{I} - \mathsf{W}_{\boldsymbol{n}}\Lambda_{\boldsymbol{n}})^{-1} W_{\boldsymbol{n}}(\boldsymbol{L}(\boldsymbol{x}))\rangle, \tag{6.3}$$

where I denotes the identity operator on the Hilbert space $\mathbb{H} = L^2(Q) \otimes \mathrm{End}(\mathcal{T})$, $\mathsf{W}_{\boldsymbol{n}}$ denotes the multiplication operator on $\mathbb{H}$: $(\mathsf{W}_{\boldsymbol{n}}\boldsymbol{H})(\boldsymbol{x}) = W_{\boldsymbol{n}}(\boldsymbol{L}(\boldsymbol{x}))\boldsymbol{H}(\boldsymbol{x})$ and $\Lambda_{\boldsymbol{n}}$ is defined by

$\widehat{\Lambda_{\boldsymbol{n}}\boldsymbol{H}}(\boldsymbol{k}) = \boldsymbol{A}_{\boldsymbol{n}}(\boldsymbol{k})\widehat{\boldsymbol{H}}(\boldsymbol{k})$, *where*

$$\boldsymbol{A}_{\boldsymbol{n}}(\boldsymbol{m}) = \begin{cases} \Gamma'(\frac{\boldsymbol{m}}{|\boldsymbol{m}|}) - \Gamma'(\boldsymbol{n}), & \text{if } \boldsymbol{m} \neq 0, \\ 0, & \text{if } \boldsymbol{m} = 0. \end{cases} \tag{6.4}$$

Using this formula we can prove the following rather messy necessary and sufficient condition for a subspace Π to correspond to an exact relation.

THEOREM **6.3** *For $\boldsymbol{v} = (\boldsymbol{l}_1, \dots, \boldsymbol{l}_k) \in (\mathbb{Z}^3)^k$ and for $\sigma \in S_k$—permutation of k elements we define*

$$\sigma(\boldsymbol{v}) = (\boldsymbol{l}_{\sigma(1)}, \dots, \boldsymbol{l}_{\sigma(k)}) \in (\mathbb{Z}^3)^k. \tag{6.5}$$

Let $\mathcal{O}(\boldsymbol{v}) = \{\sigma(\boldsymbol{v}) \in (\mathbb{Z}^3)^k \mid \sigma \in S_k\}$ be the orbit of $\boldsymbol{v}$ under the action of the permutation group S_k. The subspace $\Pi \subset \mathrm{End}(\mathcal{T})$ corresponds to an exact relation if and only if for any $k \in \mathbb{N}$, any $\boldsymbol{n} \in \mathbb{S}^2$, any $\boldsymbol{v} = (\boldsymbol{l}_1, \dots, \boldsymbol{l}_k) \in (\mathbb{Z}^3)^k$ such that

$$\sum_{i=1}^{k} \boldsymbol{l}_i = \boldsymbol{0} \tag{6.6}$$

and for any function $\boldsymbol{K} : \mathbb{Z}^3 \to \Pi$ we have

$$\sum_{(\boldsymbol{p}_1, \dots \boldsymbol{p}_k) \in \mathcal{O}(\boldsymbol{v})} \left(\prod_{s=1}^{k-1} \boldsymbol{K}(\boldsymbol{p}_s) \boldsymbol{A}_{\boldsymbol{n}}(\sum_{j=1}^{s} \boldsymbol{p}_j) \right) \boldsymbol{K}(\boldsymbol{p}_k) \in \Pi, \tag{6.7}$$

where $\boldsymbol{A}_{\boldsymbol{n}}$ is defined in (6.4) above.

The analogous theorem was formulated in [21, Theorem 3.9], where some steps of the proof were indicated. Unfortunately, the formulation of the theorem was not entirely correct. The sum in the formula [21, (3.38)] corresponding to (6.7) extended over the set S_k of all permutations, instead of the elements of the orbit $\mathcal{O}(\boldsymbol{v})$. In [21] the theorem was not used anywhere else and was included only for the purposes of a discussion. Here we give the correct formulation and a complete proof of the theorem.

PROOF: The proof is based on the formula (6.3) and the analyticity properties enjoyed by the effective tensor. In [21, Appendix] we have shown that the map $W_{\boldsymbol{n}}$ is an analytic diffeomorphism, defined everywhere on the set of positive definite matrices. Therefore, all exact relation manifolds $\mathbb{M}$, being analytic images of subspaces Π, are analytic. Also, if we take an analytic family $\boldsymbol{L}_\lambda(\boldsymbol{x})$ of local tensors, it will generate an analytic family of effective tensors $\boldsymbol{L}^*_\lambda$ [18]. The principle of analytic continuation implies that if $\boldsymbol{L}^*_\lambda \in \mathbb{M}$ for some small interval of λ then $\boldsymbol{L}^*_\lambda \in \mathbb{M}$ for all λ in the interval of analyticity of $\boldsymbol{L}^*_\lambda$. This argument shows that if we want to prove that $\mathbb{M}$ is an exact relation, then it is enough to show that $\mathbb{M}$ satisfies necessary conditions of Theorem 6.1 and that for one choice of $\boldsymbol{n} \in \mathbb{S}^2$ and for any $\boldsymbol{W}(\boldsymbol{x}) = W_{\boldsymbol{n}}(\boldsymbol{L}(\boldsymbol{x})) \in \Pi$ sufficiently small, we have $\boldsymbol{W}^* = W_{\boldsymbol{n}}(\boldsymbol{L}^*) \in \Pi$. We can ensure that

$\boldsymbol{W}(\boldsymbol{x})$ is sufficiently small if we choose the values of $\boldsymbol{L}(\boldsymbol{x})$ sufficiently close to the reference medium $\boldsymbol{L}_0 \in \mathbb{M}$.

Let $\Pi_{\mathbb{C}} = \{\boldsymbol{K}_1 + i\boldsymbol{K}_2 \mid \{\boldsymbol{K}_1, \boldsymbol{K}_2\} \subset \Pi\}$ be the complexification of Π. The analytic continuation principle also implies that if Π corresponds to an exact relation then so does $\Pi_{\mathbb{C}}$ in the sense that if $\boldsymbol{W}(\boldsymbol{x}) \in \Pi_{\mathbb{C}}$ then also $\boldsymbol{W}^* \in \Pi_{\mathbb{C}}$, where $\boldsymbol{W}^*$ is computed from $\boldsymbol{W}(\boldsymbol{x})$ by the formula (6.3). The argument is due to Milton (private communication). Let $\boldsymbol{W}_\lambda(\boldsymbol{x}) = \boldsymbol{W}_1(\boldsymbol{x}) + \lambda \boldsymbol{W}_2(\boldsymbol{x})$, where $\boldsymbol{W}_1(\boldsymbol{x}) \in \Pi$ and $\boldsymbol{W}_2(\boldsymbol{x}) \in \Pi$ are assumed to be small enough. Then, since Π corresponds to an exact relation we conclude that for real values of λ in some finite interval, we have $\boldsymbol{W}_\lambda^* \in \Pi$. Let $\boldsymbol{P} \in \mathrm{End}(\mathcal{T})$ be such that $\mathrm{Tr}\,(\boldsymbol{PK}) = 0$ for all $\boldsymbol{K} \in \Pi$. Then the function $f(\lambda) = \mathrm{Tr}\,(\boldsymbol{PW}_\lambda^*)$ is analytic in λ and is zero on an interval on the real axis. Since $\boldsymbol{W}$ is assumed to be small, the region of analyticity of $f(\lambda)$ in the complex plane includes $\lambda = i$. Thus, we conclude that $\mathrm{Tr}\,(\boldsymbol{PW}_i^*) = 0$ for all real matrices $\boldsymbol{P}$ such that $\mathrm{Tr}\,(\boldsymbol{PK}) = 0$ for all $\boldsymbol{K} \in \Pi$. But

$$\Pi_{\mathbb{C}} = \{\boldsymbol{K} \in \mathrm{End}_{\mathbb{C}}(\mathcal{T}) \mid \mathrm{Tr}\,(\boldsymbol{PK}) = 0 \ \forall \boldsymbol{P} : \mathrm{Tr}\,(\boldsymbol{PW}) = 0 \ \forall \boldsymbol{W} \in \Pi\}.$$

Thus, $\boldsymbol{W}_i^* \in \Pi_{\mathbb{C}}$.

If $\boldsymbol{W}(\boldsymbol{x})$ is complex-valued and sufficiently close to zero then we can expand (6.3) in a convergent power series:

$$\boldsymbol{W}^* = \langle \boldsymbol{W}(\boldsymbol{x}) \rangle + \langle \mathsf{W}\Lambda_{\boldsymbol{n}} \boldsymbol{W}(\boldsymbol{x}) \rangle + \ldots + \langle (\mathsf{W}\Lambda_{\boldsymbol{n}})^k \boldsymbol{W}(\boldsymbol{x}) \rangle + \ldots \tag{6.8}$$

where W is the operator of multiplication by $\boldsymbol{W}(\boldsymbol{x})$. It follows from the argument above that $\mathbb{M}$ is an exact relation if and only if each term of the expansion (6.8) belongs to $\Pi_{\mathbb{C}}$, when $\boldsymbol{W}(\boldsymbol{x}) \in \Pi_{\mathbb{C}}$ is small enough.

Let $\boldsymbol{T}_k(\boldsymbol{x}) = (\mathsf{W}\Lambda_{\boldsymbol{n}})^k \boldsymbol{W}(\boldsymbol{x})$. Taking Fourier transform of $\boldsymbol{T}_k(\boldsymbol{x}) = \boldsymbol{W}(\boldsymbol{x})(\Lambda_{\boldsymbol{n}} \boldsymbol{T}_{k-1})(\boldsymbol{x})$ and using induction in k, we can prove that

$$\widehat{\boldsymbol{T}}_k(\boldsymbol{m}) = \sum_{\boldsymbol{l}_1 + \ldots + \boldsymbol{l}_k = \boldsymbol{m}} \left(\prod_{s=1}^{k-1} \widehat{\boldsymbol{W}}(\boldsymbol{l}_s) \boldsymbol{A}_{\boldsymbol{n}}(\sum_{j=s+1}^{k} \boldsymbol{l}_j) \right) \widehat{\boldsymbol{W}}(\boldsymbol{l}_k),$$

where $\boldsymbol{m}, \boldsymbol{l}_j \in \mathbb{Z}^3$. Thus, we get

$$\langle (\mathsf{W}\Lambda_{\boldsymbol{n}})^k \boldsymbol{W}(\boldsymbol{x}) \rangle = \widehat{\boldsymbol{T}}_k(\boldsymbol{0}) = \sum_{\boldsymbol{l}_1 + \ldots + \boldsymbol{l}_k = \boldsymbol{0}} \left(\prod_{s=1}^{k-1} \widehat{\boldsymbol{W}}(\boldsymbol{l}_s) \boldsymbol{A}_{\boldsymbol{n}}(\sum_{j=1}^{s} \boldsymbol{l}_j) \right) \widehat{\boldsymbol{W}}(\boldsymbol{l}_k). \tag{6.9}$$

Observe that the sum in (6.9) can be split into parts. The summation in each part goes over all the distinct permutations of the same set of vectors $\boldsymbol{l}_j$. In order to say this more rigorously we define the action of the permutation group S_k on $(\mathbb{Z}^3)^k$ by (6.5). Thus, $(\mathbb{Z}^3)^k$ splits into the disjoint union of orbits of the group action. Let $\mathcal{Z}$ denote the set of orbits, whose elements satisfy the constraint (6.6) (invariant under the group action). Then we can write

$$\langle \boldsymbol{T}_k \rangle = \sum_{\mathcal{O} \in \mathcal{Z}} \sum_{(\boldsymbol{l}_1, \ldots, \boldsymbol{l}_k) \in \mathcal{O}} \left(\prod_{s=1}^{k-1} \widehat{\boldsymbol{W}}(\boldsymbol{l}_s) \boldsymbol{A}_{\boldsymbol{n}}(\sum_{j=1}^{s} \boldsymbol{l}_j) \right) \widehat{\boldsymbol{W}}(\boldsymbol{l}_k). \tag{6.10}$$

If $\mathbb{M}$ is an exact relation then we can choose an arbitrary k-tuple $(\boldsymbol{l}_1, \ldots, \boldsymbol{l}_k) \in (\mathbb{Z}^3)^k$ satisfying (6.6) and let

$$\boldsymbol{W}(\boldsymbol{x}) = \sum_{j=1}^{k} \boldsymbol{K}_j e^{i\boldsymbol{l}_j \cdot \boldsymbol{x}},$$

where $\{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_k\} \subset \Pi$ are arbitrary. Then we will obtain (6.7). Conversely, if (6.7) is satisfied for all $\{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_k\} \subset \Pi$ then, by the analytic continuation principle, it is satisfied for all $\{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_k\} \subset \Pi_{\mathbb{C}}$. Thus, each term under the exterior sum in (6.10) is in $\Pi_{\mathbb{C}}$. Therefore, $\boldsymbol{W}^* \in \Pi_{\mathbb{C}}$ and is real, so $\boldsymbol{W}^* \in \Pi$. The Theorem is proved. ∎

The above theorem does provide algebraic conditions for Π that guarantee that Π corresponds to an exact relation. The practical utility of such a theorem is minimal, since it is virtually impossible to check infinitely many conditions that are as complicated as (6.7). Therefore, in [21] we proved a convenient sufficient condition for Π to correspond to an exact relation. In order to formulate it we introduce the following terminology. The expression $\boldsymbol{K}_1\boldsymbol{A}_1\boldsymbol{K}_2\boldsymbol{A}_2\boldsymbol{K}_3\boldsymbol{A}_3 \ldots \boldsymbol{K}_{j-1}\boldsymbol{A}_{j-1}\boldsymbol{K}_j$ will be called a j-chain. We say that a subspace Π satisfies a j-chain property if for every $\{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_j\} \subset \Pi$ and every $\{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_{j-1}\} \subset \mathcal{A}$ we have

$$\boldsymbol{K}_1\boldsymbol{A}_1\boldsymbol{K}_2\boldsymbol{A}_2 \ldots \boldsymbol{K}_{j-1}\boldsymbol{A}_{j-1}\boldsymbol{K}_j + \boldsymbol{K}_j\boldsymbol{A}_{j-1}\boldsymbol{K}_{j-1} \ldots \boldsymbol{A}_2\boldsymbol{K}_2\boldsymbol{A}_1\boldsymbol{K}_1 \in \Pi. \tag{6.11}$$

We remark that if Π is a subspace corresponding to an exact relation then Theorem 6.1 says that Π satisfies a 2-chain property. Now the analogue of the sufficient condition established in [21] can be formulated as follows.

THEOREM **6.4** *If Π satisfies the j-chain property for $j = 2, 3$ and 4, then Π corresponds to an exact relation.*

PROOF: In [21, proof of Lemma 3.7] we proved that if Π satisfies the j-chain property for $j = 2, 3$ and 4, then Π satisfies the j-chain property for all $j \geq 2$. It remains to show that if Π satisfies the j-chain property for every $j \geq 2$ then (6.7) will be satisfied.

Let η be the permutation defined by $\eta(j) = k + 1 - j$. The permutation η is an element of order two in the group S_k. It acts on $\mathcal{O}(v)$ and splits $\mathcal{O}(v)$ in a disjoint union of orbits. If an orbit contains two elements then their sum has the form (6.11). If the orbit contains a single element then this element is $1/2$ of the sum of two copies of itself, which is again of the form (6.11). ∎

In two space dimensions, however, we can say a little bit more.

THEOREM **6.5** *In two space dimensions the 3-chain property is necessary for stability under homogenization.*

PROOF: We will show that in 2D the necessary and sufficient condition (6.7) from Theorem 6.3 for $k = 3$ implies the 3-chain property.

Let $\boldsymbol{n} = (1, 0)$. Fix $\{N, M\} \subset \mathbb{Z} \setminus \{0\}$. Choose $\boldsymbol{k} \in \mathbb{Z}^2$ to be linearly independent with $\boldsymbol{n}$. and define $\boldsymbol{l}_1 = M\boldsymbol{k}$, $\boldsymbol{l}_2 = -N\boldsymbol{n} - M\boldsymbol{k}$, so that $\boldsymbol{l}_3 = -\boldsymbol{l}_1 - \boldsymbol{l}_2 = N\boldsymbol{n}$. Observe that $\boldsymbol{l}_1$, $\boldsymbol{l}_2$

and $\boldsymbol{l}_3$ are distinct and that according to (6.4), $\boldsymbol{A}(\boldsymbol{l}_3) = \boldsymbol{A}(\boldsymbol{l}_1 + \boldsymbol{l}_2) = \boldsymbol{0}$. For simplicity we use $\boldsymbol{A}(\cdot)$ notation instead of $\boldsymbol{A}_{\boldsymbol{n}}(\cdot)$, since $\boldsymbol{n} = (1, 0)$ is fixed. Condition (6.7) for $k = 3$ then becomes

$$\boldsymbol{K}_2\boldsymbol{A}(M\boldsymbol{k} + N\boldsymbol{n})\boldsymbol{K}_3\boldsymbol{A}(\boldsymbol{k})\boldsymbol{K}_1 + \boldsymbol{K}_1\boldsymbol{A}(\boldsymbol{k})\boldsymbol{K}_3\boldsymbol{A}(M\boldsymbol{k} + N\boldsymbol{n})\boldsymbol{K}_2 \in \Pi$$

for all $\{\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3\} \subset \Pi$. Now, if we fix $\boldsymbol{k} \in \mathbb{Z}^2$, let $\boldsymbol{l}_1 = M\boldsymbol{k}$ and vary $\{M, N\} \subset \mathbb{Z} \setminus \{0\}$ then we observe, that the vectors $\{(M\boldsymbol{k} + N\boldsymbol{n})/|M\boldsymbol{k} + N\boldsymbol{n}| : \{M, N\} \subset \mathbb{Z} \setminus \{0\}\}$ form a dense subset of the unit circle. (The same construction does not yield a dense subset of the unit sphere in 3D.) Therefore, by continuity, we conclude that for any $\{\boldsymbol{q}_1, \boldsymbol{q}_2\} \subset \mathbb{S}^1$ we have

$$\boldsymbol{K}_2\boldsymbol{A}(\boldsymbol{q}_2)\boldsymbol{K}_3\boldsymbol{A}(\boldsymbol{q}_1)\boldsymbol{K}_1 + \boldsymbol{K}_1\boldsymbol{A}(\boldsymbol{q}_1)\boldsymbol{K}_3\boldsymbol{A}(\boldsymbol{q}_2)\boldsymbol{K}_2 \in \Pi$$

for all $\{\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3\} \subset \Pi$, which easily implies the 3-chain condition. ∎

We remark that in all the examples that we have worked out so far our sufficient condition was satisfied whenever the necessary conditions in Theorem 6.1 were.

We would like to conclude this section by stating the results and problems obtained in [21] for the symmetric case. For a subspace $\mathcal{X}$ of $\mathrm{End}(\mathcal{T})$ let $\mathcal{X}_{\mathrm{sym}}$ denote the set of symmetric parts of matrices in $\mathcal{X}$. If $\Pi \subset \mathrm{Sym}(\mathcal{T})$ then Theorem 6.1 can be restated as follows.

THEOREM **6.6** *If the subspace $\Pi \subset \mathrm{Sym}(\mathcal{T})$ corresponds to an exact relation then it satisfies*

$$(\Pi\mathcal{A}\Pi)_{\mathrm{sym}} \subset \Pi. \tag{6.12}$$

The sufficient condition for Π to correspond to an exact relation stated in Theorem 6.4 can be cast into a more attractive algebraic form. Let $\Pi' \subset \mathrm{End}(\mathcal{T})$ be the smallest associative algebra containing the Jordan algebra Π. In other words Π' is the smallest subspace in $\mathrm{End}(\mathcal{T})$ containing Π that satisfies

$$\Pi'\mathcal{A}\Pi' \subset \Pi'. \tag{6.13}$$

Theorem 6.4 can then be restated as follows.

THEOREM **6.7** *If the subspace $\Pi \subset \mathrm{Sym}(\mathcal{T})$ solves (6.12) and has the additional property that $\Pi = \Pi'_{\mathrm{sym}}$, then Π corresponds to the exact relation.*

The important open question is whether every Jordan algebra Π (understood in the sense of (6.12)) is the set of all symmetric matrices of the smallest associative algebra Π' (understood in the sense of (6.13)) containing Π. The question, in other words, is "Are sufficient conditions necessary?".

7 Finding exact relations

The preceding section established simple algebraic conditions that the subspace Π has to satisfy in order to correspond to an exact relation. Now the question is: can we characterize all solutions Π of

$$\boldsymbol{K}_1\boldsymbol{A}\boldsymbol{K}_2 + \boldsymbol{K}_2\boldsymbol{A}\boldsymbol{K}_1 \in \Pi, \tag{7.1}$$

for all $\{\boldsymbol{K}_1, \boldsymbol{K}_2\} \subset \Pi$ and all $\boldsymbol{A} \in \mathcal{A}$. Unfortunately, at present we do not have an efficient way to solve equation (7.1). In this section we will discuss an inefficient way of solving (7.1) for 2D Hall effect. The method simply consists of picking a matrix and computing by brute force the smallest Jordan algebra Π containing that matrix. Then we compute all other Jordan algebras extending the smallest one, again, by brute force. This method is applicable to rather small dimensional cases (or small block-dimensional as in [21]).

For 2D Hall effect the space $\mathcal{T} = \mathbb{R}^2$. Let

$$\boldsymbol{L}_0 = \boldsymbol{\sigma}_0 + \alpha \boldsymbol{S}, \tag{7.2}$$

be the positive definite reference medium, where $\boldsymbol{\sigma}_0$ is the symmetric part of $\boldsymbol{L}_0$ and

$$\boldsymbol{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{7.3}$$

We recall that for the Hall effect the matrix $\Gamma'(\boldsymbol{n})$ is given by (4.4). Therefore, the subspace $\mathcal{A}$ defined in (6.2) is

$$\mathcal{A} = \{\boldsymbol{A} \in \mathrm{End}(\mathbb{R}^2) \mid \mathrm{Tr}\, \boldsymbol{A} = 0,\ \boldsymbol{A}\boldsymbol{L}_0^T = \boldsymbol{L}_0 \boldsymbol{A}^T\}. \tag{7.4}$$

7.1 Solving (7.1)

Observe that the equation (7.1) behaves nicely with respect to the following "change of variables". Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be invertible matrices. Let $\overline{\Pi} = \boldsymbol{X}\Pi\boldsymbol{Y}$ then Π solves (7.1) if and only if $\overline{\Pi}$ solves (7.1) with $\mathcal{A}$ replaced by $\overline{\mathcal{A}} = \boldsymbol{Y}^{-1}\mathcal{A}\boldsymbol{X}^{-1}$. Applying this observation to

$$\boldsymbol{X} = \boldsymbol{\sigma}_0^{-1/2}, \quad \boldsymbol{Y} = \boldsymbol{L}_0 \boldsymbol{X} \tag{7.5}$$

we obtain that $\overline{\mathcal{A}}$ is the space of trace-free symmetric 2×2 matrices. From now on we will work only with subspaces $\overline{\mathcal{A}}$ and $\overline{\Pi}$ and therefore, we will rename them $\mathcal{A}$ and Π, respectively, for notational convenience. We will return to the original notation when we will have solved the equation (7.1). Thus, $\mathcal{A}$ now denotes the subspace of 2×2 symmetric trace-free matrices.

Observe that for $\boldsymbol{X} = \boldsymbol{R}$ and $\boldsymbol{Y} = \boldsymbol{R}^T$, where $\boldsymbol{R}$ is a 2×2 rotation matrix, the equation $\boldsymbol{K}\mathcal{A}\boldsymbol{K} \subset \Pi$, $\boldsymbol{K} \in \Pi$ remains invariant. Now fix $\boldsymbol{K} \in \Pi$. We can find a rotation $\boldsymbol{R} \in O(2)$ such that

$$\boldsymbol{R} \cdot \boldsymbol{K} = \boldsymbol{K}_0 = \begin{bmatrix} 1 & \beta \\ -\beta & \alpha \end{bmatrix}. \tag{7.6}$$

Thus, without loss of generality, Π contains a matrix $\boldsymbol{K}_0$ of the form (7.6). We have

$$\begin{bmatrix} 1 & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} s & t \\ t & -s \end{bmatrix} \begin{bmatrix} 1 & \beta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} (1+\beta^2)s & (\alpha+\beta^2)t + \beta(1-\alpha)s \\ (\alpha+\beta^2)t - \beta(1-\alpha)s & -(\alpha^2+\beta^2)s \end{bmatrix}. \tag{7.7}$$

Case I: There exists $\boldsymbol{K}_0 \in \Pi$ with $\alpha + \beta^2 \neq 0$.

In this case we easily see from (7.7) (setting $s = 0$, $t = 1$) that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \Pi$. Consequently,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s & t \\ t & -s \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -s & t \\ t & s \end{bmatrix}$$

implies that $\mathcal{A} \subset \Pi$. Simple manipulations show that the right hand side of (7.7) is equal to $(1-\alpha)s\boldsymbol{K}_0 \bmod \mathcal{A}$. Thus, we get that Π contains the subspace $\mathcal{A}$ and the matrix

$$\boldsymbol{K}_0' = \begin{bmatrix} (1+\alpha)/2 & \beta \\ -\beta & (1+\alpha)/2 \end{bmatrix}. \tag{7.8}$$

Case I(a): There exists $\boldsymbol{K}_0 \in \Pi$ such that $\alpha \neq -1$, in addition to $\alpha + \beta^2 \neq 0$.
Then we infer that

$$\begin{bmatrix} 1 & \beta' \\ -\beta' & 1 \end{bmatrix} \in \Pi,$$

where $\beta' = 2\beta/(1+\alpha)$. Again an easy calculation shows that

$$\begin{bmatrix} 1 & \beta' \\ -\beta' & 1 \end{bmatrix}\begin{bmatrix} s & t \\ t & -s \end{bmatrix}\begin{bmatrix} x & y \\ y & -x \end{bmatrix}+\begin{bmatrix} x & y \\ y & -x \end{bmatrix}\begin{bmatrix} s & t \\ t & -s \end{bmatrix}\begin{bmatrix} 1 & \beta' \\ -\beta' & 1 \end{bmatrix} = 2(sx+ty)\begin{bmatrix} 1 & \beta' \\ -\beta' & 1 \end{bmatrix} (\text{mod } \mathcal{A}).$$

Thus, the minimal subspace Π satisfying (7.1) in the Case I(a) is a 3D subspace

$$\Pi = \mathcal{A} \oplus \begin{bmatrix} 1 & \beta' \\ -\beta' & 1 \end{bmatrix} \mathbb{R}. \tag{7.9}$$

Since this subspace is of codimension 1, there are no larger proper subspaces containing Π. We note that (7.9) gives a one parameter family of solutions Π, labeled by $\beta' \in \mathbb{R}$.

Case I(b): $\alpha = -1$ for all $\boldsymbol{K}_0 \in \Pi$.
Then, returning to (7.8) we obtain that

$$\begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} \in \Pi.$$

Now if all $\boldsymbol{K}_0 \in \Pi$ are symmetric then $\beta = 0$ for all $\boldsymbol{K}_0 \in \Pi$ and $\Pi = \mathcal{A}$ or $\Pi = \text{Sym}(\mathbb{R}^2)$—the space of symmetric 2×2 matrices.

If there is $\boldsymbol{K}_0 \in \Pi$ which is non-symmetric, then

$$\Pi = \mathcal{A} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbb{R}. \tag{7.10}$$

This subspace is of codimension 1 and therefore is not contained in any other proper subspace.

Case II: $\alpha = -\beta^2$ for all $\boldsymbol{K}_0 \in \Pi$. In this case we easily see that

$$\Pi = \begin{bmatrix} 1 & \beta \\ -\beta & -\beta^2 \end{bmatrix} \mathbb{R} \tag{7.11}$$

is a one-parameter family of 1D solutions of (7.1). Now we would like to see if there are other solutions Π containing (7.11). Let $\boldsymbol{K}_1$, not a multiple of $\boldsymbol{K}_0$, be in Π. Then for all $\lambda \in \mathbb{R}$ the matrices $\boldsymbol{K}(\lambda) = \boldsymbol{K}_1 + \lambda \boldsymbol{K}_0$ belong to Π and to the Case II (we have already analyzed

Case I). Thus, $\boldsymbol{K}(\lambda)$ can be reduced to the form (7.11) with a different β by a rotation and scaling:

$$\boldsymbol{K}(\lambda) = \mu(\lambda)\boldsymbol{R}_\lambda \begin{bmatrix} 1 & \beta(\lambda) \\ -\beta(\lambda) & -\beta(\lambda)^2 \end{bmatrix} \boldsymbol{R}_\lambda^T.$$

It will be convenient to represent $\boldsymbol{K}_1$ in the form $\boldsymbol{K}_1 = \boldsymbol{P} + \gamma \boldsymbol{S}$, where $\boldsymbol{P}$ is a symmetric matrix and $\boldsymbol{S}$ is given by (7.3). Taking the antisymmetric part in the formula for $\boldsymbol{K}(\lambda)$ we obtain $\mu(\lambda)\beta(\lambda) = \gamma + \lambda\beta$, while taking the determinant of the symmetric part of $\boldsymbol{K}(\lambda)$ we get:

$$\det \boldsymbol{P} + \lambda(p_{22} - \beta^2 p_{11}) - \lambda^2\beta^2 = -\mu(\lambda)^2\beta(\lambda)^2,$$

for all $\lambda \in \mathbb{R}$. Thus, we obtain the following equations for γ and for the components p_{ij} of $\boldsymbol{P}$:

$$p_{11}p_{22} - p_{12}^2 = -\gamma^2, \quad p_{22} - \beta^2 p_{11} = -2\gamma\beta.$$

Eliminating p_{22} we get $(p_{11}\beta - \gamma)^2 = p_{12}^2$. So either

$$p_{12} = p_{11}\beta - \gamma \tag{7.12}$$

or

$$p_{12} = \gamma - p_{11}\beta \tag{7.13}$$

Assume first that $\boldsymbol{K}_1$ satisfies (7.12) Then

$$\boldsymbol{K}_1 = \begin{bmatrix} p_{11} & p_{11}\beta \\ p_{11}\gamma - 2\gamma & p_{11}\beta^2 - 2\gamma\beta \end{bmatrix}.$$

Now observe that

$$\boldsymbol{K}_1 - p_{11}\boldsymbol{K}_0 = 2(p_{11}\beta - \gamma) \begin{bmatrix} 0 & 0 \\ 1 & \beta \end{bmatrix}.$$

Notice that if $\gamma = p_{11}\beta$ then $\boldsymbol{K}_1 = p_{11}\boldsymbol{K}_0$ in contradiction to our assumption that $\boldsymbol{K}_1$ is not a multiple of $\boldsymbol{K}_0$. Thus, we conclude that

$$\Pi_1 = \operatorname{Span}\left\{ \begin{bmatrix} 0 & 0 \\ 1 & \beta \end{bmatrix}, \begin{bmatrix} 1 & \beta \\ 0 & 0 \end{bmatrix} \right\} \subset \Pi. \tag{7.14}$$

One may easily check that Π_1 is a one-parameter family of 2D solutions.

Making a similar analysis for the case (7.13) we obtain a solution $\Pi_2 \subset \Pi$, where

$$\Pi_2 = \operatorname{Span}\left\{ \begin{bmatrix} 1 & 0 \\ -\beta & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix} \right\}. \tag{7.15}$$

If Π contains Π_1 but is larger than Π_1 then our previous analysis shows that $\boldsymbol{K} \in \Pi$ must be either in Π_1 or in Π_2. But $\Pi_1 \bigcup \Pi_2$ is not a subspace.

In summary, a proper subspace Π is a solution of (7.1) if and only if it is a rotated image of either $\mathcal{A}$, $\operatorname{Sym}(\mathbb{R}^2)$, (7.9), (7.10), (7.11), (7.14) or (7.15). We can now state the result.

THEOREM **7.1** *If a proper subspace* Π *in the space of all* 2×2 *matrices satisfies (7.1) with* $\mathcal{A}$ *being the space of symmetric, trace-free* 2×2 *matrices then* Π *is a subspace from the following list.*

1. $\mathrm{Sym}(\mathbb{R}^2)$—*The space of symmetric matrices*
2. Π_0—*The space of trace-free matrices*
3. $\mathcal{A}$—*The space of symmetric, trace-free matrices*
4. $\Pi_\beta = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \; a + d = \beta(b - c) \right\}$, $\beta \neq 0$.
5. $\Pi_{\boldsymbol{a}} = \{\boldsymbol{v} \otimes \boldsymbol{a} \mid \boldsymbol{v} \in \mathbb{R}^2\}$
6. $\Pi_{\boldsymbol{a}}^T$
7. $\Pi_{\boldsymbol{a},\boldsymbol{b}} = \mathbb{R}(\boldsymbol{a} \otimes \boldsymbol{b})$

Observe that in our list items 1 and 2 are limiting cases of the item 4: $\Pi_\beta \to \Pi_0$, when $\beta \to 0$ and $\Pi_\beta \to \mathrm{Sym}(\mathbb{R}^2)$, when $\beta \to \infty$. We also observe that $\mathcal{A}$ is the intersection of Π_{β_1} and Π_{β_2} for any $\beta_1 \neq \beta_2$. Also $\Pi_{\boldsymbol{a},\boldsymbol{b}} = \Pi_{\boldsymbol{b}} \bigcap \Pi_{\boldsymbol{a}}^T$. Thus, we need to focus only on three subspaces from items 4, 5 and 6.

7.2 Checking sufficient conditions

According to Theorem 6.4 and our discussion above we need to check the 3 and 4-chain property of subspaces Π from items 4, 5 and 6 in the list above. Indeed, the intersection of subspaces satisfying these properties must also satisfy them. The subspaces $\Pi_{\boldsymbol{a}}$ and $\Pi_{\boldsymbol{a}}^T$ are closed with respect to associative multiplication: $\boldsymbol{K}_1\boldsymbol{A}\boldsymbol{K}_2 \in \Pi$ whenever $\{\boldsymbol{K}_1, \boldsymbol{K}_2\} \subset \Pi$. Thus the j-chain property is clearly satisfied for those subspaces. The actual checking needs to be done only for the subspaces Π_β. The checking can be easily done with Maple—the symbolic algebra package. And indeed, we find that Π_β does satisfy 3 and 4-chain properties.

7.3 Returning to $\boldsymbol{L}$ variables

Now that we have a list of subspaces Π corresponding to exact relations, we need to return to $\boldsymbol{L}$ variables. At the first glance the task before us is simply to compute

$$\mathbb{M} = \{\boldsymbol{L}_0 - [\boldsymbol{I} + \boldsymbol{K}\Gamma'(\boldsymbol{n})]^{-1}\boldsymbol{K}\boldsymbol{L}_0 \mid \boldsymbol{K} \in \Pi\}, \tag{7.16}$$

which is the inverse of W-transformation. However, we quickly realize that in order to get explicit results, the computation in (7.16) is not so easy. Fortunately, we can often simplify our job and sometimes avoid it altogether. One obvious observation that we have already made is that if we have computed two exact relations then there is no need to compute their intersection.

In [21] we identified an especially simple class of exact relations: uniform field relations (UFR). In general these are defined as

$$\mathbb{M} = \{\boldsymbol{L} \in \mathrm{End}(\mathcal{T}) \mid \boldsymbol{L}\boldsymbol{a}_1 = \boldsymbol{b}_1, \ldots, \boldsymbol{L}\boldsymbol{a}_s = \boldsymbol{b}_s\}, \tag{7.17}$$

for fixed uniform fields $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_s, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_s\} \subset \mathcal{T}$. These are easily recognizable at the level of subspaces Π.

THEOREM **7.2** *The uniform field relations (7.17) are in one-to-one correspondence with subspaces $V \subset \mathcal{T}$. The corresponding subspace Π is the annihilator of V:*

$$\Pi = \{\boldsymbol{K} \in \mathrm{End}(\mathcal{T}) \mid \boldsymbol{K}\boldsymbol{v} = \boldsymbol{0}\ \forall \boldsymbol{v} \in V\}.$$

The subspaces $\Pi_{\boldsymbol{a}}$ are annihilators of subspaces $V_{\boldsymbol{a}} = \mathbb{R}\boldsymbol{a}^{\perp}$, where $\boldsymbol{a}^{\perp} = \boldsymbol{S}\boldsymbol{a}$. Thus, the exact relations corresponding to subspaces $\Pi_{\boldsymbol{a}}$ are the sets of positive definite matrices $\boldsymbol{L}$ such that $\boldsymbol{L}\boldsymbol{u} = \boldsymbol{v}$ for fixed vectors $\{\boldsymbol{u}, \boldsymbol{v}\} \subset \mathbb{R}^2$. Obviously $\Pi_{\boldsymbol{a}}^T$ corresponds to the same class of exact relations, where $\boldsymbol{L}$ is replaced with $\boldsymbol{L}^T$.

The only exact relation here where we do need to compute something is Π_β. The following theorems were proved in [21] to facilitate our task.

THEOREM **7.3** *Fix $\boldsymbol{n} \in \mathbb{S}^2$. Let $\boldsymbol{M}$ be such that $\boldsymbol{K}(\Gamma'(\boldsymbol{n}) - \boldsymbol{M})\boldsymbol{K} \in \Pi$ for all $\boldsymbol{K} \in \Pi$. Then the invertible transformation $W_{\boldsymbol{M}} = [S(\boldsymbol{L}) - M]^{-1}$ maps $\mathbb{M}$ into Π.*

Recall that in our example of 2D Hall effect we found it easier not to work with subspaces Π and $\mathcal{A}$ directly, but rather with subspaces $\overline{\Pi} = \boldsymbol{X}\Pi\boldsymbol{Y}$ and $\overline{\mathcal{A}} = \boldsymbol{Y}^{-1}\mathcal{A}\boldsymbol{X}^{-1}$. Accordingly, we fix $\boldsymbol{n}$ and define $\overline{\Gamma} = \boldsymbol{Y}^{-1}\Gamma'(\boldsymbol{n})\boldsymbol{X}^{-1}$.

THEOREM **7.4** *Let $\boldsymbol{M}$ be such that*

$$\boldsymbol{K}(\overline{\Gamma} - \boldsymbol{M})\boldsymbol{K} \in \overline{\Pi} \tag{7.18}$$

for all $\boldsymbol{K} \in \overline{\Pi}$ then the invertible transformation $W_{\boldsymbol{M}} = [\boldsymbol{Y}^{-1}S(\boldsymbol{L})\boldsymbol{X}^{-1} - \boldsymbol{M}]^{-1}$ maps $\mathbb{M}$ into $\overline{\Pi}$.

Applying this theorem to the case at hand with $\boldsymbol{X}$ and $\boldsymbol{Y}$ defined by (7.5), we obtain the inversion formula that we will use:

$$\boldsymbol{L} = \boldsymbol{\sigma}_0 + \alpha\boldsymbol{S} - \boldsymbol{\sigma}_0^{1/2}[\boldsymbol{I} + \boldsymbol{K}\boldsymbol{M}]^{-1}\boldsymbol{K}\boldsymbol{\sigma}_0^{1/2}, \tag{7.19}$$

where $\boldsymbol{\sigma}_0$ and α are related to the reference medium $\boldsymbol{L}_0$ via (7.2). The utility of (7.19) is in the fact that it allows us to compute the manifold

$$\overline{\mathbb{M}} = \{[\boldsymbol{I} + \boldsymbol{K}\boldsymbol{M}]^{-1}\boldsymbol{K} \mid \boldsymbol{K} \in \overline{\Pi}\}$$

using simplified objects $\boldsymbol{M}$ and $\overline{\Pi}$. The actual exact relation is just an affine image of $\overline{\mathbb{M}}$.

In our example $\overline{\Gamma} = \boldsymbol{u} \otimes \boldsymbol{u}$, where $\boldsymbol{u} = \boldsymbol{\sigma}_0^{1/2}\boldsymbol{n}/|\boldsymbol{\sigma}_0^{1/2}\boldsymbol{n}|$. For $\overline{\Pi} = \mathrm{Sym}(\mathbb{R}^2)$ we easily see that $\boldsymbol{M} = \boldsymbol{0}$ satisfies (7.18). So the exact relation $\mathbb{M}$ says that if we mix materials with the same Hall coefficient r_0 then the mixture will have the same Hall coefficient r_0.

Now let us compute the exact relation corresponding to Π_β. Here $\boldsymbol{M} = \boldsymbol{0}$ does not work, so we choose $\boldsymbol{M} = \overline{\Gamma} = \boldsymbol{u} \otimes \boldsymbol{u}$. Observe that the choice $\boldsymbol{M} = \overline{\Gamma}$ always satisfies conditions of Theorem 7.4. In our simple example of 2D Hall effect the choice $\boldsymbol{M} = \overline{\Gamma}$ is still simple enough for practical purposes. In other contexts such as elasticity, thermo-electricity or other coupled problems the matrix $\overline{\Gamma}$ is not so simple, and other choices for $\boldsymbol{M}$ work (see [21] for such formidable examples as 3D thermo-piezo-electricity). Observe that the subspace Π_β is a hyperplane defined by the equation $\operatorname{Tr}(\boldsymbol{KQ}) = 0$, where $\boldsymbol{Q} = \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix}$. Solving (7.19) for $\boldsymbol{K}$, we obtain

$$\boldsymbol{K} = \left[(\boldsymbol{P} - \boldsymbol{L}')^{-1} - \boldsymbol{u} \otimes \boldsymbol{u}\right]^{-1}, \tag{7.20}$$

where $\boldsymbol{L}' = \boldsymbol{\sigma}_0^{-1/2} \boldsymbol{L} \boldsymbol{\sigma}_0^{-1/2}$, $\boldsymbol{P} = \boldsymbol{I} + \alpha' \boldsymbol{S}$ and $\alpha' = \alpha \det \boldsymbol{\sigma}_0^{-1/2}$. Since the answer does not depend on the choice of the vector $\boldsymbol{u}$, we can set $\boldsymbol{u} = (1, 0)$ and simplify the equation $\operatorname{Tr}(\boldsymbol{KQ}) = 0$ with Maple. The result is written most conveniently in terms of the conductivity tensor $\boldsymbol{\sigma}$ and Hall coefficient r, so that $\boldsymbol{L} = \boldsymbol{\sigma} + r\boldsymbol{S}$:

$$\det \boldsymbol{\sigma} + (r - \alpha - \beta\sqrt{\det \boldsymbol{\sigma}_0})^2 = (1 + \beta^2) \det \boldsymbol{\sigma}_0. \tag{7.21}$$

The exact relation we obtain can be written most concisely as

$$\mathbb{M} = \{(\boldsymbol{\sigma}, r) \mid \det \boldsymbol{\sigma} + (r - r_0)^2 = \text{const}\}. \tag{7.22}$$

This relation was first derived by Milton [37] (see also [14]). When $\beta \to 0$ the exact relation (7.21) still retains the form (7.22). If $\beta \to \infty$ the exact relation (7.21) becomes $r = r_0$, first derived by Stroud and Bergman [47]. We note that the intersection of exact relations (7.22) and $r = 0$ results in the well-known Keller-Dykhne-Mendelson exact relation [13, 26, 36].

7.4 Exact relations with volume fractions

Very often exact relations are supplemented by other relations involving volume averages. For example, Hill's exact relation for elasticity mentioned in the Introduction has an extra part:

$$(3\kappa^* + 4\mu)^{-1} = \langle (3\kappa(\boldsymbol{x}) + 4\mu)^{-1} \rangle, \tag{7.23}$$

where μ is the common shear modulus, $\kappa(\boldsymbol{x})$ and κ^* are the local and effective bulk moduli respectively. In order to obtain these additional relations we need to compute the *derived Jordan ideals* for each subspace Π satisfying (7.1).

Definition 3 *The derived Jordan ideal of the solution* Π *of (7.1) is the subspace*

$$\Pi^2 = \operatorname{Span}\{\boldsymbol{K}_1 *_{\boldsymbol{A}} \boldsymbol{K}_2 \mid \{\boldsymbol{K}_1, \boldsymbol{K}_2\} \subset \Pi,\ \boldsymbol{A} \in \mathcal{A}\},$$

where the Jordan multiplication $*_{\boldsymbol{A}}$ *is defined in (6.1).*

THEOREM **7.5** *Let* Π^2 *be the derived Jordan ideal of* Π *and suppose* $\Pi^2 \neq \Pi$. *Let* $\mathcal{N}$ *be the orthogonal complement of* Π^2 *in* Π. *Then we have*

$$\mathcal{P}_{\mathcal{N}} W_{\boldsymbol{M}}(\boldsymbol{L}^*) = \mathcal{P}_{\mathcal{N}} \langle W_{\boldsymbol{M}}(\boldsymbol{L}(\boldsymbol{x}))\rangle, \tag{7.24}$$

where $\mathcal{P}_{\mathcal{N}}$ *denotes the orthogonal projection onto* $\mathcal{N}$ *and* $\boldsymbol{M}$ *satisfies conditions of Theorem 7.3.*

We have actually computed the derived Jordan ideals for each subspace Π when we were computing all the solutions of the equation (7.1). The reader may go back and verify that for each solution Π for the 2D Hall effect we have $\Pi^2 = \Pi$. The simplest context where nontrivial derived Jordan ideals appear is 2D elasticity, which is a bit more involved, and therefore less suitable for the purposes of the present review than 2D Hall effect. We refer the reader to [21], where there are plenty of exact relations with nontrivial derived Jordan ideals and corresponding volume average relations.

7.5 Links between uncoupled problems

Now we can ask the following question: Is there a link between conducting and thermal properties of a composite? More generally, is there a link between $\boldsymbol{L}^*$ and $\boldsymbol{F}^*$—the effective tensor for $\boldsymbol{f}(\boldsymbol{L}(\boldsymbol{x}))$. This question has also been investigated in [21], where $\boldsymbol{f}$ is some nonlinear map between $\mathrm{End}(\mathcal{T}_1)$ and $\mathrm{End}(\mathcal{T}_2)$. The answer uses the concept of Jordan ideal.

Definition 4 *A subspace* $\mathcal{K}$ *of a solution* Π *of (7.1) is called a Jordan ideal in* Π *if* $\boldsymbol{K}_1 *_{\boldsymbol{A}} \boldsymbol{K}_2 \in \mathcal{K}$ *for all* $\boldsymbol{A} \in \mathcal{A}$, *all* $\boldsymbol{K}_1 \in \mathcal{K}$ *and all* $\boldsymbol{K}_2 \in \Pi$.

For example, the derived Jordan ideal is a Jordan ideal. As is customary in algebra, an ideal is good for factoring over it. Let $\mathcal{F} = \Pi/\mathcal{K}$ be the factor space in the sense of vector spaces. Then $\mathcal{F}$ has a set of well-defined Jordan multiplications:

$$\overline{\boldsymbol{K}_1} *_{\boldsymbol{A}} \overline{\boldsymbol{K}_2} \stackrel{\mathrm{def}}{=} \overline{\boldsymbol{K}_1 *_{\boldsymbol{A}} \boldsymbol{K}_2},$$

for all $\boldsymbol{A} \in \mathcal{A}$ and all $\{\boldsymbol{K}_1, \boldsymbol{K}_2\} \subset \Pi$, where $\overline{\boldsymbol{K}} \in \mathcal{F}$ denotes the equivalence class of $\boldsymbol{K}$.

In order to treat the links between uncoupled problems we need to discuss coupled problems, at least in passing. As we have argued in Section 3, each physical context gives rise to a vector space $\mathcal{T}$ where the pair of relevant physical fields take their values. If we have two problems with spaces $\mathcal{T}_1$ and $\mathcal{T}_2$ then the vector space $\mathcal{T}$ corresponding to a coupled problem is $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$. Exact relations for the coupled problem would correspond to subspaces of $\mathrm{End}(\mathcal{T}_1 \oplus \mathcal{T}_2)$.

Definition 5 *The links between uncoupled problems are those exact relations for the coupled problem whose subspaces* Π *lie in the "block-diagonal" part* $\mathrm{End}(\mathcal{T}_1) \oplus \mathrm{End}(\mathcal{T}_2)$ *of* $\mathrm{End}(\mathcal{T}_1 \oplus \mathcal{T}_2)$.

The technical difficulty is that in general $\mathcal{A} \neq \mathcal{A}_1 \oplus \mathcal{A}_2$. In [21] we were able to avoid this difficulty by focusing on polycrystals, where we were able to replace the subspace $\mathcal{A}$ by a single matrix $\tilde{\Gamma}$. Nevertheless, we can still state the general theorem.

In what follows we use notation $[\boldsymbol{A}, \boldsymbol{B}]$ for $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{bmatrix}$ in order to save space.

THEOREM **7.6** *Let* $\Pi \subset \mathrm{End}(\mathcal{T}_1) \oplus \mathrm{End}(\mathcal{T}_2)$ *be an exact relation. Define*

$$\begin{aligned}
\Pi_1 &= \{\boldsymbol{K}_1 \in \mathrm{End}(\mathcal{T}_1) : [\boldsymbol{K}_1, \boldsymbol{K}_2] \in \Pi \text{ for some } \boldsymbol{K}_2 \in \mathrm{End}(\mathcal{T}_2)\},\\
\Pi_2 &= \{\boldsymbol{K}_2 \in \mathrm{End}(\mathcal{T}_2) : [\boldsymbol{K}_1, \boldsymbol{K}_2] \in \Pi \text{ for some } \boldsymbol{K}_1 \in \mathrm{End}(\mathcal{T}_1)\}.\\
\mathcal{K}_1 &= \{\boldsymbol{K}_1 \in \mathrm{End}(\mathcal{T}_1) : [\boldsymbol{K}_1, \boldsymbol{0}] \in \Pi\},\\
\mathcal{K}_2 &= \{\boldsymbol{K}_2 \in \mathrm{End}(\mathcal{T}_2) : [\boldsymbol{0}, \boldsymbol{K}_2] \in \Pi\}.
\end{aligned} \tag{7.25}$$

Then the subspaces Π_j *of* $\mathrm{End}(\mathcal{T}_j)$, $j = 1, 2$ *are Jordan algebras in the sense of equation (7.1). The subspaces* $\mathcal{K}_j \subset \Pi_j$, $j = 1, 2$ *are Jordan ideals. There is a natural linear isomorphism* $\Phi : \mathcal{F}_1 = \Pi_1/\mathcal{K}_1 \to \Pi_2/\mathcal{K}_2 = \mathcal{F}_2$ *defined by the rule* $\overline{\boldsymbol{K}_2} = \Phi(\overline{\boldsymbol{K}_1})$, *whenever* $[\boldsymbol{K}_1, \boldsymbol{K}_2] \in \Pi$. *The map* Φ *is well-defined and satisfies an important additional condition:*

$$\Phi(\overline{\boldsymbol{K}_1} *_{\boldsymbol{A}_1} \overline{\boldsymbol{K}_1'}) = \Phi(\overline{\boldsymbol{K}_1}) *_{\boldsymbol{A}_2} \Phi(\overline{\boldsymbol{K}_1'}) \tag{7.26}$$

for every $\{\boldsymbol{K}_1, \boldsymbol{K}_1'\} \subset \Pi_1$ *and every* $[\boldsymbol{A}_1, \boldsymbol{A}_2] \in \mathcal{A}$.

Please, note that if $[\boldsymbol{A}_1, \boldsymbol{A}_2] \in \mathcal{A}$ then, in general, $\boldsymbol{A}_2$ is not determined by $\boldsymbol{A}_1$ uniquely, nor $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are independent. Therefore, the linear isomorphism Φ with the property (7.26) is Jordan algebras homomorphism.

For the specific example of 2D Hall effect we simply used the brute force Maple computation to figure out the subspace $\mathcal{A}$. Let $\boldsymbol{L}_0 = [\boldsymbol{L}_1, \boldsymbol{L}_2]$ be the reference medium through which the link $\mathbb{M}$ passes. As before it will be convenient to work not with the subspace $\mathcal{A}$ directly but with the subspace $\overline{\mathcal{A}} = \boldsymbol{Y}^{-1}\mathcal{A}\boldsymbol{X}^{-1}$, where $\boldsymbol{X} = [\boldsymbol{\sigma}_1^{-1/2}, \boldsymbol{\sigma}_2^{-1/2}]$ and $\boldsymbol{Y} = [\boldsymbol{L}_1, \boldsymbol{L}_2]\boldsymbol{X}$. Then $\overline{\mathcal{A}}$ is described in terms of $\mathcal{A}_0 = \{\boldsymbol{A} \in \mathrm{Sym}(\mathbb{R}^2) \mid \mathrm{Tr}\,(\boldsymbol{A}) = 0\}$, depending on what $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are.

THEOREM **7.7** *Let* $\boldsymbol{\sigma}_j$, $j = 1, 2$ *be the symmetric parts of* $\boldsymbol{L}_j$. *If there is a scalar* $s > 0$ *such that* $\boldsymbol{\sigma}_2 = s\boldsymbol{\sigma}_1$ *then* $\overline{\mathcal{A}} = \{[\boldsymbol{A}, \boldsymbol{A}] : \boldsymbol{A} \in \mathcal{A}_0\}$. *Otherwise,* $\overline{\mathcal{A}} = \mathcal{A}_0 \oplus \mathcal{A}_0$.

Now that we know the subspace $\overline{\mathcal{A}}$, we can apply Theorem 7.6. First consider the case when $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are not multiples of one another. Then we can set $\boldsymbol{A}_2 = \boldsymbol{0}$ in (7.26) and, recalling that Φ is a bijection, conclude that for every $\{\boldsymbol{K}_1, \boldsymbol{K}_1'\} \subset \Pi_1$ and every $\boldsymbol{A}_1 \in \mathcal{A}_0$ we have $\overline{\boldsymbol{K}_1} *_{\boldsymbol{A}_1} \overline{\boldsymbol{K}_1'} = \overline{\boldsymbol{0}}$. The meaning of that last relation is that the ideal $\mathcal{K}_1$ contains the derived ideal Π_1^2. But we have already verified that there are no non-trivial derived ideals in the context of 2D Hall effect. In this case we have to conclude that $\mathcal{K}_1 = \Pi_1$ resulting in the trivial link: $[\boldsymbol{L}_1(\boldsymbol{x}), \boldsymbol{L}_2(\boldsymbol{x})]^* = [\boldsymbol{L}_1^*, \boldsymbol{L}_2^*]$ with no relation between $\boldsymbol{L}_1^*$ and $\boldsymbol{L}_2^*$.

Now let us assume that $\boldsymbol{\sigma}_2 = s\boldsymbol{\sigma}_1$. Here we will get some interesting results. We begin by finding all pairs of nested exact relations from the list in Theorem 7.1 (to what the space $\mathrm{End}(\mathbb{R}^2)$ is added) and checking if the smaller subspace in each pair is a proper Jordan ideal of the larger. The checking is routine and may be automated with Maple even for problems of large size. We find that none of the subspaces from Theorem 7.1 is a Jordan ideal in any of the other subspaces, including $\mathrm{End}(\mathbb{R}^2)$. Thus, we must necessarily have that $\mathcal{K}_1$ and $\mathcal{K}_2$ are zero subspaces and that Φ is a Jordan isomorphism $\Phi : \Pi_1 \to \Pi_2$

$$\Phi(\boldsymbol{K}_1 *_{\boldsymbol{A}} \boldsymbol{K}_2) = \Phi(\boldsymbol{K}_1) *_{\boldsymbol{A}} \Phi(\boldsymbol{K}_2) \tag{7.27}$$

for all $\boldsymbol{A} \in \mathcal{A}_0$.

We will now describe how to find all the Jordan isomorphisms Φ between the subspaces Π_1 and Π_2 satisfying (7.1). The method here is based on a nice relation between 2×2 matrix algebra and complex arithmetic. To a complex number $z = a + ib$ we associate a vector $\pi(z) = (a, b) \in \mathbb{R}^2$ and two 2×2 matrices:

$$\phi(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ and } \psi(z) = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}.$$

In the remaining part of this section the bold lower case letter will denote a 2D vector corresponding to a complex number denoted by the same non-bold letter. In situations where such simplified notation is inadequate we will use notation $\pi(\cdot)$ defined above.

The functions ϕ and ψ enjoy many special properties. For example,

$$\phi(z)\boldsymbol{u} = \pi(zu), \quad \psi(z)\boldsymbol{u} = \pi(z\overline{u}).$$

As a corollary we have the following multiplicative identities:

$$\phi(z_1)\phi(z_2) = \phi(z_1 z_2),\ \phi(z_1)\psi(z_2) = \psi(z_1 z_2),\ \psi(z_1)\phi(z_2) = \psi(z_1\overline{z_2}),\ \psi(z_1)\psi(z_2) = \phi(z_1\overline{z_2}).$$

Observe that if Φ is the Jordan isomorphism between Π_1 and Π_2 then $\dim \Pi_1 = \dim \Pi_2$. Therefore, our strategy is to go through every dimension 1 through 4 and determine all Jordan isomorphisms between subspaces Π from Theorem 7.1 of that dimension.

Dimension 1. All 1D solutions of (7.1) have the form $\Pi_{\boldsymbol{a},\boldsymbol{b}}$.

PROPOSITION **1** *The linear mapping $\Phi : \Pi_{\boldsymbol{a},\boldsymbol{b}} \to \Pi_{\boldsymbol{c},\boldsymbol{d}}$, defined by $\Phi(\boldsymbol{a} \otimes \boldsymbol{b}) = \boldsymbol{c} \otimes \boldsymbol{d}$, is a Jordan isomorphism between $\Pi_{\boldsymbol{a},\boldsymbol{b}}$ and $\Pi_{\boldsymbol{c},\boldsymbol{d}}$ if and only if $ab = cd$.*

PROOF: For every $\boldsymbol{A} \in \mathcal{A}_0$ we have

$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{A}(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{A}\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{a} \otimes \boldsymbol{b}.$$

Applying the mapping Φ to this relation and using the property (7.27) we obtain

$$(\boldsymbol{c} \otimes \boldsymbol{d})\boldsymbol{A}(\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{A}\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \otimes \boldsymbol{d}.$$

Thus, we must have

$$\boldsymbol{A}\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{A}\boldsymbol{c} \cdot \boldsymbol{d} \tag{7.28}$$

for every $\boldsymbol{A} \in \mathcal{A}_0$. Now observe that $\mathcal{A}_0 = \{\psi(z) \mid z \in \mathbb{C}\}$. Therefore equation (7.28) becomes $\Re(\overline{z}(ab - cd)) = 0$ for every $z \in \mathbb{C}$. Thus, Φ is a Jordan isomorphism if and only if $ab = cd$. ∎

Dimension 2. There are three classes of 2D solutions of (7.1). We denoted them $\mathcal{A}_0$, $\Pi_{\boldsymbol{a}}$ and $\Pi_{\boldsymbol{a}}^T$.

PROPOSITION **2** *There are no Jordan isomorphisms between $\mathcal{A}_0$ and $\Pi_{\boldsymbol{a}}$ and between $\mathcal{A}_0$ and $\Pi_{\boldsymbol{a}}^T$. The only Jordan isomorphism between $\mathcal{A}_0$ and $\mathcal{A}_0$ is the identity mapping.*

PROOF: Suppose Φ is the Jordan isomorphism between $\mathcal{A}_0$ and $\Pi_{\boldsymbol{a}}$. Then there exists a non-singular matrix $\boldsymbol{V}$ such that $\Phi(\psi(u)) = \boldsymbol{V}\boldsymbol{u} \otimes \boldsymbol{a}$ for all $u \in \mathbb{C}$. Applying the Jordan isomorphism Φ to

$$\psi(u)\psi(z)\psi(u) = \psi(u^2\overline{z}), \tag{7.29}$$

we obtain

$$(\boldsymbol{V}\boldsymbol{u} \otimes \boldsymbol{a})\psi(z)(\boldsymbol{V}\boldsymbol{u} \otimes \boldsymbol{a}) = \boldsymbol{V}\pi(u^2\overline{z}) \otimes \boldsymbol{a}.$$

Therefore $(\psi(z)\boldsymbol{a}\cdot\boldsymbol{V}\boldsymbol{u})u = u^2\overline{z}$, which implies that $u\overline{z}$ must be real for all $\{u, z\} \subset \mathbb{C}$. This is impossible, and therefore, the Jordan isomorphism Φ between $\mathcal{A}_0$ and $\Pi_{\boldsymbol{a}}$ does not exist. The Jordan isomorphism between $\mathcal{A}_0$ and $\Pi_{\boldsymbol{a}}^T$ is also impossible. Indeed the mapping $\boldsymbol{K} \rightarrow \boldsymbol{K}^T$ is the Jordan isomorphism between $\Pi_{\boldsymbol{a}}$ and $\Pi_{\boldsymbol{a}}^T$. If there was a Jordan isomorphism between $\mathcal{A}_0$ and $\Pi_{\boldsymbol{a}}^T$ then there would be a Jordan isomorphism between $\mathcal{A}_0$ and $\Pi_{\boldsymbol{a}}$, which, we proved, does not exist.

Now let Φ be the Jordan isomorphism from $\mathcal{A}_0$ to itself. Then there exists a real-linear invertible mapping $\vartheta : \mathbb{C} \rightarrow \mathbb{C}$ such that $\Phi(\psi(u)) = \psi(\vartheta(u))$. Applying Φ to (7.29) we get $(\vartheta(u))^2\overline{z} = \vartheta(u^2\overline{z})$. The linear map ϑ has the form $\vartheta(u) = \theta_1 u + \theta_2\overline{u}$ for some complex numbers θ_1 and θ_2. Equating the coefficients at z and $\overline{z}$ in

$$(\theta_1 u + \theta_2\overline{u})^2\overline{z} = \theta_1 u^2\overline{z} + \theta_2\overline{u}^2 z,$$

we obtain that $\theta_2 = 0$ and $\theta_1 = 1$. Thus, $\Phi(\psi(u)) = \psi(u)$ for all $u \in \mathbb{C}$. ∎

PROPOSITION **3** *The unique Jordan isomorphism Φ between $\Pi_{\boldsymbol{a}}$ and $\Pi_{\boldsymbol{b}}$ is given by*

$$\Phi(\boldsymbol{v} \otimes \boldsymbol{a}) = \phi(a/b)\boldsymbol{v} \otimes \boldsymbol{b}. \tag{7.30}$$

The unique Jordan isomorphism Φ between $\Pi_{\boldsymbol{a}}$ and $\Pi_{\boldsymbol{b}}^T$ is given by

$$\Phi(\boldsymbol{v} \otimes \boldsymbol{a}) = \boldsymbol{b} \otimes \phi(a/b)\boldsymbol{v}. \tag{7.31}$$

The unique Jordan isomorphism Φ between $\Pi_{\boldsymbol{a}}^T$ and $\Pi_{\boldsymbol{b}}^T$ is given by

$$\Phi(\boldsymbol{a} \otimes \boldsymbol{v}) = \boldsymbol{b} \otimes \phi(a/b)\boldsymbol{v}. \tag{7.32}$$

PROOF: For a linear isomorphism Φ between $\Pi_{\boldsymbol{a}}$ and $\Pi_{\boldsymbol{b}}$ there exists a non-singular matrix $\boldsymbol{V}$ such that $\Phi(\boldsymbol{v} \otimes \boldsymbol{a}) = \boldsymbol{V}\boldsymbol{v} \otimes \boldsymbol{b}$. Applying Φ to

$$(\boldsymbol{v} \otimes \boldsymbol{a})\psi(z)(\boldsymbol{v} \otimes \boldsymbol{a}) = (\psi(z)\boldsymbol{a} \cdot \boldsymbol{v})\boldsymbol{v} \otimes \boldsymbol{a},$$

we obtain

$$(\boldsymbol{V}\boldsymbol{v} \otimes \boldsymbol{b})\psi(z)(\boldsymbol{V}\boldsymbol{v} \otimes \boldsymbol{b}) = (\psi(z)\boldsymbol{a} \cdot \boldsymbol{v})\boldsymbol{V}\boldsymbol{v} \otimes \boldsymbol{b}.$$

Therefore, for every $z \in \mathbb{C}$ we have $\boldsymbol{V}^T \psi(z)\boldsymbol{b} = \psi(z)\boldsymbol{a}$. The map $\boldsymbol{V}$ as a map on $\mathbb{C}$ can be written as follows

$$\pi^{-1}(\boldsymbol{V}\boldsymbol{u}) = v_1 u + v_2 \overline{u}$$

for some complex numbers v_1 and v_2. Therefore, we have

$$\overline{v_1} z\overline{b} + v_2 \overline{z} b = z\overline{a}.$$

Equating coefficients at z and $\overline{z}$ we obtain that $v_2 = 0$ and $v_1 = a/b$. Thus, (7.30) gives the unique Jordan isomorphism between $\Pi_{\boldsymbol{a}}$ and $\Pi_{\boldsymbol{b}}$.

Now, if Φ is a Jordan isomorphism between $\Pi_{\boldsymbol{a}}$ and $\Pi_{\boldsymbol{b}}^T$ then $\tau \circ \Phi$ is the Jordan isomorphism between $\Pi_{\boldsymbol{a}}$ and $\Pi_{\boldsymbol{b}}$, where $\tau(\boldsymbol{K}) = \boldsymbol{K}^T$. Hence there is a unique Jordan isomorphism between $\Pi_{\boldsymbol{a}}$ and $\Pi_{\boldsymbol{b}}^T$ given by (7.31). Similarly, if Φ is a Jordan isomorphism between $\Pi_{\boldsymbol{a}}^T$ and $\Pi_{\boldsymbol{b}}^T$, then $\tau \circ \Phi \circ \tau$ is a Jordan isomorphism between $\Pi_{\boldsymbol{a}}$ and $\Pi_{\boldsymbol{b}}$. ∎

Dimension 3. The 3D solutions of (7.1) are the subspaces Π_β from Theorem 7.1.

PROPOSITION 4 *For any pair $\{\beta_1, \beta_2\} \subset \mathbb{R} \cup \{\infty\}$ there are two Jordan isomorphisms between Π_{β_1} and Π_{β_2} given by*

$$\Phi(\boldsymbol{K}) = \phi(e^{i\alpha/2})\boldsymbol{K}\phi(e^{i\alpha/2}), \text{ and } \Phi(\boldsymbol{K}) = \phi(ie^{i\alpha/2})\boldsymbol{K}\phi(ie^{i\alpha/2}), \tag{7.33}$$

where α is the unique solution in $[0, \pi)$ of

$$\tan(\alpha) = \frac{\beta_2 - \beta_1}{1 + \beta_1\beta_2}.$$

PROOF: Suppose Φ is the Jordan isomorphism between Π_{β_1} and Π_{β_2}. Observe that $\mathcal{A}_0 \subset \Pi_\beta$ for any β. We conclude that $\Phi(\mathcal{A}_0) = \mathcal{A}_0$, since $\mathcal{A}_0$ is not isomorphic to any other Jordan algebra but itself. Moreover, the restriction of Φ to $\mathcal{A}_0$ must be the identity map. Observe that any matrix $\boldsymbol{K} \in \Pi_{\beta_1}$ can be written as $\boldsymbol{K} = x\phi(e^{i\gamma_1}) + \psi(u)$, where $\beta_1 = -\cot\gamma_1$. Any linear map Φ from Π_{β_1} into Π_{β_2} that is identity on $\mathcal{A}_0$ has the form

$$\Phi(x\phi(e^{i\gamma_1}) + \psi(u)) = ax\phi(e^{i\gamma_2}) + \psi(u + xp).$$

for some $a \in \mathbb{R}$ and $p \in \mathbb{C}$. Here $\beta_2 = -\cot\gamma_2$. Applying the Jordan isomorphism Φ to

$$(x\phi(e^{i\gamma_1}) + \psi(u))\psi(z)(x\phi(e^{i\gamma_1}) + \psi(u)) = 2x\Re(z\overline{u})\phi(e^{i\gamma_1}) + \psi(x^2 z + u^2\overline{z}),$$

we obtain

$$(ax\phi(e^{i\gamma_2})+\psi(u+xp))\psi(z)(ax\phi(e^{i\gamma_2})+\psi(u+xp)) = 2xa\Re(z\overline{u})\phi(e^{i\gamma_2})+\psi(x^2z+u^2\overline{z}+2x\Re(z\overline{u})p).$$

Since any 2×2 matrix can be uniquely written as $\phi(z_1) + \psi(z_2)$ we obtain, by equating arguments of ϕ,

$$\Re(z\overline{u}) = \Re(z\overline{u} + xz\overline{p}).$$

It follows, therefore, that $p = 0$. Equating arguments of ψ we get

$$x^2 z + u^2\overline{z} = a^2x^2 z + u^2\overline{z}.$$

It follows that either $a = 1$ or $a = -1$, corresponding to the two Jordan isomorphisms given by (7.33). ∎

Dimension 4.

THEOREM **7.8** *If Φ is a Jordan isomorphism of* $\mathrm{End}(\mathbb{R}^2)$*, then either* $\Phi(\boldsymbol{K}) = \phi(e^{i\alpha})\boldsymbol{K}\phi(e^{i\alpha})$ *or* $\Phi(\boldsymbol{K}) = \phi(e^{i\alpha})\boldsymbol{K}^T\phi(e^{i\alpha})$*. Moreover, all Jordan isomorphisms between* Π_1 *and* Π_2*—solutions of (7.1) are restrictions of the Jordan isomorphisms of* $\mathrm{End}(\mathbb{R}^2)$ *to* Π_1*.*

Before proving the theorem. We remark that all Jordan isomorphisms Φ of $\mathrm{End}(\mathbb{R}^2)$ have the property

$$\Phi(\boldsymbol{K}_1\boldsymbol{A}\boldsymbol{K}_2) = \Phi(\boldsymbol{K}_1)\boldsymbol{A}\Phi(\boldsymbol{K}_2),$$

for all $\boldsymbol{A} \in \mathcal{A}_0$. Therefore, all the j-chain properties are obviously satisfied. Thus, all the links that we find are stable under homogenization.

PROOF: Let Φ be the Jordan isomorphism of $\mathrm{End}(\mathbb{R}^2)$. If we restrict Φ to a 3D subspace $\{x\phi(e^{i\gamma}) + \psi(u) \mid x \in \mathbb{R},\ u \in \mathbb{C}\}$ then we have, according to Proposition 4

$$\Phi(x\phi(e^{i\gamma}) + \psi(u)) = x\phi(e^{i\alpha}e^{i\gamma}) + \psi(u).$$

Therefore, for every $\gamma \in [0, \pi)$, every $x \in \mathbb{R}$ and $u \in \mathbb{C}$

$$\Phi(x\phi(e^{i\gamma}) + \psi(u)) = x\phi(e^{i\alpha(\gamma)}e^{i\gamma}) + \psi(u).$$

But Φ is linear and therefore there are complex numbers p_1 and p_2 such that $\Phi(\phi(z)) = \phi(p_1 z + p_2\overline{z})$. Consequently, $e^{i\alpha(\gamma)}e^{i\gamma} = p_1e^{i\gamma} + p_2e^{-i\gamma}$. Thus, for every $\gamma \in [0, \pi)$ we have $|p_1 + p_2e^{-2i\gamma}| = 1$. This equation says geometrically that the circle centered at p_1 with radius $|p_2|$ is a subset of the unit circle. There are only two possibilities: $p_1 = 0$, $|p_2| = 1$ or $p_2 = 0$, $|p_1| = 1$. Thus, either $\Phi(\phi(v) + \psi(u)) = \phi(e^{i\alpha}v) + \psi(u)$ or $\Phi(\phi(v) + \psi(u)) = \phi(e^{i\alpha}\overline{v}) + \psi(u)$. In the matrix notation, either $\Phi(\boldsymbol{K}) = \phi(e^{i\alpha/2})\boldsymbol{K}\phi(e^{i\alpha/2})$, or $\Phi(\boldsymbol{K}) = \phi(e^{i\alpha/2})\boldsymbol{K}^T\phi(e^{i\alpha/2})$, and the first part of the theorem is proved.

We have already described all the Jordan isomorphisms between solutions of (7.1). It is now a simple matter to check that each one of the Jordan isomorphisms described in Propositions 1–4 is indeed a restriction of one of the Jordan isomorphisms of $\mathrm{End}(\mathbb{R}^2)$. ∎

We remark that the group of all Jordan isomorphisms of $\mathrm{End}(\mathbb{R}^2)$ is isomorphic to $O(2)$. Another remark is that Theorem 7.8 saves us a lot of work. All we have to do is to compute the links between uncoupled problems for $\mathrm{End}(\mathbb{R}^2)$. Any other link is just a restriction of the global link to one of the exact relations computed before. We also observe that the Jordan isomorphism $\tau(\boldsymbol{K}) = \boldsymbol{K}^T$ corresponds to the link $(\boldsymbol{L}(\boldsymbol{x})^T)^* = (\boldsymbol{L}^*)^T$, which is a theorem of Murat and Tartar [40] that says that if $\boldsymbol{L}^\epsilon(\boldsymbol{x})$ H-converges to $\boldsymbol{L}^*(\boldsymbol{x})$ then $(\boldsymbol{L}^\epsilon(\boldsymbol{x}))^T$ H-converges to $(\boldsymbol{L}^*(\boldsymbol{x}))^T$. If we combine this link with the links corresponding to $\Phi(\boldsymbol{K}) = \phi(e^{i\alpha})\boldsymbol{K}\phi(e^{i\alpha})$ we will obtain the links corresponding to $\Phi(\boldsymbol{K}) = \phi(e^{i\alpha})\boldsymbol{K}^T\phi(e^{i\alpha})$. All we have to do now is to apply the method of Section 7.3 to the subspace

$$\Pi = \{[\boldsymbol{K}, \phi(e^{i\alpha})\boldsymbol{K}\phi(e^{i\alpha})] :\ \boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2)\}.$$

According to the formula (7.20) we define $\boldsymbol{L}_1' = \boldsymbol{\sigma}_1^{-1/2}\boldsymbol{L}_1\boldsymbol{\sigma}_1^{-1/2}$, $\boldsymbol{P}_1 = \boldsymbol{I} + \alpha_1\boldsymbol{S}$ and $\boldsymbol{L}_2' = (1/s)\boldsymbol{\sigma}_1^{-1/2}\boldsymbol{L}_2\boldsymbol{\sigma}_1^{-1/2}$, $\boldsymbol{P}_2 = \boldsymbol{I} + \alpha_2\boldsymbol{S}$. Then the link between $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ has the form:

$$\left[(\boldsymbol{P}_2 - \boldsymbol{L}_2')^{-1} - \boldsymbol{u}\otimes\boldsymbol{u}\right]^{-1} = \phi(e^{i\alpha})\left[(\boldsymbol{P}_1 - \boldsymbol{L}_1')^{-1} - \boldsymbol{u}\otimes\boldsymbol{u}\right]^{-1}\phi(e^{i\alpha}). \tag{7.34}$$

The relation between $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ does not depend on the choice of the unit vector $\boldsymbol{u}$. Indeed, we can rewrite (7.34) as

$$(\boldsymbol{P}_2 - \boldsymbol{L}_2')^{-1} = \phi(e^{-i\alpha})(\boldsymbol{P}_1 - \boldsymbol{L}_1')^{-1}\phi(e^{-i\alpha}) + \boldsymbol{u}\otimes\boldsymbol{u} - \phi(e^{-i\alpha})\boldsymbol{u}\otimes\phi(e^{i\alpha})\boldsymbol{u}.$$

Now we use our complex calculus to show that $\boldsymbol{u}\otimes\boldsymbol{u} - \phi(e^{-i\alpha})\boldsymbol{u}\otimes\phi(e^{i\alpha})\boldsymbol{u}$ does not depend on the choice of the unit vector $\boldsymbol{u}$. Indeed, a rank-1 matrix $\boldsymbol{a}\otimes\boldsymbol{b}$ can be written as $(\phi(a\bar{b}) + \psi(ab))/2$. Thus, we have

$$\boldsymbol{u}\otimes\boldsymbol{u} - \phi(e^{-i\alpha})\boldsymbol{u}\otimes\phi(e^{i\alpha})\boldsymbol{u} = (\phi(1) + \psi(u^2))/2 - (\phi(e^{-i\alpha}u\overline{e^{i\alpha}u}) + \psi(e^{-i\alpha}ue^{i\alpha}u))/2,$$

and therefore

$$\boldsymbol{L}_2' = \boldsymbol{P}_2 - \left[\phi(e^{-i\alpha})(\boldsymbol{P}_1 - \boldsymbol{L}_1')^{-1}\phi(e^{-i\alpha}) + \sin(\alpha)\phi(ie^{-i\alpha})\right]^{-1}. \tag{7.35}$$

We examine the right hand side of (7.35) with Maple and after a lengthy investigation obtain the link in terms of the conductivity tensors $\boldsymbol{\sigma}$ and the Hall coefficient r:

$$\boldsymbol{\sigma}_2 = c_0\frac{\boldsymbol{\sigma}_1}{(r_0 - r_1)^2 + \det\boldsymbol{\sigma}_1}, \quad r_2 = c_0\frac{r_0 - r_1}{(r_0 - r_1)^2 + \det\boldsymbol{\sigma}_1} + q_0, \tag{7.36}$$

where $c_0 > 0$, r_0 and q_0 are constants. This link was first obtained by Dykhne [12] in a particular case of an isotropic composite made with two isotropic phases. In its present form the link was first derived by Milton [37]. The links corresponding the the remaining Jordan isomorphisms can be obtained by combining (7.36) with the link $\boldsymbol{L}_2 = \boldsymbol{L}_1^T$. In other words, the other set of links has the form (7.36) but with r_1 replaced by $-r_1$:

$$\boldsymbol{\sigma}_2 = c_0\frac{\boldsymbol{\sigma}_1}{(r_0 + r_1)^2 + \det\boldsymbol{\sigma}_1}, \quad r_2 = c_0\frac{r_0 + r_1}{(r_0 + r_1)^2 + \det\boldsymbol{\sigma}_1} + q_0, \tag{7.37}$$

Following Milton [37], we observe that the trivial exact relation "$r_1(\boldsymbol{x}) = 0$ implies $r_1^* = 0$" transforms into the relation (7.22) according to the link (7.36) (or (7.37)).

Restricting the links to the case of conductivity we obtain the well-known result [36, 39]

$$(\boldsymbol{\sigma}(\boldsymbol{x})/\det\boldsymbol{\sigma}(\boldsymbol{x}))^* = \boldsymbol{\sigma}^*/\det\boldsymbol{\sigma}^*.$$

It is possible to play further with the links and exact relations, applying them to the case of two-phase composites, polycrystals and such. We will not pursue this and stop here.

8 Conclusions

Our theory of exact relations permitted us to derive the following *microstructure independent* results for effective properties of composites in the context of 2D Hall effect. All of them were previously known (see [12, 14, 37, 47]). In a weak magnetic field the local conducting properties of a 2D composite are described by the non-symmetric positive definite 2×2 matrix field $\boldsymbol{L}(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x}) + r(\boldsymbol{x})\boldsymbol{S}$. We have obtained the following *complete* list of results.

1. If $\boldsymbol{L}(\boldsymbol{x})\boldsymbol{a} = \boldsymbol{b}$ for all $\boldsymbol{x} \in Q$ then $\boldsymbol{L}^*\boldsymbol{a} = \boldsymbol{b}$.

2. If $r(\boldsymbol{x}) = r_0$ then $r^* = r_0$.

3. If $\boldsymbol{L}^*$ is the effective tensor for $\boldsymbol{L}(\boldsymbol{x})$ then $(\boldsymbol{L}^*)^T$ is the effective tensor for $\boldsymbol{L}(\boldsymbol{x})^T$.

4. If $(r(\boldsymbol{x}) - r_0)^2 + \det \boldsymbol{\sigma}(\boldsymbol{x}) = c_0$ for all $\boldsymbol{x} \in Q$ then $(r^* - r_0)^2 + \det \boldsymbol{\sigma}^* = c_0$.

5. If $\boldsymbol{L}^* = \boldsymbol{\sigma}^* + r^*\boldsymbol{S}$ is the effective tensor for $\boldsymbol{L}(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x}) + r(\boldsymbol{x})\boldsymbol{S}$ then $\widehat{\boldsymbol{L}}^* = \widehat{\boldsymbol{\sigma}}^* + \widehat{r}^*\boldsymbol{S}$ is the effective tensor for $\widehat{\boldsymbol{L}}(\boldsymbol{x}) = \widehat{\boldsymbol{\sigma}}(\boldsymbol{x}) + \widehat{r}(\boldsymbol{x})\boldsymbol{S}$, where

$$\widehat{\boldsymbol{\sigma}}(\boldsymbol{x}) = c_0 \frac{\boldsymbol{\sigma}(\boldsymbol{x})}{(r_0 - r(\boldsymbol{x}))^2 + \det \boldsymbol{\sigma}(\boldsymbol{x})}, \quad \widehat{r}(\boldsymbol{x}) = c_0 \frac{r_0 - r(\boldsymbol{x})}{(r_0 - r(\boldsymbol{x}))^2 + \det \boldsymbol{\sigma}(\boldsymbol{x})} + q_0$$

and

$$\widehat{\boldsymbol{\sigma}}^* = c_0 \frac{\boldsymbol{\sigma}^*}{(r_0 - r^*)^2 + \det \boldsymbol{\sigma}^*}, \quad \widehat{r}^* = c_0 \frac{r_0 - r^*}{(r_0 - r^*)^2 + \det \boldsymbol{\sigma}^*} + q_0.$$

Technically speaking, there are other results, but all of them are consequences of the ones listed above. Our theory also guarantees that there are no other microstructure-independent equalities.

Throughout this paper we have described several important open questions in the theory. We summarize them here.

- The polycrystalline G-closures possess an important convexity property that was the cornerstone of the present paper. Can it be used to obtain a finite geometric algorithm for constructing G-closures? Such construction was found by Francfort and Milton [15] for 2D conducting polycrystals, suggesting the positive answer to our question. For more detailed discussion see Section 5.

- The equation (7.1) has to be solved in each physical setting to determine all the exact relations there. So far we have been able to solve this equations for the problems of modest size, where the brute force approach works. Yet, the equation (7.1) says that the subspace Π has a special structure of Jordan algebra. Can this structure be used to help solve the equation? In fact, our result [21, Theorem 5.2] suggests that the question is meaningful.

- Finally, there is a question of whether the sufficient conditions for a manifold to be an exact relation are also necessary. See the more detailed formulation of this question at the end of Section 6.

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