Exact relations for effective tensors of composites: Necessary conditions and sufficient conditions.

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Abstract

Typically, the elastic and electrical properties of composite materials are strongly microstructure dependent. So it comes as a nice surprise to come across exact formulae for effective moduli that are universally valid no matter what the microstructure. Such exact formulae provide useful benchmarks for testing numerical and actual experimental data, and for evaluating the merit of various approximation schemes. They can be also regarded as fundamental invariances existing in a given physical context. Classic examples include, Hill's formulae for the effective bulk modulus of a two-phase mixture when the phases have equal shear moduli, Levin's formulae linking the effective thermal expansion coefficient and effective bulk modulus of two-phase mixtures, and Dykhne's result for the effective conductivity of an isotropic two-dimensional polycrystalline material. Here we present a systematic theory of exact relations embracing the known exact relations and establishing new ones. The search for exact relations is reduced to a search for matrix subspaces having a structure of special Jordan algebras. One of many new exact relations is for the effective shear modulus of a class of three-dimensional polycrystalline materials. We present complete lists of exact relations for 3D thermo-electricity and for 3D thermo-piezo-electric composites which includes all exact relations for elasticity, thermo-elasticity and piezo-electricity as particular cases.

Contents

1 Introduction.

The main difficulty in predicting effective properties of composite materials is a (sometimes strong) dependence on the microstructure. During this century there has been a sustained effort to understand the nature of that dependence. To this end powerful bounding techniques and various approximation schemes have been developed. Yet, we cannot claim that we have a complete understanding of the question. We can divide the area into two branches: geometry-independent and geometry-dependent results. The first tries to predict effective properties of a composite without using any information about the geometric arrangement of the phases, aside from perhaps the volume fractions occupied by the phases, while the second tries to improve the results of the first when some information about the microstructure is taken into account. Therefore, the first branch serves as a base for the more practically important second branch. Thus, if we want to make progress in our ability to predict the effective properties of composites it is crucial that we gain more insight into the more fundamental aspects of the problem.

Our basic paradigm is the computation of G-closures—the set of all possible effective tensors obtainable by mixing a given set of materials (possibly in given volume fractions) [34, 37, 53]. Thus far the general G-closure problem has been regarded as hopelessly complicated. This paper, however will create a tiny hope for further fundamental progress, as it presents a simple new formula for the map $L(\boldsymbol{x}) \longrightarrow L^*$, where $L(\boldsymbol{x})$ and L^* denote the local tensor and the effective tensor respectively. Using this formula we were able to answer an even more basic question: When does the G-closure have an interior and when does it degenerate into a surface, creating what we call an *exact relation.* In fact, generically, the G-closure has a non-empty interior. It is only in a few exceptional cases that the G-closure collapses to a surface. In this paper we will produce complete lists of such cases for many physically important examples. These include among others: elasticity, piezo-electricity, thermo-electricity and thermo-elasticity. The two most widely known examples of exact relations are due to Hill [25, 26] in elasticity and Keller [28], Dykhne [18] and Mendelson [40] (KDM) in 2D conductivity. Hill's exact relation says that a mixture of isotropic materials with constant shear modulus is isotropic and has the same shear modulus. The KDM exact relation says in particular that the effective conductivity of an isotropic 2D conducting polycrystal $\sigma^* = \sqrt{\det \sigma_0}$, where σ_0 is the conductivity tensor of a pure crystal. More generally KDM reads as follows. If σ^* is the effective conductivity corresponding to the local tensor $\sigma(\boldsymbol{x})$ then $\sigma^*/\det \sigma^*$ is the effective conductivity for the local tensor $\sigma(\boldsymbol{x})/\det \sigma(\boldsymbol{x})$.

The question that we address is whether there are any more beautiful relations like these for other physical phenomena. The answer is an emphatic "yes" as evidenced by results of Benveniste [4, 5, 6], Benveniste and Dvorak [8], Dunn [15], Dvorak [16, 17], Hashin [23], Levin [33], Milgrom [41], Milgrom and Shtrikman [42, 43], Rosen and Hashin [47], Schulgasser [49] and many others (see a review by Milton [45]). The purpose of this paper is to indicate how *all* of these exact relations could be harvested in any physical context by applying our general theory of exact relations, which we continue to develop following the earlier papers of Grabovsky [19] and Grabovsky and Sage [22]. These papers established necessary and sufficient conditions for stability of an exact relation under lamination and provided a method for getting all such relations for problems of modest size. The stability under lamination criterion served the purpose of proving that in many contexts there were no other exact relations beyond the already established ones. In this paper we prove an algebraic sufficient condition for stability under homogenization (not just lamination) and sharpen the method of [22] to be applicable to much larger sized problems. In addition we extend the general theory to exact relations which are not "rotationally invariant", and to exact relations which incorporate the volume fractions of the phases.

In order to place this paper in context, let us review some previous work on exact relations. We already mentioned the KDM exact relation for 2D conductivity. In [19] we have shown that there are no other exact relations there. We also showed that there are no exact relations for 3D conductivity. In the same paper we proved that for 2D elasticity there are exactly four exact relations, all previously known, see [1, 21, 25, 26, 35, 39]. Two among them [25, 26] (see also [39]) were known to hold in 3D. In [22] we showed that the 3D analogue of a third one is stable under lamination and that no other exact relations were possible. In this paper we establish stability under homogenization of that third exact relations which we term "rank-one plus a null-Lagrangian", or RPN for short. We describe it a little later in the introduction. Other exact relations are found in settings such as piezo-electricity [4, 6, 8], thermo-elasticity [17, 23, 33, 47, 49], thermo-electricity [3, 41, 42, 43, 51] and thermo-piezo-electricity [5, 15]. These papers suggested to us the immodest idea of going after *all* exact relations in a general coupled field problem of the form (1.4) below, of which the above examples are particular cases. In this paper we address all of these settings and produce complete lists of exact relations in each of them.

The structure of the paper is the following. In Section 2 we provide several complete lists of exact relations as evidence of the power of our methods. In Section 3 we develop a general theory of exact relations, and in Section 4 we explore some of the corollaries that are universally valid regardless of the particular physical context to which the theory applies. Unfortunately it is not that easy to pass from the the general theory to particular applications. In Sections 5 and 6 we develop the tools needed to make that transition. In order to keep this article of manageable size we do not treat 2D problems in Sections 5 and 6 (up to that point the analysis is the same in any number of space dimensions). It is possible to develop an equally effective methodology for 2D based on the commutativity of the rotation group $SO(2)$. We illustrate our theory with several simple examples. The most interesting one in our opinion is the "rank-one plus a null-Lagrangian" or RPN exact relation for 3D elasticity. We state it now.

Let $\tau(\boldsymbol{x})$ denote the stress field and $\epsilon(\boldsymbol{x})$ denote the strain field. Let $C(\boldsymbol{x})$ denote the Hooke's law relating the two tensor fields:

$$\tau(\boldsymbol{x}) = C(\boldsymbol{x})\epsilon(\boldsymbol{x}).$$

Suppose $C(\boldsymbol{x})$ has the form

$$C_{ijkl}(\boldsymbol{x}) = 2\mu T_{ijkl} + B_{ij}(\boldsymbol{x})B_{kl}(\boldsymbol{x}),$$

where

$$T_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) - \delta_{ij}\delta_{kl} \tag{1.1}$$

is the null-Lagrangian (satisfying $\langle \nabla u \cdot T\nabla u\rangle = \langle \nabla u\rangle \cdot T\langle \nabla u\rangle$ for all periodic $\nabla u(\boldsymbol{x})$, where $\langle\cdot\rangle$ denotes the average over the period cell). Then C^* has the same form

$$C^*_{ijkl} = 2\mu T_{ijkl} + B^*_{ij}B^*_{kl}. \tag{1.2}$$

The analysis of this exact relation has appeared earlier in [20][1], where an abridged version of our theory was presented and applied to the setting of 3D elasticity. The 2D analogue of this exact relation was first discovered in [1] for the special case of an isotropic polycrystal, with the local elasticity matrix $C(\boldsymbol{x}) = R(\boldsymbol{x}) \cdot C_0$ and $R(\boldsymbol{x}) \in SO(2)$. Later we generalized this 2D result in [21] uncovering its RPN structure. The 3D result (1.2) was anticipated in [22], where we proved its stability under lamination. The stability under homogenization established here is completely new.

In order to approach our goals from the unifying point of view we need to have a general framework that comprises such phenomena as conductivity, elasticity, thermo-electricity, etc. Such a framework has been developed by Milton [44]. We will cover as much of it as needed to introduce our notation.

If we look at the formulations of conductivity and elasticity problems we may realize that they have a very similar structure. Namely, both have a pair of physical fields: an electric field and a current density for conductivity, and a strain and a stress fields for elasticity. In general we may consider a pair of abstract fields E—the intensity field and J—the flux field. They take values from a finite dimensional tensor space $\mathcal{T}$ and are related by a linear map L:

$$J = LE,$$

[1] Full text is available at http://www.mathematik.uni-bielefeld.de/documenta/xvol-icm/16/16.html

where $L \in \mathrm{Sym}^+(\mathcal{T})$—the space of symmetric positive definite linear maps on $\mathcal{T}$. For conductivity $\mathcal{T} = \mathbb{R}^3$ for elasticity $\mathcal{T} = \mathrm{Sym}(\mathbb{R}^3)$. In our two examples the pair of fields must satisfy differential constraints: kinematic compatibility for the intensity field and balance equations for the flux field. These differential constraints can be transformed into local constraints in Fourier space. In order to write this fact in terms of our general fields we will restrict ourselves to the set of periodic composites. This presents no loss of generality because of the unpublished localization theorem of Kohn and Dal Maso and because all we are after are the G-closed sets of tensors in $\mathrm{Sym}^+(\mathcal{T})$.

Let $Q = [0,1]^3$ denote the period cell in $\mathbb{R}^3$. Then $\{E, J\} \subset \mathcal{H} = L^2(Q) \otimes \mathcal{T}$, where the tensor product is taken over $\mathbb{R}$. Let $\mathcal{U} = \mathbb{R} \otimes \mathcal{T}$ denote the set of constant fields. Then the differential constraints can be stated as follows: there exist an orthogonal splitting of $\mathcal{H} = \mathcal{E} \oplus \mathcal{J} \oplus \mathcal{U}$ such that $E \in \mathcal{E} \oplus \mathcal{U}$, $J \in \mathcal{J} \oplus \mathcal{U}$ and the orthogonal projection $\boldsymbol{\Gamma}$ onto the subspace $\mathcal{E}$ is local in Fourier space:

$$\widehat{\boldsymbol{\Gamma} h}(\boldsymbol{k}) = \Gamma\left(\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\right)\hat{h}(\boldsymbol{k}),$$

if $\boldsymbol{k} \neq 0$, and $\widehat{\boldsymbol{\Gamma} h}(0) = 0$ for any $h \in \mathcal{H}$. Here $\Gamma(\boldsymbol{n})$ is the orthogonal projection onto a subspace $\mathcal{E}_{\boldsymbol{n}}$ of $\mathcal{T}$. For conductivity $\Gamma(\boldsymbol{n})$ is given by $\Gamma(\boldsymbol{n})e = (e \cdot \boldsymbol{n})\boldsymbol{n}$ for any $e \in \mathcal{T} = \mathbb{R}^3$, and for elasticity $\Gamma(\boldsymbol{n})$ is given by

$$\Gamma(\boldsymbol{n})\varepsilon = \varepsilon\boldsymbol{n} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \varepsilon\boldsymbol{n} - (\varepsilon\boldsymbol{n} \cdot \boldsymbol{n})\boldsymbol{n} \otimes \boldsymbol{n}$$

for any $\varepsilon \in \mathcal{T} = \mathrm{Sym}(\mathbb{R}^3)$. In all physical settings the function $\Gamma(\boldsymbol{n})$ has a rotational invariance property:

$$R \cdot \Gamma(\boldsymbol{n}) = \Gamma(R\boldsymbol{n}) \tag{1.3}$$

for any rotation $R \in SO(3)$. Here $R\cdot$ denotes the natural action of the rotation group on an appropriate tensor space. For example, for any $\boldsymbol{x} \in \mathbb{R}^3$ $R \cdot \boldsymbol{x} = R\boldsymbol{x}$, for any $\xi \in \mathrm{Sym}(\mathbb{R}^3)$ $R \cdot \xi = R\xi R^{-1}$, and so on. The rotational invariance property will turn out to be extremely useful later on.

An ultimate example that we have in mind is the coupled problem involving n_1 electric fields, n_2 elastic fields and n_0 temperature fields:

$$\begin{aligned} E(\boldsymbol{x}) &= (\epsilon_1(\boldsymbol{x}), \ldots, \epsilon_{n_2}(\boldsymbol{x}), d_1(\boldsymbol{x}), \ldots, d_{n_1}(\boldsymbol{x}), \theta_1, \ldots, \theta_{n_0}), \\ J(\boldsymbol{x}) &= (\tau_1(\boldsymbol{x}), \ldots, \tau_{n_2}(\boldsymbol{x}), e_1(\boldsymbol{x}), \ldots, e_{n_1}(\boldsymbol{x}), \zeta_1(\boldsymbol{x}), \ldots, \zeta_{n_0}(\boldsymbol{x})), \\ J(\boldsymbol{x}) &= L(\boldsymbol{x})E(\boldsymbol{x}), \end{aligned} \tag{1.4}$$

The fields d_i are divergence free, while the fields e_i are curl-free. The elastic strains ϵ_i and stresses τ_i satisfy the usual differential constraints. The scalar entropy fields ζ_i do not obey any differential constraints, and the temperature increments θ_i from some reference temperature are constant scalars (independent of $\boldsymbol{x}$). In this case the space $\mathcal{T}$ is a direct sum of n_0 copies of $\mathbb{R}$, n_1 copies of $\mathbb{R}^3$ and n_2 copies of $\mathrm{Sym}(\mathbb{R}^3)$. The projection operator $\Gamma(\boldsymbol{n})$ is a direct sum of $n_0 + n_1 + n_2$ projection operators corresponding to the individual blocks in $E(\boldsymbol{x})$.

2 A plethora of exact relations.

In this section we focus on 3D exact relations. We have organized our lists in the order of increasing complexity. We start with pyro-electricity and conclude with thermo-piezo-electric composites and a link between two uncoupled elasticity problems. We do not address 3D conductivity and elasticity in this section. There are no exact relations for 3D conductivity [19], while the case of 3D elasticity is worked out in Examples 5.8 and 6.4 in this paper as an illustration of our methods (see also [20]).

The number of exact relations (or rather the number of infinite families of exact relations) explodes as we increase the size of the problem. For the simplest examples we are able to list all exact relations, but it would be a cumbersome task to do so for the larger ones. In this latter instance we omit the obvious exact relations (for example those that follow from the fact that the thermal expansion tensor can not influence the effective elasticity matrix in a thermo-elastic composite). We also omit some (but not all) exact relations that can be obtained as intersections of the other ones. In fact most exact relations in this section are particular cases of the exact relations for thermo-piezo-electric composites listed in Section 2.5. Still we included some of them for the convenience of the reader. At the same time we took care that all the omitted exact relations are reconstructible from the listed ones.

2.1 Pyro-electric composites.

There is a single (and almost trivial) exact relation for pyro-electric composites. Suppose that the constitutive equation is

$$e(\boldsymbol{x}) = \rho_0 d(\boldsymbol{x}) + p(\boldsymbol{x})\theta,$$

where the notation is consistent with (1.4). We assume that the local dielectric tensor ρ_0^{-1} is constant, so that the inhomogeneity is only in the pyro-electric coupling moduli $p(\boldsymbol{x}) = [p_1(\boldsymbol{x}), p_2(\boldsymbol{x}), p_3(\boldsymbol{x})]$. Taking the averages we obtain a formula for the effective pyro-electric tensor:

$$\langle e \rangle = \rho_0 \langle d \rangle + \langle p \rangle \theta,$$

where $\langle \cdot \rangle$ denotes the average over the period cell Q, so that $\rho^* = \rho_0$ and $p^* = \langle p \rangle$.

2.2 Thermo-electric composites.

The constitutive equation for the linear thermo-electric effect can be formulated as a linear relation between a pair of curl-free intensity fields $e_1(\boldsymbol{x})$ and $e_2(\boldsymbol{x})$ and a pair of divergence-free flux fields $d_1(\boldsymbol{x})$ and $d_2(\boldsymbol{x})$ (see [13] for the physical meaning of intensity fields and fluxes) effected by a *symmetric* tensor $L(\boldsymbol{x})$.

$$\begin{bmatrix} d_1(\boldsymbol{x}) \\ d_2(\boldsymbol{x}) \end{bmatrix} = L(\boldsymbol{x}) \begin{bmatrix} e_1(\boldsymbol{x}) \\ e_2(\boldsymbol{x}) \end{bmatrix}, \tag{2.1}$$

where $L(\boldsymbol{x})$ is the two by two symmetric positive definite block-matrix. Each block of $L(\boldsymbol{x})$ is a three by three matrix. If the material is isotropic then all four blocks are scalar multiples of the three by three identity matrix.

We have the following four exact relations.

1. If
$$L(\boldsymbol{x}) = \begin{pmatrix} \sigma(\boldsymbol{x}) & 0 \\ 0 & \sigma(\boldsymbol{x}) \end{pmatrix},$$
where $\sigma(\boldsymbol{x})$ is symmetric positive definite three by three matrix. Then
$$L^* = \begin{pmatrix} \sigma^* & 0 \\ 0 & \sigma^* \end{pmatrix}.$$
This exact relation is trivial.

2. If
$$L(\boldsymbol{x}) = \begin{pmatrix} \sigma(\boldsymbol{x}) & -\nu(\boldsymbol{x}) \\ \nu(\boldsymbol{x}) & \sigma(\boldsymbol{x}) \end{pmatrix},$$
where $\nu(\boldsymbol{x})$ is a *skew-symmetric* three by three matrix, such that $L(\boldsymbol{x})$ is positive definite. Then
$$L^* = \begin{pmatrix} \sigma^* & -\nu^* \\ \nu^* & \sigma^* \end{pmatrix},$$
where ν^* is a skew-symmetric matrix again. This exact relation corresponds to complex conductivity since (2.1) can be written as
$$d_1(\boldsymbol{x}) + id_2(\boldsymbol{x}) = (\sigma(\boldsymbol{x}) + i\nu(\boldsymbol{x}))(e_1(\boldsymbol{x}) + ie_2(\boldsymbol{x})).$$
See the review article [10] and references therein for a discussion of the physical interpretation of complex conductivity.

3. Let $\boldsymbol{n} = [n_1, n_2]$ be a fixed unit vector in $\mathbb{R}^2$ and let
$$L(\boldsymbol{x}) = \begin{pmatrix} \sigma_0 I + n_1^2\sigma(\boldsymbol{x}) & \nu_0 I + n_1 n_2 \sigma(\boldsymbol{x}) \\ \nu_0 I + n_1 n_2 \sigma(\boldsymbol{x}) & \gamma_0 I + n_2^2 \sigma(\boldsymbol{x}) \end{pmatrix},$$
or using a tensor product notation
$$L(\boldsymbol{x}) = \begin{pmatrix} \sigma_0 & \nu_0 \\ \nu_0 & \gamma_0 \end{pmatrix} \otimes I + (\boldsymbol{n} \otimes \boldsymbol{n}) \otimes \sigma(\boldsymbol{x}),$$
where $\sigma(\boldsymbol{x})$ is the three by three symmetric matrix field such that $L(\boldsymbol{x})$ is positive definite. Then
$$L^* = \begin{pmatrix} \sigma_0 & \nu_0 \\ \nu_0 & \gamma_0 \end{pmatrix} \otimes I + (\boldsymbol{n} \otimes \boldsymbol{n}) \otimes \sigma^*.$$

This is a family of uniform field relations parameterized by $\boldsymbol{n}$ (see Section 4.1) called that way because for any uniform field $e \in \mathbb{R}^3$ and any $\boldsymbol{x} \in Q$

$$L(\boldsymbol{x}) \begin{bmatrix} -n_2 e \\ n_1 e \end{bmatrix} = L_0 \begin{bmatrix} -n_2 e \\ n_1 e \end{bmatrix},$$

where

$$L_0 = \begin{pmatrix} \sigma_0 I & \nu_0 I \\ \nu_0 I & \gamma_0 I \end{pmatrix}.$$

Thus, taking averages over Q we get the result. In the special case of an isotropic composite made of two isotropic materials we can get a simpler form of this exact relation. Suppose that L_1 and L_2 are isotropic tensors of the two constituents and that $\det(L_1 - L_2) = 0$. Let L^* denote the isotropic effective tensor of the composite. Then $\det(L^* - L_1) = 0$ and $\det(L^* - L_2) = 0$.

4. Suppose that the four blocks comprising $L(\boldsymbol{x})$ are linearly dependent, i.e. there are four constants c_{ij}, $i, j = 1, 2$ such that

$$c_{11} L_{11}(\boldsymbol{x}) + c_{12} L_{12}(\boldsymbol{x}) + c_{21} L_{12}^T(\boldsymbol{x}) + c_{22} L_{22}(\boldsymbol{x}) = 0.$$

Then the same relation holds for L^* as well:

$$c_{11} L_{11}^* + c_{12} L_{12}^* + c_{21} (L_{12}^*)^T + c_{22} L_{22}^* = 0.$$

This exact relation is due to Milgrom and Shtrikman [43] (see also Milgrom [41]). See Examples 4.14 and 5.1 for a more general discussion.

Let us assume that we have a composite made with two isotropic thermo-electric materials

$$L_i = \begin{pmatrix} \sigma_i I & \nu_i I \\ \nu_i I & \gamma_i I \end{pmatrix},$$

where I is a three by three identity matrix, $i = 1, 2$. We can easily find three numbers c_1, c_2 and c_3 such that the vector $\boldsymbol{c} = [c_1, c_2, c_3]$ is orthogonal to the two vectors $\boldsymbol{l}_1 = [\sigma_1, \gamma_1, \nu_1]$ and $\boldsymbol{l}_2 = [\sigma_2, \gamma_2, \nu_2]$. Then our exact relation tells us that the vector $\boldsymbol{l}^* = (\sigma^*, \gamma^*, \nu^*)$ made with components of the isotropic effective tensor L^* is also orthogonal to $\boldsymbol{c}$. In other words, the three vectors $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ and $\boldsymbol{l}^*$ are linearly dependent, i.e.

$$\det \begin{vmatrix} \sigma^* & \gamma^* & \nu^* \\ \sigma_1 & \gamma_1 & \nu_1 \\ \sigma_2 & \gamma_2 & \nu_2 \end{vmatrix} = 0.$$

2.3 Thermo-elastic composites.

The properties of thermo-elastic materials can be represented by a block-vector $L(\boldsymbol{x}) = [C(\boldsymbol{x}), \alpha(\boldsymbol{x})]$, so that $\tau(\boldsymbol{x}) = C(\boldsymbol{x})\epsilon(\boldsymbol{x}) + \alpha(\boldsymbol{x})\theta$, where $\epsilon(\boldsymbol{x})$ and $\tau(\boldsymbol{x})$ denote the elastic strain and stress fields respectively, while the uniform field θ represents a constant temperature change from some reference temperature. Thus $C(\boldsymbol{x})$ has the meaning of an elasticity tensor and $\alpha(\boldsymbol{x})$ is a three by three matrix of thermal stress coefficients. We would like to mention a curious remark made by R. Lakes that if one allows void volume or slip interfaces, one can obtain arbitrarily high thermal expansion coefficients as well as negative ones [29, 30, 50].

1. Let $C(\boldsymbol{x}) = 2\mu_0 T + B(\boldsymbol{x}) \otimes B(\boldsymbol{x})$ and $\alpha(\boldsymbol{x}) = \alpha_0 I + q(\boldsymbol{x})B(\boldsymbol{x})$ then C^* and α^* have the same form.

2. Another exact relation is obtained if we replace $q(\boldsymbol{x})$ in item 1 by $\nu_0 \mathbf{Tr} B(\boldsymbol{x})$. This exact relation is due to Hashin [23], Rosen and Hashin [47] and Schulgasser [49].

3. Suppose that $C(\boldsymbol{x})I = \kappa_0 I + a_0\alpha(\boldsymbol{x})$, then $C^* = \kappa_0 I + a_0\alpha^*$. This exact relation is due to Laws [32] see also [7, 17].

4. Suppose that $C(\boldsymbol{x}) = 2\mu_0 I_s + \kappa(\boldsymbol{x}) I \otimes I$ and $\alpha(\boldsymbol{x})$ is isotropic (but still depends on $\boldsymbol{x}$), where I_s is defined on the next page. As we have mentioned at the beginning of this section, the effective elastic tensor C^* does not depend on the local thermo-elastic coefficient $\alpha(\boldsymbol{x})$. Therefore, according to Hill's relation [25, 26] mentioned in the introduction C^* will be isotropic with the same shear modulus μ_0 and the effective bulk modulus κ^* given by the famous Hill's formula

$$(3\kappa^* + 4\mu)^{-1} = \langle (3\kappa(\boldsymbol{x}) + 4\mu)^{-1} \rangle. \tag{2.2}$$

   Our new thermo-elastic exact relation says that the effective thermal stress coefficient α^* will always be isotropic and

$$\frac{\alpha^*}{3\kappa^* + 4\mu_0} = \langle \frac{\alpha(\boldsymbol{x})}{3\kappa(\boldsymbol{x}) + 4\mu_0} \rangle. \tag{2.3}$$

5. If in the previous item we assume in addition that there are constants a_0 b_0 and c_0 such that $a_0\alpha(\boldsymbol{x}) + b_0\kappa(\boldsymbol{x}) = c_0$ then $a_0\alpha^* + b_0\kappa^* = c_0$. In particular for a two-phase composite this exact relation can be written as

$$\det \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha^* \\ \kappa_1 & \kappa_2 & \kappa^* \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

   This exact relation is due to Levin [33].

2.4 Piezo-electric composites.

A 3D piezo-electric material is described by a symmetric nine by nine matrix $L(\boldsymbol{x})$, which can be represented as a two by two block matrix

$$L(\boldsymbol{x}) = \begin{pmatrix} C(\boldsymbol{x}) & P(\boldsymbol{x}) \\ P^T(\boldsymbol{x}) & \rho(\boldsymbol{x}) \end{pmatrix},$$

so that

$$\begin{bmatrix} \tau(\boldsymbol{x}) \\ e(\boldsymbol{x}) \end{bmatrix} = L(\boldsymbol{x}) \begin{bmatrix} \epsilon(\boldsymbol{x}) \\ d(\boldsymbol{x}) \end{bmatrix},$$

where $\epsilon(\boldsymbol{x})$ and $\tau(\boldsymbol{x})$ denote the elastic strain and stress fields respectively, while $e(\boldsymbol{x})$ and $d(\boldsymbol{x})$ denote the electric field and the electric displacement respectively. Thus $\rho(\boldsymbol{x})^{-1}$ has the meaning of a dielectric tensor, while $P(\boldsymbol{x})$ is a six by three matrix of piezo-electric coupling moduli. The remaining notation is consistent with the previous section.

Our exact relations have a nice block representation. Two of them are relatives of the elastic exact relation RPN (1.2), while the other two are uniform field relations (UFR) discussed in Section 4.1.

1. Let
$$L(\boldsymbol{x}) = \begin{pmatrix} 2\mu T + A(\boldsymbol{x}) \otimes A(\boldsymbol{x}) & A(\boldsymbol{x}) \otimes w(\boldsymbol{x}) \\ w(\boldsymbol{x}) \otimes A(\boldsymbol{x}) & \rho_0 I + w(\boldsymbol{x}) \otimes w(\boldsymbol{x}) \end{pmatrix}.$$
Then L^* has the same form:
$$L^* = \begin{pmatrix} 2\mu T + A^* \otimes A^* & A^* \otimes w^* \\ w^* \otimes A^* & \rho_0 I + w^* \otimes w^* \end{pmatrix}.$$

2. Let
$$L(\boldsymbol{x}) = \begin{pmatrix} 2\mu T + A(\boldsymbol{x}) \otimes A(\boldsymbol{x}) & A(\boldsymbol{x}) \otimes w(\boldsymbol{x}) \\ w(\boldsymbol{x}) \otimes A(\boldsymbol{x}) & \rho(\boldsymbol{x}) \end{pmatrix}.$$
Then L^* has the same form:
$$L^* = \begin{pmatrix} 2\mu T + A^* \otimes A^* & A^* \otimes w^* \\ w^* \otimes A^* & \rho^* \end{pmatrix}.$$

Our next two exact relations say that if we choose $A(\boldsymbol{x}) = a(\boldsymbol{x})I$ in items 1 and 2 then $A^* = a^* I$.

3. Let
$$L(\boldsymbol{x}) = \begin{pmatrix} 2\mu I_s + \kappa(\boldsymbol{x}) I \otimes I & I \otimes w(\boldsymbol{x}) \\ w(\boldsymbol{x}) \otimes I & \rho_0 I + \dfrac{3(w(\boldsymbol{x}) \otimes w(\boldsymbol{x}))}{4\mu + 3\kappa(\boldsymbol{x})} \end{pmatrix}.$$

Then

$$L^* = \begin{pmatrix} 2\mu I_s + \kappa^* I \otimes I & I \otimes w^* \\ w^* \otimes I & \rho_0 I + \dfrac{3(w^* \otimes w^*)}{4\mu + 3\kappa^*} \end{pmatrix}.$$

Moreover

$$\frac{w^*}{4\mu + 3\kappa^*} = \langle \frac{w(\boldsymbol{x})}{4\mu + 3\kappa(\boldsymbol{x})} \rangle. \tag{2.4}$$

4. Let

$$L(\boldsymbol{x}) = \begin{pmatrix} 2\mu I_s + \kappa(\boldsymbol{x}) I \otimes I & I \otimes w(\boldsymbol{x}) \\ w(\boldsymbol{x}) \otimes I & \rho(\boldsymbol{x}) \end{pmatrix}.$$

   Then

$$L^* = \begin{pmatrix} 2\mu I_s + \kappa^* I \otimes I & I \otimes w^* \\ w^* \otimes I & \rho^* \end{pmatrix},$$

   where I_s denotes the orthogonal projector onto the space of symmetric trace-free three by three matrices and I always denotes a three by three identity matrix. It is curious to note that when $\rho(\boldsymbol{x})$ does not have the form as in item 3 no volume average relations like (2.4) hold.

5. Let $C(\boldsymbol{x})$ have I as an eigenvector with a fixed eigenvalue κ: $C(\boldsymbol{x})I = \kappa I$ and assume that $P^T(\boldsymbol{x})I = 0$ then $C^* I = \kappa I$ and $(P^*)^T I = 0$.

2.5 Thermo-piezo-electric composites.

The physical properties of such materials are characterized by ten by ten symmetric positive definite matrices. We will focus, however only on those components that relate elastic stress $\tau(\boldsymbol{x})$ and electric field $e(\boldsymbol{x})$ to strain $\epsilon(\boldsymbol{x})$, electric displacement $d(\boldsymbol{x})$ and a temperature increment θ from a reference temperature.

$$\begin{aligned} \tau(\boldsymbol{x}) &= C(\boldsymbol{x})\epsilon(\boldsymbol{x}) + P(\boldsymbol{x})d(\boldsymbol{x}) + \alpha(\boldsymbol{x})\theta \\ e(\boldsymbol{x}) &= P^T(\boldsymbol{x})\epsilon(\boldsymbol{x}) + \rho(\boldsymbol{x})d(\boldsymbol{x}) + p(\boldsymbol{x})\theta, \end{aligned} \tag{2.5}$$

where the notation is consistent with the previous sections.
Group 1. Group 1 contains four exact relations. One of them is the following.

$$\begin{aligned} &C(\boldsymbol{x}) = 2\mu_0 T + B(\boldsymbol{x}) \otimes B(\boldsymbol{x}), && P(\boldsymbol{x}) = B(\boldsymbol{x}) \otimes w(\boldsymbol{x}), \\ &\alpha(\boldsymbol{x}) = \alpha_0 I + q(\boldsymbol{x}) B(\boldsymbol{x}), && p(\boldsymbol{x}) = q(\boldsymbol{x}) w(\boldsymbol{x}), \\ &\rho(\boldsymbol{x}) = \rho_0 I + w(\boldsymbol{x}) \otimes w(\boldsymbol{x}), \end{aligned} \tag{2.6}$$

where T is given by (1.1) and $B \otimes B$ denotes the linear map from $\mathrm{Sym}(\mathbb{R}^3)$ to $\mathrm{Sym}(\mathbb{R}^3)$ acting by $(B \otimes B)A = \mathbf{Tr}(AB)B$. Similarly, $B \otimes u$ denotes the linear map from $\mathbb{R}^3$ to $\mathrm{Sym}(\mathbb{R}^3)$ acting by $(B \otimes u)v = (v \cdot u)B$. The quantities μ_0, q, α_0 and ρ_0 are scalars. This exact relation says that the effective constitutive relation corresponding to (2.5) will have the form (2.6) provided the local tensors have the same form. In order to obtain the three other exact relations in this group we may choose $p(\boldsymbol{x})$ and/or $\rho(\boldsymbol{x})$ to be arbitrary, independent of $w(\boldsymbol{x})$.

Group 2. Group 2 contains two exact relations. One of them is the following.

$$\begin{aligned} C(\boldsymbol{x}) &= 2\mu_0 T + B(\boldsymbol{x}) \otimes B(\boldsymbol{x}), & P(\boldsymbol{x}) &= B(\boldsymbol{x}) \otimes w(\boldsymbol{x}), \\ \alpha(\boldsymbol{x}) &= \alpha_0 I + \nu_0 \mathbf{Tr}(B(\boldsymbol{x}))B(\boldsymbol{x}), & p(\boldsymbol{x}) &= \nu_0 \mathbf{Tr}(B(\boldsymbol{x}))w(\boldsymbol{x}), \\ \rho(\boldsymbol{x}) &= \rho_0 I + w(\boldsymbol{x}) \otimes w(\boldsymbol{x}), \end{aligned} \tag{2.7}$$

The other exact relation in this group is obtained by choosing $\rho(\boldsymbol{x})$ independent of $w(\boldsymbol{x})$. The latter exact relation was first obtained by Dunn for two-phase composites [15]. In Dunn's paper the relation is not given in the form (2.7), partly because Dunn used different notation:

$$\begin{aligned} \epsilon(\boldsymbol{x}) &= \quad S(\boldsymbol{x})\tau(\boldsymbol{x}) + g(\boldsymbol{x})d(\boldsymbol{x}) + \Delta(\boldsymbol{x})\theta \\ e(\boldsymbol{x}) &= -g^T(\boldsymbol{x})\tau(\boldsymbol{x}) + \beta(\boldsymbol{x})d(\boldsymbol{x}) + \gamma(\boldsymbol{x})\theta. \end{aligned} \tag{2.8}$$

In these variables Dunn's relation has a very simple form. If

$$\Delta(\boldsymbol{x}) = a_0 S(\boldsymbol{x}) I + b_0 I \quad \text{and} \quad \gamma(\boldsymbol{x}) = 0$$

then the same relation holds for the effective tensors. It is curious that a non-linear exact relation of type (2.7) can be transformed into a linear one (the effective tensors lie on an affine subspace) in new but physically meaningful variables.

We would like to note that there are no formulas in the two groups above that express effective tensors through the volume averages of local tensors. Such formulas will be present in the next group of six exact relations. Since these formulas distinguish the individual members of this group, we are forced to list them separately.

Group 3.

1. $$\begin{aligned} C(\boldsymbol{x}) &= 2\mu_0 I_s + \kappa(\boldsymbol{x}) I \otimes I, & P(\boldsymbol{x}) &= I \otimes w(\boldsymbol{x}), \\ \alpha(\boldsymbol{x}) &= (\alpha_0 + q(\boldsymbol{x})(\kappa(\boldsymbol{x}) + 4\mu_0/3))I, & p(\boldsymbol{x}) &= q(\boldsymbol{x})w(\boldsymbol{x}), \\ \rho(\boldsymbol{x}) &= \rho_0 I + \frac{3w(\boldsymbol{x}) \otimes w(\boldsymbol{x})}{4\mu_0 + 3\kappa(\boldsymbol{x})}. \end{aligned} \tag{2.9}$$

   Moreover,

   $$\langle \frac{w(\boldsymbol{x})}{3\kappa(\boldsymbol{x}) + 4\mu_0} \rangle = \frac{w^*}{3\kappa^* + 4\mu_0}, \qquad \langle q(\boldsymbol{x}) \rangle = q^*. \tag{2.10}$$

2. If we replace $\rho(\boldsymbol{x})$ by an arbitrary tensor field $\rho(\boldsymbol{x})$ then the first relation (2.10) disappears, while the second one still holds.

3. If instead we replace $p(\boldsymbol{x})$ by an independent choice $p(\boldsymbol{x})$, leaving $\rho(\boldsymbol{x})$ as in (2.9) then the first relation (2.10) is still true but the second scalar formula is replaced by a new vectorial one:

   $$\langle \frac{3\alpha(\boldsymbol{x})}{3\kappa(\boldsymbol{x}) + 4\mu_0} w(\boldsymbol{x}) \rangle - \langle p(\boldsymbol{x}) \rangle = \frac{3\alpha^*}{3\kappa^* + 4\mu_0} w^* - p^*. \tag{2.11}$$

4. If however we combine changes proposed in items 2 and 3, then we will obtain a new exact relation but with no volume average formulas like (2.10).

5. $$\begin{aligned} &C(\boldsymbol{x}) = 2\mu_0 I_s + \kappa(\boldsymbol{x}) I \otimes I, && P(\boldsymbol{x}) = I \otimes w(\boldsymbol{x}), \\ &\alpha(\boldsymbol{x}) = (\alpha_0 + q_0\kappa(\boldsymbol{x}))I, && p(\boldsymbol{x}) = q_0 w(\boldsymbol{x}), \\ &\rho(\boldsymbol{x}) = \rho_0 I + \frac{3w(\boldsymbol{x}) \otimes w(\boldsymbol{x})}{4\mu_0 + 3\kappa(\boldsymbol{x})}. \end{aligned} \tag{2.12}$$

   Moreover the formulas (2.10) are still valid (with the second formula being trivially true).

6. If we replace in (2.12) the dielectric tensor $\rho(\boldsymbol{x})^{-1}$ by an arbitrary tensor field independent of $w(\boldsymbol{x})$ and $\kappa(\boldsymbol{x})$ then we get a new exact relation, but there are no longer any volume average formulas to supplement (2.12).

There are two other exact relations that do not belong to any of the groups above.
Group 4.

1. Suppose that for all $\boldsymbol{x}$

   $$C(\boldsymbol{x})I = a_0\alpha(\boldsymbol{x}) + \kappa_0 I, \qquad P^T(\boldsymbol{x})I = a_0 p(\boldsymbol{x}), \tag{2.13}$$

   then the effective tensors C^*, P^*, α^* and p^* satisfy the same relation. Here a_0 and κ_0 are scalar constants. This exact relation belongs to the class of uniform field relations described in Section 4.1.

2. Let $C(\boldsymbol{x}) = C_0$ be isotropic and constant and let $P(\boldsymbol{x}) = 0$. Then the effective tensors have the same properties. Moreover, $\alpha^* = \langle \alpha(\boldsymbol{x}) \rangle$.

2.6 Links between two uncoupled elasticity problems.

In two space dimensions we would like to mention the link based on two-dimensional duality due to Berdichevsky [9] (see also [24] for some extensions). In three space dimensions there is a single and different link based on the RPN exact relation (1.2). Let $C_1(\boldsymbol{x}) = 2\mu_1 T + A_1(\boldsymbol{x}) \otimes A_1(\boldsymbol{x})$ and let $C_2(\boldsymbol{x}) = 2\mu_2 T + A_2(\boldsymbol{x}) \otimes A_2(\boldsymbol{x})$. For a three by three symmetric matrix A let $A' = A - I(\mathbf{Tr}A)/3$. Assume that for any $\boldsymbol{x} \in Q$ $A_1'(\boldsymbol{x})$ is a scalar multiple of $A_2'(\boldsymbol{x})$. According to the RPN exact relation for 3D elasticity $C_i^* = 2\mu_i T + A_i^* \otimes A_i^*$, $i = 1, 2$. Then our new result says that $A_1^{*\prime}$ is a scalar multiple of $A_2^{*\prime}$.

3 A general theory of exact relations.

3.1 Lamination formula.

In this section we derive a linear lamination formula which is best suited to our needs. This formula first appeared in the papers of Milton [44] and Zhikov [54]. Other linear lamination formulas were derived by Bacus [2] and Tartar [52] based on the idea to rewrite the constitutive relation so that the continuous and discontinuous components of the elastic fields are separated. In what follows we will use bold font to denote operators on a Hilbert space.

Recall the "cell problem" for finding effective properties for periodic composites characterized by the local tensor of physical properties $L(\boldsymbol{x})$ [44]: Find a pair of fields $e \in \mathcal{E}$, $J \in \mathcal{J} \oplus \mathcal{U}$ such that

$$J = \boldsymbol{L}(e + E_0), \tag{3.1}$$

where the operator $\boldsymbol{L} : \mathcal{H} \to \mathcal{H}$ is given by $\boldsymbol{L}h = L(\boldsymbol{x})h(\boldsymbol{x})$ and E_0 is the given mean value of the intensity field $E = e + E_0$. Then the effective tensor L^* is defined by

$$J_0 = L^* E_0 = \langle L(\boldsymbol{x})E(\boldsymbol{x})\rangle, \tag{3.2}$$

where $J_0 = \langle J\rangle$ is the mean value of the flux field J.

It is not very convenient to have such a definition of the effective tensor L^*, where we have to remember what the subspaces $\mathcal{E}$ and $\mathcal{J}$ are. Instead we are going to rewrite the equations relating L^* to L directly. Before we do so, we are going to derive a lamination formula that will turn out to be the cornerstone of our analysis. To this end let us fix a reference medium $L_0 \in \mathrm{Sym}^+(\mathcal{T})$ and let $\boldsymbol{\Gamma}'$ be the orthogonal projection onto the subspace $L_0^{1/2}\mathcal{E} \subset \mathcal{H}$. We still have

$$\widehat{\boldsymbol{\Gamma}' h}(\boldsymbol{k}) = \Gamma'(\frac{\boldsymbol{k}}{|\boldsymbol{k}|})\hat{h}(\boldsymbol{k}),$$

if $\boldsymbol{k} \neq 0$, and $\widehat{\boldsymbol{\Gamma}' h}(0) = 0$ for any $h \in \mathcal{H}$. Here $\Gamma'(\boldsymbol{n})$ is the orthogonal projection onto a subspace $\mathcal{E}'_{\boldsymbol{n}} = L_0^{1/2}\mathcal{E}_{\boldsymbol{n}}$ of $\mathcal{T}$. If we choose the reference medium L_0 to be isotropic then the rotational invariance property (1.3) is preserved for $\Gamma'(\boldsymbol{n})$:

$$R \cdot \Gamma'(\boldsymbol{n}) = \Gamma'(R\boldsymbol{n}) \tag{3.3}$$

We will not need the rotational invariance for awhile, so L_0 can be completely general until then.

Now we consider a laminate material with parameters L_1, L_2, θ_1, $\theta_2 = 1 - \theta_1$ and $\boldsymbol{n}$ (see fig. 1). Let us consider the W-transform of $L(\boldsymbol{x})$ first introduced in [44]:

$$W_{\boldsymbol{n}}(L) = [(I - L_0^{-1/2} L L_0^{-1/2})^{-1} - \Gamma'(\boldsymbol{n})]^{-1}. \tag{3.4}$$

In (3.4) we may choose L_0 to be any symmetric and positive definite operator on $\mathcal{T}$. In fact it is sufficient to require only that L_0 be positve definite on $\mathcal{E}_{\boldsymbol{n}}$ for each $\boldsymbol{n}$. However, in this paper we use only positive definite reference media. Under such a choice of L_0 the transformation $W_{\boldsymbol{n}}(L)$ is well defined and injective on all of $\mathrm{Sym}^+(\mathcal{T})$. We prove this statement in Theorem A.1 in the Appendix.

THEOREM **3.1** *If $L(\boldsymbol{x})$ is the laminate with parameters L_1, L_2, θ_1, $\theta_2 = 1 - \theta_1$ and $\boldsymbol{n}$ as in fig. 1 and L^* is its effective tensor, then*

$$W_{\boldsymbol{n}}(L^*) = \theta_1 W_{\boldsymbol{n}}(L_1) + \theta_2 W_{\boldsymbol{n}}(L_2).$$

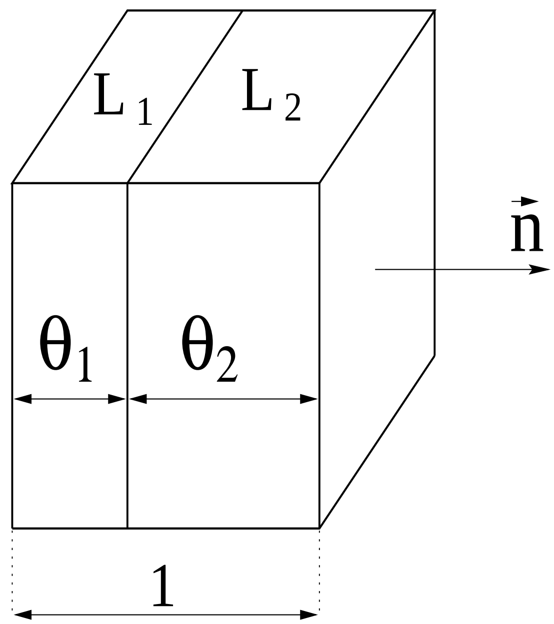

Figure 1: The period cell of the laminate.

PROOF: The most important observation is that in the laminate both $L(\boldsymbol{x})$ and the fields $E(\boldsymbol{x})$ and $J(\boldsymbol{x})$ depend on $\boldsymbol{x}\cdot\boldsymbol{n}$ only. Then we have

$$\mathbf{\Gamma}'F = \Gamma'(\boldsymbol{n})f(\boldsymbol{x}) = \Gamma'(\boldsymbol{n})(F(\boldsymbol{x}) - F_0), \tag{3.5}$$

provided F depends only on $\boldsymbol{x}\cdot\boldsymbol{n}$. We use subscript 0 to denote the mean value of a field and lower case to denote mean zero part of a field (e.g. $F = F_0 + f$, etc.).

Now we can begin the proof by introducing the polarization field

$$P(\boldsymbol{x}) = (L(\boldsymbol{x}) - L_0)E(\boldsymbol{x}). \tag{3.6}$$

If we take the average of (3.6) we get from (3.2)

$$P_0 = (L^* - L_0)E_0. \tag{3.7}$$

Now let us change coordinates by $L_0^{1/2}$. We let $E' = L_0^{1/2}E$, $J' = L_0^{-1/2}J$, $P' = L_0^{-1/2}P$, $L' = L_0^{-1/2}LL_0^{-1/2}$. Then (3.6) and (3.7) become

$$P'(\boldsymbol{x}) = (L'(\boldsymbol{x}) - I)E'(\boldsymbol{x}) = J'(\boldsymbol{x}) - E'(\boldsymbol{x}) \tag{3.8}$$

and

$$P_0' = (L^{*\prime} - I)E_0', \tag{3.9}$$

where $L^{*\prime} = L_0^{-1/2}L^*L_0^{-1/2}$. Applying the operator $\mathbf{\Gamma}'$ to (3.8) we get

$$\mathbf{\Gamma}'P' = -e'(\boldsymbol{x}), \tag{3.10}$$

since $J' \in (L_0^{1/2}\mathcal{E})^\perp$. Now we apply (3.5) to (3.10) and obtain using (3.9)

$$E'(\boldsymbol{x}) = E_0' + e'(\boldsymbol{x}) = (L^{*\prime} - I)^{-1}P_0' - \Gamma'(\boldsymbol{n})(P'(\boldsymbol{x}) - P_0'). \tag{3.11}$$

We also have from (3.8)

$$E'(\boldsymbol{x}) = (L'(\boldsymbol{x}) - I)^{-1}P'(\boldsymbol{x}). \tag{3.12}$$

In order to simplify our formulas we use Bergman-Milton S-transformation

$$S(L) = (I - L_0^{-1/2}LL_0^{-1/2})^{-1}. \tag{3.13}$$

Then equating the right hand sides in (3.11) and (3.12) we get

$$[S(L^*) - \Gamma'(\boldsymbol{n})]\, P_0' = [S(L(\boldsymbol{x})) - \Gamma'(\boldsymbol{n})]\, P'(\boldsymbol{x}). \tag{3.14}$$

Then we get by solving (3.14) for $P'(\boldsymbol{x})$ and taking averages:

$$P_0' = \langle P'(\boldsymbol{x})\rangle = \langle W_{\boldsymbol{n}}(L(\boldsymbol{x}))\rangle\, [S(L^*) - \Gamma'(\boldsymbol{n})]\, P_0'.$$

Since P_0' can be arbitrary, we obtain

$$W_{\boldsymbol{n}}(L^*) = \langle W_{\boldsymbol{n}}(L(\boldsymbol{x}))\rangle,$$

and the theorem follows. ∎

Corollary 3.2 *If a set $X \subset \mathrm{Sym}^+(\mathcal{T})$ is closed under lamination then $W_{\boldsymbol{n}}(X)$ is a convex set for any direction $\boldsymbol{n} \in \mathbb{S}^2$.*

3.2 A new formula for the effective tensor of a composite.

If we have a general, not necessarily laminar microstructure, then we can easily adjust the calculations in the proof of Theorem 3.1 in order to get a formula relating $W(L^*)$ and $W(L(\boldsymbol{x}))$, where $W(L) = W_{\boldsymbol{e}_1}(L)$ with $\boldsymbol{e}_1 = (1,0,0)$. We will start with equations (3.11) and (3.12), except that (3.11) will now read

$$E'(\boldsymbol{x}) = (L^{*\prime} - I)^{-1}P_0' - \boldsymbol{\Gamma}'P'. \tag{3.15}$$

So we get from (3.12) and (3.15)

$$S(L^*)P_0' = [\boldsymbol{S} - \boldsymbol{\Gamma}']P',$$

where $\boldsymbol{S}$ is an operator on $\mathcal{H}$ acting on an arbitrary $h \in \mathcal{H}$ by $\boldsymbol{S}h = S(L(\boldsymbol{x}))h(\boldsymbol{x})$. We now pass to the W variable by

$$W(L) = [S(L) - \Gamma']^{-1}, \tag{3.16}$$

where $\Gamma' = \Gamma'(\boldsymbol{e}_1)$. We obtain

$$([W(L^*)]^{-1} + \Gamma')P_0' = (\boldsymbol{W}^{-1} + \Gamma' - \boldsymbol{\Gamma}')P',$$

where

$$\boldsymbol{W}h = W(L(\boldsymbol{x}))h(\boldsymbol{x}).$$

Alternatively, introducing a Fourier multiplier $\boldsymbol{\Lambda}$ acting on an arbitrary $h \in \mathcal{H}$ by

$$\widehat{\boldsymbol{\Lambda} h}(\boldsymbol{k}) = (\Gamma'(\frac{\boldsymbol{k}}{|\boldsymbol{k}|}) - \Gamma')\hat{h}(\boldsymbol{k}), \tag{3.17}$$

if $\boldsymbol{k} \neq 0$, and $\widehat{\boldsymbol{\Lambda} h}(0) = 0$, we get

$$[W(L^*)]^{-1}P_0' = (\boldsymbol{W}^{-1} - \boldsymbol{\Lambda})P'. \tag{3.18}$$

Then, solving (3.18) for P' and taking averages we arrive at the following theorem.

Theorem **3.3** *Let L^* be the effective tensor for the composite described by the local tensor $L(\boldsymbol{x})$ then*

$$W(L^*) = \langle (\boldsymbol{I} - \boldsymbol{W}\boldsymbol{\Lambda})^{-1} W(L(\boldsymbol{x}))\rangle, \tag{3.19}$$

where $\boldsymbol{I}$ denotes the identity operator on the Hilbert space $\mathbb{H} = L^2(Q) \otimes End(\mathcal{T})$, and where $\boldsymbol{\Lambda}$ is defined by (3.17), while $\boldsymbol{W}$ denotes the operator on $\mathbb{H}$ acting by left multiplication:

$$\boldsymbol{W}H = W(L(\boldsymbol{x}))H(\boldsymbol{x})$$

for all $H \in \mathbb{H}$. Here $End(\mathcal{T})$ denotes the algebra of all linear operators on $\mathcal{T}$.

The Theorem 3.3 is equivalent to our original formulation (3.1), (3.2) but appears to be much more convenient for our purposes.

3.3 Exact relations.

Now we are ready to define the notion of an exact relation and develop a general theory.

Definition 3.4 *A G-closed smooth submanifold $\mathbb{M} \subset \mathrm{Sym}^+(\mathcal{T})$ is called an* exact relation*.*

Throughout this paper we will use the term "surface" to refer to this smooth submanifold, in order to invoke the intuitive imagery.

Let $\mathbb{M}$ be an exact relation, then by Corollary 3.2 $W_{\boldsymbol{n}}(\mathbb{M})$ must be a convex surface of the same dimension. Thus, $W_{\boldsymbol{n}}(\mathbb{M})$ is a convex subset of an affine subspace $\Pi_{\boldsymbol{n}}$. Now, let us take the reference medium $L_0 \in \mathbb{M}$. Then an easy calculation shows that $W_{\boldsymbol{n}}(L_0) = 0$. Thus, the affine subspaces $\Pi_{\boldsymbol{n}}$ are linear subspaces.

Theorem **3.5** *The subspaces $\Pi_{\boldsymbol{n}}$ do not depend on $\boldsymbol{n}$:*

$$\Pi_{\boldsymbol{n}} = \Pi.$$

Let

$$\mathcal{A} = \mathrm{Span}(\Gamma'(\boldsymbol{n}) - \Gamma' : \ |\boldsymbol{n}| = 1). \tag{3.20}$$

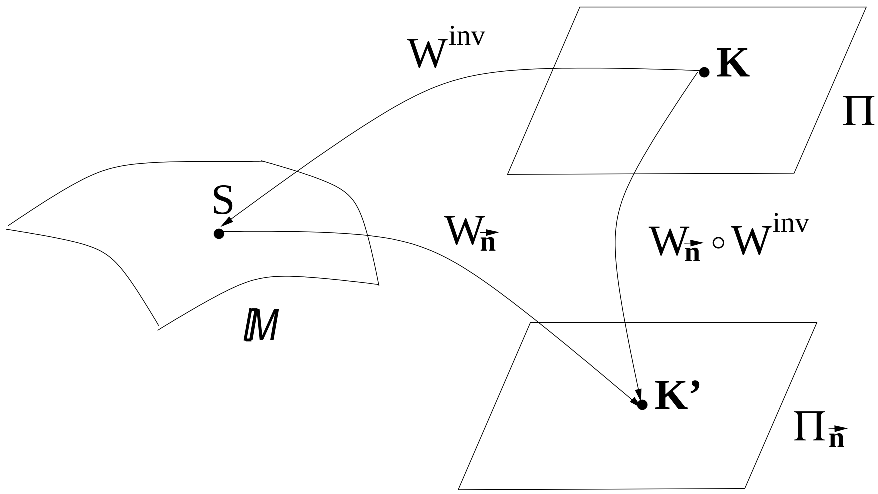

Figure 2: Proof of uniqueness of the image subspace.

Then Π *solves the following equation*

$$(\Pi\mathcal{A}\Pi)_{\mathrm{sym}} \subset \Pi, \tag{3.21}$$

where, for any subspace $\mathcal{X}$*,*

$$\mathcal{X}_{\mathrm{sym}} \stackrel{\mathrm{def}}{=} (\mathcal{X} + \mathcal{X}^T) \cap \mathrm{Sym}(\mathcal{T})$$

and

$$\mathcal{X}\mathcal{Y} = \mathrm{Span}(xy:\ x \in \mathcal{X}, y \in \mathcal{Y}),$$

for all subspaces $\mathcal{X}$ *and* $\mathcal{Y}$ *of* $End(\mathcal{T})$*.*

PROOF: Let $\Pi = \Pi_{\boldsymbol{e}_1}$. Fix an arbitrary direction $\boldsymbol{n}$ and consider a map from Π to $\Pi_{\boldsymbol{n}}$ obtained by applying W^{inv} (inverse of $W(L)$) followed by $W_{\boldsymbol{n}}$ as in fig. 2. Let us choose $K \in \Pi$ close to zero. Then there is a unique $L \in \mathbb{M}$ such that

$$K = [S(L) - \Gamma']^{-1}. \tag{3.22}$$

Let

$$K' = W_{\boldsymbol{n}}(L) = [S(L) - \Gamma'(\boldsymbol{n})]^{-1}. \tag{3.23}$$

Then, expressing $S(L)$ from (3.22) and substituting into (3.23), we get

$$K' = [I - KA(\boldsymbol{n})]^{-1}K, \tag{3.24}$$

where

$$A(\boldsymbol{n}) = \Gamma'(\boldsymbol{n}) - \Gamma'. \tag{3.25}$$

The function in (3.24) maps a small neighborhood $\mathcal{O}$ of zero of the subspace Π into a small neighborhood $\mathcal{O}_{\boldsymbol{n}}$ of zero of the subspace $\Pi_{\boldsymbol{n}}$. But K' is not a linear function of K. Therefore, the subspaces $\Pi_{\boldsymbol{n}}$ should be rather special to be non-linear images of subspaces. In order to see just how special they should be we expand (3.24) in powers of K:

$$K' = K + KA(\boldsymbol{n})K + \ldots + K(A(\boldsymbol{n})K)^n + \ldots \tag{3.26}$$

Since the image of $\mathcal{O}$ lies in a subspace, then each term of the expansion must belong to $\Pi_{\boldsymbol{n}}$. The first term shows that $K \in \Pi_{\boldsymbol{n}}$. Thus $\Pi \subset \Pi_{\boldsymbol{n}}$. But all the subspaces $\Pi_{\boldsymbol{n}}$ are of the same dimension. Thus $\Pi = \Pi_{\boldsymbol{n}}$, and the subspaces $\Pi_{\boldsymbol{n}}$ do not really depend on $\boldsymbol{n}$.

The second term in (3.26) says that the subspace Π satisfies

$$KA(\boldsymbol{n})K \subset \Pi \tag{3.27}$$

for all $K \in \Pi$. It is easy to see that all other terms in (3.26) will be in Π if Π satisfies (3.27). This is proved by induction in the order n of the term since

$$K(A(\boldsymbol{n})K)^n = \frac{1}{2}\{KA(\boldsymbol{n})[K(A(\boldsymbol{n})K)^{n-1}] + [K(A(\boldsymbol{n})K)^{n-1}]A(\boldsymbol{n})K\}.$$

We can reformulate (3.27) as follows. If we set $K = K_1 + K_2$ in (3.27) then we get that for any $\{K_1, K_2\} \subset \Pi$ and any direction $\boldsymbol{n}$

$$K_1A(\boldsymbol{n})K_2 + K_2A(\boldsymbol{n})K_1 \in \Pi. \tag{3.28}$$

Since Π is a subspace we can make all possible linear combinations of expressions in (3.28) with same K_1 and K_2 and different $\boldsymbol{n}$:

$$K_1AK_2 + K_2AK_1 \in \Pi \tag{3.29}$$

for any $A \in \mathcal{A}$, where $\mathcal{A}$ is given by (3.20). Using the fact that both Π and $\mathcal{A}$ are subspaces of $\mathrm{Sym}(\mathcal{T})$ we get the theorem. ∎

The equation (3.21) is a necessary condition for a surface $\mathbb{M}$ to be an exact relation. In fact it is equivalent to $\mathbb{M}$ being closed under lamination. Unfortunately, we do not know if (3.21) is sufficient for stability under homogenization. In general, stability under lamination does not imply stability under homogenization as was shown recently by Milton [46]. Our next theorem provides a sufficient condition. In order to formulate it we will need to introduce some new notation. Let

$$\mathrm{Skew}(\mathcal{T}) = \{A \in \mathrm{End}(\mathcal{T}) : A^T = -A\}.$$

Let

$$\mathcal{L} = \Pi\mathcal{A}\Pi \cap \mathrm{Skew}(\mathcal{T}). \tag{3.30}$$

THEOREM **3.6** *Let a subspace $\Pi \subset \mathrm{Sym}(\mathcal{T})$ solve (3.21) and the following two conditions are satisfied*

$$(i)\ \ (\mathcal{L}\mathcal{A}\Pi)_{\mathrm{sym}} \subset \Pi$$

$$(ii)\ (\mathcal{L}\mathcal{A}\mathcal{L})_{\mathrm{sym}} \subset \Pi,$$

where $\mathcal{L}$ is given by (3.30). Then there exists a neighborhood $\mathcal{O}$ of $0 \in \Pi$ such that $\mathbb{M} = W^{\mathrm{inv}}(\mathcal{O})$ is stable under homogenization.

Even though we are almost certain that our sufficient conditions do not follow from (3.21), in all the examples from our list in Section 2 the sufficient conditions are satisfied.

PROOF: We will split the proof into a sequence of lemmas.

Lemma 3.7 *Suppose* Π *satisfies all hypotheses of Theorem 3.6. Then for any* $j \geq 0$ $((\Pi\mathcal{A})^j\Pi)_{\rm sym} \subset \Pi$.

PROOF: Let

$$\mathcal{S} = (\Pi\mathcal{A}\Pi)_{\rm sym} \subset \Pi.$$

Since $\Pi\mathcal{A}\Pi$ is closed under transposition we have $\Pi\mathcal{A}\Pi = \mathcal{L} \oplus \mathcal{S}$, where $\mathcal{L}$ is given by (3.30). Then

$$((\Pi\mathcal{A})^2\Pi)_{\rm sym} = ((\mathcal{L} \oplus \mathcal{S})\mathcal{A}\Pi)_{\rm sym} = (\mathcal{L}\mathcal{A}\Pi)_{\rm sym} + (\mathcal{S}\mathcal{A}\Pi)_{\rm sym} \subset \Pi \tag{3.31}$$

by (i) and (3.21). Also

$$((\Pi\mathcal{A})^3\Pi)_{\rm sym} \subset ((\mathcal{L} \oplus \mathcal{S})\mathcal{A}(\mathcal{L} \oplus \mathcal{S}))_{\rm sym} \subset \Pi \tag{3.32}$$

by (i), (ii) and (3.21).

Lastly, we are going to show that (3.21), (3.31) and (3.32) imply

$$((\Pi\mathcal{A})^j\Pi)_{\rm sym} \subset \Pi \tag{3.33}$$

for any $j \geq 0$. The proof is an induction in j. If $j \leq 3$ the statement is true. Let $\{A_1, \ldots, A_{j-1}\} \subset \mathcal{A}$ and $\{K_1, \ldots, K_j\} \subset \Pi$. The expression $K_1A_1K_2A_2K_3A_3\ldots K_{j-1}A_{j-1}K_j$ will be called a j-chain. Obviously (3.33) is equivalent to the "j-chain property" that the symmetrized j-chain lies in Π:

$$K_1A_1K_2A_2\ldots K_{j-1}A_{j-1}K_j + K_jA_{j-1}K_{j-1}\ldots A_2K_2A_1K_1 \in \Pi. \tag{3.34}$$

In order to prove the j-chain property we rewrite the 2-chain property (3.29) as

$$K_1AK_2 = -K_2AK_1 + K' \tag{3.35}$$

for some $K' \in \Pi$ and any $A \in \mathcal{A}$. This identity allows us to swap successively the positions of adjacent K's in any j-chain, leaving $(j-1)$-chains as remainder. With the 3-chain property (i) it implies

$$K_1A_1K_2A_2K_3 = K_1A_2K_2A_1K_3 + K'' + \text{ a sum of 2}-\text{chains}$$

for some $K'' \in \Pi$. This allows us to swap successively the positions of adjacent A's in any j-chain, leaving $(j-1)$-chains and $(j-2)$-chains as remainder. By first reversing the order of the A's and then reversing the order of the K's we see that

$$\begin{aligned} &K_1A_1K_2A_2K_3A_3\ldots K_{j-1}A_{j-1}K_j = \\ &(-1)^{j(j-1)/2}K_jA_{j-1}K_{j-1}\ldots A_3K_3A_2K_2A_1K_1 \\ + &\text{ a sum of } (j-1)-\text{chains and } (j-2)-\text{chains} \end{aligned}$$

in which $j(j-1)/2$ is the number of swaps of adjacent K's needed to achieve this reordering. If the sign of $(-1)^{j(j-1)/2}$ is positive (as it is when $j=4$) then we apply the 4-chain property once, replacing the chain header

$$K_1A_1K_2A_2K_3A_3K_4 \quad \text{with} \quad -K_4A_3K_3A_2K_2A_1K_1 + K'$$

where $K' \in \Pi$ before swapping the A's and K's to obtain minus the reversed order chain plus shorter chains. This allows us to identify the symmetrized j-chain with a sum of symmetrized $(j-1)$-chains, $(j-2)$-chains and $(j-3)$-chains. By induction the j-chain property (3.34), or, equivalently (3.33), is then satisfied for all j. ∎

Lemma 3.7 ensures that $\Pi' \cap \mathrm{Sym}(\mathcal{T}) = \Pi$, where

$$\Pi' = \sum_{j=0}^{\infty} (\Pi\mathcal{A})^j \Pi. \tag{3.36}$$

The subspace Π' has the structure of an associative algebra with respect to the multiplication $K_1 * K_2 = K_1AK_2$ for any $A \in \mathcal{A}$ since it satisfies $\Pi'\mathcal{A}\Pi' \subset \Pi'$. For our purposes we will need to establish a structure of the associative algebra on $L^\infty(Q) \otimes \Pi'$.

Lemma 3.8 *Let* $\{W_1(\boldsymbol{x}), W_2(\boldsymbol{x})\} \subset L^\infty(Q) \otimes \Pi'$ *then* $\boldsymbol{W}_1\boldsymbol{\Lambda}W_2 \in L^\infty(Q) \otimes \Pi'$.

Proof: Let $K(\boldsymbol{x}) = \boldsymbol{W}_1\boldsymbol{\Lambda}W_2$. Then by definition of $\boldsymbol{\Lambda}$ (3.17) we have

$$\hat{K}(\boldsymbol{k}) = \sum_{\boldsymbol{p}\neq 0} \hat{W}_1(\boldsymbol{k}-\boldsymbol{p}) A(\frac{\boldsymbol{p}}{|\boldsymbol{p}|}) \hat{W}_2(\boldsymbol{p}).$$

Now, for each $\boldsymbol{k} \in \mathbb{Z}^3$ $\hat{W}_j(\boldsymbol{k}) \in \Pi' \otimes \mathbb{C} \stackrel{\text{def}}{=} \tilde{\Pi}'$, $j=1,2$. Obviously $\tilde{\Pi}'\mathcal{A}\tilde{\Pi}' \subset \tilde{\Pi}'$, therefore $\hat{K}(\boldsymbol{k}) \in \tilde{\Pi}'$ for each $\boldsymbol{k} \in \mathbb{Z}^3$. Thus $K(\boldsymbol{x}) \in \tilde{\Pi}'$ for almost every $\boldsymbol{x} \in Q$. But $K(\boldsymbol{x})$ is real and so the lemma is proved. ∎

Now we can easily finish the proof of the theorem. Let $L(\boldsymbol{x}) \in \mathbb{M}$. It will be sufficient to show that $W(L^*) \in \Pi$. Let $W(\boldsymbol{x}) = W(L(\boldsymbol{x})) \in \mathcal{O}$ and let $\boldsymbol{W}$ be the corresponding operator on the Hilbert space $\mathbb{H}$. Then, expanding (3.19) in powers of $\boldsymbol{W}$ we get

$$W(L^*) = \langle W(\boldsymbol{x})\rangle + \langle \boldsymbol{W\Lambda}W(\boldsymbol{x})\rangle + \ldots + \langle (\boldsymbol{W\Lambda})^n W(\boldsymbol{x})\rangle + \ldots \tag{3.37}$$

The series will converge if the neighborhood $\mathcal{O} \subset \Pi$ is sufficiently small. We will show that each term of that expansion is an element of Π' given by (3.36). Let $K_n(\boldsymbol{x}) = (\boldsymbol{W\Lambda})^{n-1}W$. We prove that $K_n \in L^\infty(Q) \otimes \Pi'$ by induction. For $n=1$ $K_1 = W \in L^\infty(Q) \otimes \Pi'$. Suppose that $K_n \in L^\infty(Q) \otimes \Pi'$ then $K_{n+1}(\boldsymbol{x}) = \boldsymbol{W\Lambda}K_n \in L^\infty(Q) \otimes \Pi'$ by Lemma 3.8. Thus for every $n \geq 1$ $\langle K_n(\boldsymbol{x})\rangle \in \Pi'$ and $W(L^*) \in \Pi' \cap \mathrm{Sym}(\mathcal{T}) = \Pi$, so that $L^* \in W^{\mathrm{inv}}(\mathcal{O}) = \mathbb{M}$. The theorem is proved. ∎

We give now necessary and sufficient conditions for Π to correspond to an exact relation. The theorem is of theoretical rather than practical utility.

THEOREM **3.9** *The subspace* $\Pi \subset Sym(\mathcal{T})$ *corresponds to an exact relation if and only if for any* $k \geq 1$, *any* $\{\boldsymbol{n}_1, \ldots, \boldsymbol{n}_k\} \subset \mathbb{Z}^3 \times \ldots \times \mathbb{Z}^3$ *such that*

$$\sum_{i=1}^{k} \boldsymbol{n}_i = 0$$

and for any $\{K_1, \ldots, K_k\} \subset \Pi$ *we have*

$$\sum_{\sigma \in S_k} \left(\prod_{s=1}^{k-1} K_{\sigma(s)} A(\sum_{j=1}^{s} \boldsymbol{n}_{\sigma(j)}) \right) K_{\sigma(k)} \in \Pi, \tag{3.38}$$

where S_k *is the set of all permutations of* k *elements and we define*

$$A(\boldsymbol{m}) = \begin{cases} \Gamma'(\frac{\boldsymbol{m}}{|\boldsymbol{m}|}) - \Gamma', & \text{if } \boldsymbol{m} \neq 0, \\ 0, & \text{if } \boldsymbol{m} = 0. \end{cases}$$

The proof is a straightforward application of Fourier convolution formula to (3.37) with subsequent choice of $K(\boldsymbol{x})$ to be a trigonometric polynomial of degree k.

For example, if $k = 3$ we get that

$$\begin{aligned} & K_1 A(\boldsymbol{q}_1) K_2 A(\boldsymbol{q}_3) K_3 + K_3 A(\boldsymbol{q}_3) K_2 A(\boldsymbol{q}_1) K_1 \\ + \; & K_2 A(\boldsymbol{q}_2) K_3 A(\boldsymbol{q}_1) K_1 + K_1 A(\boldsymbol{q}_1) K_3 A(\boldsymbol{q}_2) K_2 \\ + \; & K_3 A(\boldsymbol{q}_3) K_1 A(\boldsymbol{q}_2) K_2 + K_2 A(\boldsymbol{q}_2) K_1 A(\boldsymbol{q}_3) K_3 \in \Pi. \end{aligned}$$

This condition, which must hold for any vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ such that $\boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}_3 = 0$ and any $K_1, K_2, K_3 \in \Pi$ is necessary for stability under homogenization. It does not appear to be a consequence of the conditions for the stability under lamination, although it remains an open question as to whether there exists a subspace Π satisfying (3.21) but not satisfying the above constraint.

So far we have developed a local theory, i.e. we have a way of describing the exact relation surface near the point L_0. A look at our formulas shows that an exact relation must be an analytic surface, since it is an image of a subspace under an analytic map W^{inv}. Thus we can use an analytic continuation argument to prove the global result.

THEOREM **3.10** *Let a subspace* $\Pi \subset Sym(\mathcal{T})$ *satisfy conditions of Theorem 3.9 and let* $\mathbb{M} = Sym^+(\mathcal{T}) \cap \overline{W^{\text{inv}}(\Pi)}$. *And let* $L(\boldsymbol{x}) \in L^\infty(Q; Sym(\mathcal{T}))$ *belong pointwise to a compact subset of* $\mathbb{M}$. *Then* $L^* \in \mathbb{M}$.

PROOF: Let $L_\lambda(\boldsymbol{x})$ be the local field analytic in λ and such that $L_\lambda(\boldsymbol{x}) \in \mathbb{M}$ for all $\boldsymbol{x} \in Q$. Assume that $L_0(\boldsymbol{x}) = L_0$ and $L_1(\boldsymbol{x}) = L(\boldsymbol{x})$. We can choose $L_\lambda(\boldsymbol{x})$ such that it is uniformly bounded and uniformly positive definite in $\lambda \in [0, 1]$. Then, L^*_λ is analytic in $\lambda \in [0, 1]$. By our local theory we have $L^*_\lambda \in \mathbb{M}$ for small λ. But $\mathbb{M}$ is an analytic manifold, and so $L^*_\lambda \in \mathbb{M}$ for all $\lambda \in [0, 1]$. Setting $\lambda = 1$ we get $L^* \in \mathbb{M}$. ∎

4 Consequences of the general theory.

Even though all exact relations are described by a very simple equation (3.21), it is far from obvious how we can solve such an equation in settings more complicated that 2-D conductivity. We will address this question directly in Section 5. Here we will make some general remarks and describe some important classes of exact relations visible at our context-independent level. One key observation that was made in [19] is that the rotational invariance of polycrystalline G-closure allows us to reduce the size of the problem significantly. In Section 5 we are going to develop these ideas to a much higher level than in the two papers cited above. Much of this section, though, can be done without invoking rotational invariance. Nevertheless we will assume it from now on. Our assumption amounts to the fact that $R \cdot L \in \mathbb{M}$ whenever $L \in \mathbb{M}$ for any $R \in SO(3)$. We also choose $L_0 \in \mathbb{M}$ isotropic (i.e. $R \cdot L_0 = L_0$). It is intuitively obvious that given any $L \in \mathbb{M}$ we can make an isotropic polycrystal out of it. Since our surface $\mathbb{M}$ is assumed to be G-closed, the effective tensor of our polycrystal should lie in $\mathbb{M}$. Thus L_0 always exists. Our next theorem is a direct consequence of rotational invariance of $\mathbb{M}$. Let $\overline{\Gamma}$ be the orthogonal projection of Γ' onto the subspace of isotropic tensors in $\mathrm{Sym}(\mathcal{T})$ (with respect to the inner product $(A, B) = \mathbf{Tr}(AB)$ for $\{A, B\} \subset \mathrm{Sym}(\mathcal{T})$).

THEOREM **4.1** *The subspaces* Π *and* $\mathcal{A}$ *are rotationally invariant. Moreover*

$$\mathcal{A} = \mathrm{Span}(R \cdot \tilde{\Gamma} : R \in SO(3)), \tag{4.1}$$

where $\tilde{\Gamma} = \Gamma' - \overline{\Gamma}$.

PROOF: The map $W(L)$ provides a diffeomorphism between a neighborhood of L_0 in $\mathbb{M}$ and a neighborhood of zero in Π by Theorem A.1. Thus, for any $K \in \Pi$ near zero there is $L \in \mathbb{M}$ such that $K = [S(L) - \Gamma']^{-1}$. Then $R \cdot K = [S(R \cdot L) - R \cdot \Gamma']^{-1}$. But $R \cdot \Gamma' = \Gamma'(\boldsymbol{n})$ by (3.3), where $\boldsymbol{n} = R\boldsymbol{e}_1$. Thus $R \cdot K = W_{\boldsymbol{n}}(R \cdot L) \in W_{\boldsymbol{n}}(\mathbb{M}) \subset \Pi$, by Theorem 3.5. So Π is invariant under rotations.

The rotational invariance of $\mathcal{A}$ is a direct consequence of (3.3). For any unit vector $\boldsymbol{n} \in \mathbb{R}^3$

$$R \cdot A(\boldsymbol{n}) = \Gamma'(R\boldsymbol{n}) - \Gamma'(R\boldsymbol{e}_1) = A(R\boldsymbol{n}) - A(R\boldsymbol{e}_1) \in \mathcal{A}.$$

Let $\mathcal{A}' = \mathrm{Span}(R \cdot \tilde{\Gamma} : R \in SO(3))$. We want to show that $\mathcal{A} = \mathcal{A}'$. By (3.3) we can describe $\mathcal{A}$ as

$$\mathcal{A} = \mathrm{Span}(R \cdot \Gamma' - \Gamma' : R \in SO(3)) = \mathrm{Span}(R \cdot \tilde{\Gamma} - \tilde{\Gamma} : R \in SO(3)),$$

since $R \cdot \overline{\Gamma} = \overline{\Gamma}$. Choosing $R = I$ in the definition of $\mathcal{A}'$ shows that $\tilde{\Gamma} \in \mathcal{A}'$. Thus $\mathcal{A} \subset \mathcal{A}'$. Now we need to show the reverse inclusion. Let dR denote the invariant Haar measure on $SO(3)$. By our construction of $\tilde{\Gamma}$, we have

$$\int_{SO(3)} R \cdot \tilde{\Gamma} dR = 0.$$

Thus

$$\tilde{\Gamma} = \int_{SO(3)} (\tilde{\Gamma} - R \cdot \tilde{\Gamma}) dR \in \mathcal{A},$$

and $\mathcal{A}' \subset \mathcal{A}$. The theorem is proved. ∎

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Remark 4.2 *If $\mathcal{X}$ and $\mathcal{Y}$ are two rotationally invariant subspaces of* $\mathrm{End}(\mathcal{T})$ *then the subspaces $\mathcal{X}\mathcal{Y}$ and $\mathcal{X}_{\mathrm{sym}}$ are also rotationally invariant.*

4.1 Uniform field relations.

There is a simple observation made by Lurie and Cherkaev [36, 38] (although in the context of thermal expansion it dates back to the work of Cribb [14] that for any given uniform fields $\{u, v\} \subset \mathcal{T}$ the surface

$$\mathcal{U}(u, v) = \{L \in \mathrm{Sym}^{+}(\mathcal{T}) : Lu = v\} \tag{4.2}$$

is G-closed. However this surface is not generally rotationally invariant. Our idea then, is to look for rotationally invariant surfaces $\mathbb{M}$ of the form

$$\mathbb{M} = \bigcap_{\alpha \in \mathcal{I}} \mathcal{U}(u_{\alpha}, v_{\alpha}), \tag{4.3}$$

where $\mathcal{I}$ is an arbitrary index set.

Definition 4.3 *An exact relation $\mathbb{M}$ is called a* uniform field relation *(UFR) if it can be represented in the form (4.3).*

We remind the reader that we are considering only rotationally invariant exact relations. Let us construct an example of a UFR. Let $N \subset \mathcal{T}$ be a rotationally invariant subspace. Let L_0 be an arbitrary isotropic tensor and let

$$\mathrm{Ann}(N) = \{L \in \mathrm{Sym}(\mathcal{T}) : Lu = 0, \text{ for all } u \in N\}$$

be the annihilator of N in $\mathrm{Sym}(\mathcal{T})$. Then

$$\mathbb{M} = (L_0 + \mathrm{Ann(N)}) \cap \mathrm{Sym}^{+}(\mathcal{T}) \tag{4.4}$$

is a UFR. We remark that $\mathbb{M}$ is obviously rotationally invariant.

Lemma 4.4 *Let $\mathbb{M}$ be given by (4.4). Then*

$$\mathbb{M} = \bigcap_{u \in N} \mathcal{U}(u, L_0 u).$$

PROOF: Let

$$\mathbb{M}' = \bigcap_{u \in N} \mathcal{U}(u, L_0 u).$$

For any $u \in N$ and for any $L \in \mathbb{M}$ we have $Lu = L_0 u$. Thus $\mathbb{M} \subset \mathbb{M}'$. Now, let $L \in \mathbb{M}'$. Then for any $u \in N$ we have $(L - L_0)u = 0$. Thus $L - L_0 \in \mathrm{Ann}(N)$ and so $L \in \mathbb{M}$. The lemma is proved. ∎

THEOREM **4.5** *The set of UFR passing through an isotropic tensor L_0 is in one-to-one correspondence with invariant subspaces N of $\mathcal{T}$ via (4.4)*

PROOF: Let $\mathbb{M}$ be given by (4.3). Define

$$N = \mathrm{Span}(u_\alpha : \alpha \in \mathcal{I}) \subset \mathcal{T}$$

and let $L_0 \in \mathbb{M}$ be isotropic. Let

$$\mathbb{M}' = (L_0 + \mathrm{Ann(N)}) \cap \mathrm{Sym}^+(\mathcal{T}).$$

For any $L \in \mathbb{M}$ and for any $\alpha \in \mathcal{I}$ $(L - L_0)u_\alpha = 0$. Thus $\mathbb{M} \subset \mathbb{M}'$. Conversely, for any $L \in \mathbb{M}'$ and for any $\alpha \in \mathcal{I}$ $Lu_\alpha = L_0 u_\alpha = v_\alpha$. Thus $\mathbb{M}' \subset \mathbb{M}$. It remained to prove that N is invariant. We have the following characterization of N:

$$N = \{u \in \mathcal{T} : (L - L_0)u = 0 \text{ for all } L \in \mathbb{M}\}.$$

Now it is easy to see why N must be rotationally invariant. Pick any $u \in N$ and any $R \in SO(3)$. Then, since $\mathbb{M}$ is rotationally invariant,

$$(R^{-1} \cdot L - L_0)u = 0. \tag{4.5}$$

Then, we apply R to (4.5) and use that L_0 is isotropic, to get $(L - L_0)(R \cdot u) = 0$. Thus $R \cdot u \in N$. It is an obvious fact that if $N_1 \neq N_2$ then $\mathrm{Ann}(N_1) \neq \mathrm{Ann}(N_2)$. ∎

Remark 4.6 *The UFR $\mathbb{M}$ given by (4.4) corresponds to $\Pi = \mathrm{Ann}(L_0^{1/2}N)$. As N runs over all invariant subspaces of $\mathcal{T}$ then so does $L_0^{1/2}N$.*

PROOF: Since $W(L)$ is a local diffeomorphism near L_0, the differential $dW(L_0)$ maps tangent space $\mathbb{T}_{L_0}\mathbb{M}$ onto Π. An easy calculation shows that $dW(L_0)\xi = -L_0^{-1/2}\xi L_0^{-1/2}$ for any $\xi \in \mathbb{T}_{L_0}\mathbb{M}$. The statement follows from the fact that $\mathbb{T}_{L_0}\mathbb{M} = \mathrm{Ann}(N)$. ∎

Example 4.7

Consider 3D elasticity. In this case $\mathcal{T} = \mathrm{Sym}(\mathbb{R}^3)$. There are only two rotationally invariant subspaces N of $\mathcal{T}$: N_0 consisting of scalar multiples of a three by three identity matrix, and N_2 comprising all trace-free symmetric three by three matrices. The UFR corresponding to N_2 is the well known Hill's exact relation that we have mentioned in the introduction. The UFR corresponding to N_0 says that the set of all Hooke's laws C such that $CI = \kappa I$, where κ is a given constant, is stable under homogenization. This exact relation is due to Hill [26] and Lurie, Cherkaev and Fedorov [39]. We will refer to it as H-LCF.

4.2 An extension of the covariance principle of Milgrom and Shtrikman.

In Section 3 we have developed a general theory of exact relations passing through a *fixed* isotropic tensor L_0. We have also seen that the structure of UFRs depends only on the structure of $\mathcal{T}$ and not on the choice of L_0. Thus it is a natural question to try to relate the G-closed surfaces passing through L_1 to similar surfaces passing through L_2. We will show that this is possible in general under some constraints. These constraints are satisfied in many physically important situations.

We can ensure that the equation (3.21) has the same structure of solutions at L_1 and L_2 if the algebraic structure of the ambient space $\mathrm{Sym}(\mathcal{T})$ is the same at L_1 and L_2. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ correspond to L_1 and L_2 respectively. It will be convenient to make the following definition.

Definition 4.8 *The tensor $M \in \mathcal{A}$ is called a* generating tensor *for $\mathcal{A}$ if*

$$\mathcal{A} = \mathrm{Span}(R \cdot M : R \in SO(3)).$$

We are looking for invertible linear transformations

$$\phi : \mathrm{Sym}(\mathcal{T}) \to \mathrm{Sym}(\mathcal{T}),$$

such that they commute with the action of $SO(3)$

$$R \cdot \phi(X) = \phi(R \cdot X) \tag{4.6}$$

and such that for any $\{X, Y\} \subset \mathrm{Sym}(\mathcal{T})$

$$\phi(XM_1Y + YM_1X) = \phi(X)M_2\phi(Y) + \phi(Y)M_2\phi(X), \tag{4.7}$$

for some generating tensors $M_1 \in \mathcal{A}_1$, $M_2 \in \mathcal{A}_2$. Then, if Π satisfies (3.21) with $\mathcal{A} = \mathcal{A}_1$ then $\phi(\Pi)$ satisfies (3.21) with $\mathcal{A} = \mathcal{A}_2$. A map ϕ satisfying (4.6) and (4.7) is called an $SO(3)$ Jordan isomorphism.

Our next theorem exhibits the possible choices for ϕ.

THEOREM **4.9** *Assume that* $\dim \mathcal{T} \geq 3$. *An $SO(3)$ Jordan isomorphism ϕ satisfying (4.7) exists if and only if there exists an isotropic tensor $C \in \mathrm{GL}_{SO(3)}(\mathcal{T})$ such that*

$$\mathcal{A}_1 = C\mathcal{A}_2C^T. \tag{4.8}$$

In this case we let

$$\mathcal{C} = \{C \in \mathrm{GL}_{SO(3)}(\mathcal{T}) : \mathcal{A}_1 = C\mathcal{A}_2C^T\}, \tag{4.9}$$

where $\mathrm{GL}_{SO(3)}(\mathcal{T})$ denotes the space of all invertible isotropic operators on $\mathcal{T}$. Then the set of all $SO(3)$ Jordan isomorphisms ϕ is given by

$$\phi(X) = C^TXC, \quad C \in \mathcal{C}. \tag{4.10}$$

PROOF: We will split the proof into several steps. We ignore the $SO(3)$ action until the very end of the proof. The key lemma below is due to Etingof (personal communication). In order to formulate the lemma we recall the notion of the signature of a symmetric matrix. The matrix A is said to have a signature (p, q) if it has exactly p positive eigenvalues and q negative ones.

Lemma 4.10 *Suppose there is a map $\phi : \mathrm{Sym}(\mathcal{T}) \to \mathrm{Sym}(\mathcal{T})$ such that (4.7) is satisfied for some tensors M_1 and M_2 then M_2 or $-M_2$ has the same signature as M_1.*

PROOF: We will consider the case when M_1 and M_2 are invertible (the case of singular M_1 and M_2 is similar). Consider the map

$$F_1 : \mathrm{Sym}(\mathcal{T}) \to \mathrm{Sym}(\mathcal{T}),$$

given by

$$F_1(X) = XM_1X. \tag{4.11}$$

Consider the fixed points of the map F_1. It turns out that the geometry of these fixed points tells us the signature of M_1 up to a sign.

Fixed points are solutions of $X = XM_1X$. This is equivalent to $(M_1X)^2 = M_1X$. Thus, $P = M_1X$ is a projector on $\mathcal{T}$. Symmetry of X is equivalent to the symmetry of P with respect to the inner product given by M_1:

$$(u, v)_1 = (M_1^{-1}u, v).$$

A symmetric (orthogonal) projector on an inner product space is completely determined by its image W, and exists if and only if W is a non-degenerate subspace (i.e. has zero intersection with its orthogonal complement). Thus, the space Y_1 of fixed points of F_1 is homeomorphic to the disjoint union of spaces Y_1^k, $0 \le k \le \dim(\mathcal{T})$, and Y_1^k is homeomorphic to the open subset of the Grassmannian $G(k, \mathcal{T})$ consisting of non-degenerate subspaces of $\mathcal{T}$ of dimension k (or orthogonal projectors on $\mathcal{T}$ of rank k). The fact that they are separated from each other topologically follows from the fact that on Y_1^k, the trace of P is k.

The spaces Y_1^k are manifolds of dimensions $k(n-k)$, $n = \dim(\mathcal{T})$ (open sets in Grassmannians). So if we have any homeomorphism ϕ conjugating F_1 and F_2 (defined by (4.11) with index 1 replaced by index 2), it must induce a homeomorphism, for each k, between the union of Y_1^k and Y_1^{n-k} and the same union for M_2. Note that Y_1^k and Y_1^{n-k} are isomorphic by ($W \to$ orthogonal complement of W). Thus, if F_1 is equivalent to F_2, we must have that Y_1^k is homeomorphic to Y_2^k for all k.

For us it is enough to look at $k = 1$. Then Y_1^1 is the space of lines on which the form M_1^{-1} is nonzero. It consists of two parts Y_+ and Y_-, corresponding to lines on which the form is positive and negative. Let the signature of M_1^{-1} be (p, q), and we assume $p, q > 0$. Non-degenerate lines are permuted by $SO(p, q)$. In Y_+, the stabilizer is $O(p-1, q)$, and in Y_- the stabilizer is $O(p, q-1)$ Thus, $Y_+ \cong SO(p, q)/O(p-1, q)$ and $Y_- \cong SO(p, q)/O(p, q-1)$. These spaces are connected and their universal covering spaces are homotopy equivalent to spheres $\mathbb{S}^{p-1}$ and $\mathbb{S}^{q-1}$. Thus, p and q are uniquely determined by F_1 up to permutation $p, q \to q, p$, as desired. If $p = 0$ or $q = 0$ then $F_1(X) = 0$ has only the trivial solution. Then so does $F_2(X) = 0$. But this is possible only when M_2 has the signature $(n, 0)$ or $(0, n)$. ∎

Now we assume that the tensors M_1 and M_2 have the same signature, i.e. there exists $C \in \mathrm{GL}(\mathcal{T})$ such that $M_1 = CM_2C^T$.

Lemma 4.11 *Let*

$$\mathcal{C}(M_1, M_2) = \{C \in \mathrm{GL}(\mathcal{T}) : M_1 = CM_2C^T\}.$$

Then the set of all invertible linear maps ϕ satisfying (4.7) is

$$\mathcal{F} = \{\phi : \mathrm{Sym}(\mathcal{T}) \to \mathrm{Sym}(\mathcal{T}) | \phi(X) = C^T XC,\ C \in \mathcal{C}(M_1, M_2)\}. \tag{4.12}$$

PROOF: From the proof of the previous lemma we have seen that our Jordan isomorphism ϕ maps Y_1^1 into either Y_2^1 or Y_2^{n-1}. An easy calculation shows that for $j = 1, 2$

$$Y_j^1 = \{X \in \mathrm{Sym}(\mathcal{T}) : X = \frac{a \otimes a}{(M_j a, a)},\ a \in \mathcal{T},\ (M_j a, a) \neq 0\},$$

$$Y_j^{n-1} = \{X \in \mathrm{Sym}(\mathcal{T}) : X = M_j^{-1} - \frac{a \otimes a}{(M_j a, a)},\ a \in \mathcal{T},\ (M_j a, a) \neq 0\}.$$

Let us suppose that $\phi : Y_1^1 \to Y_2^1$. In other words ϕ maps symmetric rank-one tensors into symmetric rank-one tensors. Thus we need to characterize the symmetry group of the cone of symmetric rank-one matrices.

Lemma 4.12 *Let $\phi \in \mathrm{GL}(\mathrm{Sym}(\mathcal{T}))$ be such that it maps symmetric rank-one matrices into symmetric rank-one matrices. Then there exists $C \in \mathrm{GL}(\mathcal{T})$ such that*

$$\phi(X) = CXC^T \text{ or } \phi(X) = -CXC^T.$$

PROOF: A general symmetric rank-one matrix has the form $X = \epsilon a \otimes a$, where $a \in \mathcal{T}$, $\epsilon \in \{1, -1\}$. Then $\phi(a \otimes a) = \epsilon(a)\psi(a) \otimes \psi(a)$. Thus $|\psi(a)|^2 = |\mathbf{Tr}\phi(a \otimes a)|$. So, $|\psi(a)|^2$ is a continuous function on $\mathcal{T}$. If $\psi(a) = 0$ for $a \neq 0$ then $\phi(a \otimes a) = 0$, which contradicts invertibility of ϕ. Then for any $a \in \mathcal{T} \setminus \{0\}$

$$\epsilon(a) = \frac{\mathbf{Tr}\phi(a \otimes a)}{|\psi(a)|^2}.$$

Thus $\epsilon(a)$ is continuous on $\mathcal{T} \setminus \{0\}$. So, $\epsilon(a)$ is constant by connectedness of $\mathcal{T} \setminus \{0\}$ (since $\dim \mathcal{T} \geq 3$). Let us assume that $\epsilon(a) = 1$. Now it is an easy application of the "lifting theorem" to prove that there is a continuous choice of $\psi(a)$. Indeed, the set $\{a \otimes a : a \neq 0\}$ is homeomorphic to $\mathbb{R}P^{n-1} \times \mathbb{R}$ with the double cover $p : \mathcal{T} \setminus \{0\} \to \mathbb{R}P^{n-1} \times \mathbb{R}$, $p(a) = a \otimes a$. There is another continuous map $f : \mathcal{T} \setminus \{0\} \to \mathbb{R}P^{n-1} \times \mathbb{R}$, $f(a) = \phi(a \otimes a)$. Since the space $\mathcal{T} \setminus \{0\}$ is simply connected (we are assuming that $\dim \mathcal{T} \geq 3$) we have the existence of the continuous lifted map $\psi : \mathcal{T} \setminus \{0\} \to \mathcal{T} \setminus \{0\}$, such that $p(\psi(a)) = f(a)$, or equivalently, $\psi(a) \otimes \psi(a) = \phi(a \otimes a)$. Now we will prove that ψ is linear. If a and b are linearly independent then so are $\psi(a)$ and $\psi(b)$. Indeed, if $\psi(b) = \mu\psi(a)$ then $\phi(b\otimes b - \mu^2 a\otimes a) = 0$, which contradicts invertibility of ϕ. So, there exists $c \in \mathcal{T}$ such that $(c, \psi(b)) = 0$, while $(c, \psi(a)) \neq 0$. Then we have

$$\psi(a+\lambda b)\otimes\psi(a+\lambda b) = \psi(a)\otimes\psi(a) + 2\lambda\phi(a\odot b) + \lambda^2\psi(b)\otimes\psi(b), \tag{4.13}$$

where $a\odot b = \frac{1}{2}(a\otimes b + b\otimes a)$. Thus

$$0 \leq (c, \psi(a+\lambda b))^2 = (c, \psi(a))^2 + 2\lambda(c, \phi(a\odot b)c)$$

for all $\lambda \in \mathbb{R}$. The above inequality is possible for all λ only if $(c, \phi(a\odot b)c) = 0$. Thus, by continuity of ψ, $(c, \psi(a+\lambda b)) = (c, \psi(a))$ independent of λ. But then, letting (4.13) act on c we get

$$\psi(a+\lambda b) = \psi(a) + \frac{2\lambda\phi(a\odot b)c}{(c, \psi(a))}.$$

So $\psi(a+\lambda b)$ is affine in λ:

$$\psi(a+\lambda b) = \psi(a) + \lambda u. \tag{4.14}$$

Substituting this form in (4.13) we get by comparing terms at λ^2 that $u = \psi(b)$ or $u = -\psi(b)$. Since u does not depend on λ, then neither does the choice of sign. Then continuity of ψ implies that the choice of the sign does not depend on the values of a and b. The minus sign contradicts the non-degeneracy of ψ when we pass to the limit as $b \to a$ in (4.14). Thus $\psi(x)$ is linear and non-singular. Therefore, there exists $C \in \mathrm{GL}(\mathcal{T})$ such that $\psi(x) = Cx$. Thus for any $a \in \mathcal{T}$ $\phi(a\otimes a) = C(a\otimes a)C^T$. Since ϕ is linear we have $\phi(X) = CXC^T$ for any $X \in \mathrm{Sym}(\mathcal{T})$. We could also get $\phi(X) = -CXC^T$ had we assumed $\epsilon = -1$ at the beginning of the proof. ∎

Now we can finish the proof of Lemma 4.11. It is an easy calculation to verify that ϕ given by $\phi(X) = CXC^T$ is a Jordan isomorphism if and only if $C^T \in \mathcal{C}(M_1, M_2)$.

We have not considered another possibility that $\phi : Y_1^1 \to Y_2^{n-1}$ yet. We are going to show that this is impossible. Let us choose an arbitrary $C \in \mathcal{C}(M_1, M_2)$ and consider the map $\phi_C : \mathrm{Sym}(\mathcal{T}) \to \mathrm{Sym}(\mathcal{T})$,

$$\phi_C(X) = \mathbf{Tr}(M_1X)M_2^{-1} - C^TXC.$$

Observe that ϕ_C maps Y_1^1 onto Y_2^{n-1}. Let $\psi : Y_1^1 \to Y_1^1$ be given by $\psi = \phi_C^{-1}\circ\phi$. By Lemma 4.12 there exists $B \in \mathrm{GL}(\mathcal{T})$ such that $\psi(X) = BXB^T$ or $\psi(X) = -BXB^T$. Let us consider the plus sign (the minus sign is treated similarly). Then

$$\phi(X) = \phi_C(BXB^T) = \mathbf{Tr}(M_2C^TBXB^TC)M_2^{-1} - C^TBXB^TC.$$

Let $f(X) = C^TBXB^TC$ and let

$$\phi_0(X) = \mathbf{Tr}(M_2X)M_2^{-1} - X.$$

Then $\phi = \phi_0 \circ f$. Thus $\phi_0 = \phi \circ f^{-1}$. It follows then that ϕ_0 is a Jordan isomorphism satisfying

$$\phi_0(XMX) = \phi_0(X)M_2\phi_0(X), \tag{4.15}$$

where M is defined by the relation $M_1 = B^T CMC^T B$. Writing (4.15) explicitly we get

$$\mathbf{Tr}(M_2XMX)M_2^{-1} - XMX = [\mathbf{Tr}(M_2X)]^2 M_2^{-1} - 2X\mathbf{Tr}(M_2X) + XM_2X. \tag{4.16}$$

Multiplying the last equation by M_2 and choosing $X = M_2^{-1}$ we get

$$\mathbf{Tr}(M_2^{-1}M)I - M_2^{-1}M = (n-1)^2 I, \tag{4.17}$$

where $n = \dim \mathcal{T} \geq 3$. Taking traces in (4.17) we obtain $\mathbf{Tr}(M_2^{-1}M) = n(n-1)$. Substitution of this into (4.17) results in the relation $M = (n-1)M_2$. Substituting the value of M in (4.16) we get an equation for a matrix $Y = XM_2$:

$$Y^2 - \frac{2}{n}Y\mathbf{Tr}Y + \frac{1}{n}\Big((\mathbf{Tr}Y)^2 - (n-1)\mathbf{Tr}(Y^2)\Big)I = 0. \tag{4.18}$$

This equation implies that for any $X \in \mathrm{Sym}(\mathcal{T})$ the matrix Y has at most two distinct eigenvalues. However it is easy to construct a symmetric matrix X such that Y has n distinct prescribed eigenvalues $\alpha_1, \ldots, \alpha_n$. This statement becomes obvious if we choose a basis in $\mathcal{T}$ in which M_2 is diagonal. Therefore, equation (4.18) can not be true. It is curious to note that equation (4.18) becomes true if $n = 2$. ▮

Now we can easily finish the proof of the theorem. Assume that ϕ satisfying all conditions of the theorem exists. Then by Lemma 4.10 there are $M_1 \in \mathcal{A}_1$ and $M_2 \in \mathcal{A}_2$ with the same signature. Then there is $C \in \mathrm{GL}(\mathcal{T})$ such that $M_1 = CM_2C^T$. But then, by Lemma 4.11 $\phi(X) = B^T XB$ for some $B \in \mathcal{C}(M_1, M_2)$. Since ϕ commutes with the action of $SO(3)$ we conclude that B is isotropic. Indeed, if

$$(R \cdot B)(R \cdot X)(R \cdot B)^T = B(R \cdot X)B^T$$

for all symmetric X then for all symmetric Y we have

$$[B^{-1}(R \cdot B)]Y[B^{-1}(R \cdot B)]^T = Y.$$

Thus $R \cdot B = B$ or $R \cdot B = -B$ for all $R \in SO(3)$. The second equation is not possible because if $R = I$ we get $B = -B$, which is impossible. Now, since B is isotropic and belongs to $\mathcal{C}(M_1, M_2)$ we get that (4.8) holds.

Conversely, if (4.8) holds then we can choose $M_2 = \tilde{\Gamma}_2$ and $M_1 = C\tilde{\Gamma}_2C^T$. Then the set of maps $\mathcal{F}$ given by (4.12) with $C \in \mathcal{C}$ given by (4.9) is the set of Jordan $SO(3)$ isomorphisms. Lemma 4.11 guarantees that there are no other isomorphisms. The theorem is established. ▮

Remark 4.13 *If $L_1 = L_2 = L_0$ we can still have a non-trivial set $\mathcal{C}$. Then our theorem would relate different exact relations passing through L_0. This becomes useful when an obvious exact relation is mapped onto a non-trivial one. Another application of Theorem 4.9 is simplification of $\mathcal{A}$. The idea is to find an isotropic tensor C such that $C\mathcal{A}C^T$ becomes as simple as possible.*

Example 4.14

Let $\mathcal{T} = \mathbb{R}^n \otimes N_1$ where N_1 is $\mathbb{R}^3$ with the standard action of $SO(3)$, while $\mathbb{R}^n$ denotes an n-dimensional vector space with the trivial action of $SO(3)$. Physically, $\mathcal{T}$ corresponds to n coupled conductivity problems. In this situation $\mathcal{A}$ does not depend on L_0 and is given by $\mathcal{A} = I \otimes N_2$. Then $\mathcal{C} = \{Q \otimes I_{N_2} : Q \in O(n)\}$. Thus if Π is an exact relation then so is $(Q \otimes I_{N_2})\Pi(Q^T \otimes I_{N_2})$ for any $Q \in O(n)$. Milgrom and Shtrikman observed this fact and called it the covariance principle [42]. Applying this principle they obtained a family of new exact relations that are the images of $\Pi = \mathcal{D}_n \otimes \mathrm{Sym}(N_1)$, where $\mathcal{D}_n$ is the space of $n \times n$ diagonal matrices. This Π corresponds to the obvious fact that the composite with n decoupled electric fields will not produce coupling. The images of this Π appear to be interesting and non-trivial. We will show in Example 5.1 that the set of all exact relations in the present context is given by $\Pi = (\mathcal{B} \otimes \mathrm{End}(N_1))_{\mathrm{sym}}$, where $\mathcal{B}$ is an associative subalgebra of $\mathrm{End}(\mathbb{R}^n)$, closed under transposition. If $\mathcal{B}$ is conjugate to the algebra of all n by n diagonal matrices $\mathcal{D}_n$ we obtain the Milgrom-Shtrikman family of exact relations.

4.3 Links between problems.

In this section we are going to explore a special situation where $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$ and we restrict our attention only to surfaces stable under lamination (which we still call exact relations) generated by subspaces of $\mathcal{V} = \mathrm{Sym}(\mathcal{T}_1) \oplus \mathrm{Sym}(\mathcal{T}_2) \subset \mathrm{Sym}(\mathcal{T}_1 \oplus \mathcal{T}_2)$. An arbitrary element of $\mathcal{V}$ has the form $V = [A, B]$, where $A \in \mathrm{Sym}(\mathcal{T}_1)$ and $B \in \mathrm{Sym}(\mathcal{T}_2)$. In this case $\tilde{\Gamma} = [\tilde{\Gamma}_1, \tilde{\Gamma}_2]$, and for any $V \in \mathcal{V}$ and any $R \in SO(3)$ $R \cdot V = [R \cdot A, R \cdot B]$.

Let us define the Jordan product (with a slight abuse of notation)

$$[A_1, A_2] * [B_1, B_2] = [A_1 * B_1, A_2 * B_2],$$

where

$$A_j * B_j = \frac{1}{2}(A_j \tilde{\Gamma}_j B_j + B_j \tilde{\Gamma}_j A_j), \quad j = 1, 2.$$

Then an exact relation $\Pi \subset \mathcal{V}$ is an algebra with respect to the $*$ operation. Such an algebra is called a Jordan algebra [27]. It is a commutative non-associative algebra. Now we define the essential objects of our analysis. Let

$$\begin{aligned} \Pi_1 &= \{A \in \mathrm{Sym}(\mathcal{T}_1) : [A, B] \in \Pi \text{ for some } B \in \mathrm{Sym}(\mathcal{T}_2)\}, \\ \Pi_2 &= \{B \in \mathrm{Sym}(\mathcal{T}_2) : [A, B] \in \Pi \text{ for some } A \in \mathrm{Sym}(\mathcal{T}_1)\}. \\ \mathcal{K}_1 &= \{A \in \mathrm{Sym}(\mathcal{T}_1) : [A, 0] \in \Pi\}, \\ \mathcal{K}_2 &= \{B \in \mathrm{Sym}(\mathcal{T}_2) : [0, B] \in \Pi\}. \end{aligned} \tag{4.19}$$

Obviously $\mathcal{K}_j \subset \Pi_j$, $j = 1, 2$.

Suppose that we know everything about exact relations in $\mathrm{Sym}(\mathcal{T}_1)$ and $\mathrm{Sym}(\mathcal{T}_2)$. The objective then is to describe all exact relations in $\mathcal{V}$. There is a class of such exact relations which we are *not* interested in.

Definition 4.15 *Let Π_j be an exact relation in* $\mathrm{Sym}(\mathcal{T}_j)$*, $j = 1,2$ then $\Pi = \Pi_1 \oplus \Pi_2$ is obviously an exact relation in $\mathcal{V}$. If Π has the above form then we say that Π* splits.

Physically split exact relations express no information about any links that may exist between physical problems. Obviously, Π splits if and only if $\Pi_1 = \mathcal{K}_1$ (or $\Pi_2 = \mathcal{K}_2$). Thus we will assume in what follows that $\Pi_j \neq \mathcal{K}_j$, $j = 1,2$.

Definition 4.16 *Let $\mathcal{B}$ be a Jordan algebra with multiplication $*$. The subspace $\mathcal{S} \subset \mathcal{B}$ is called a Jordan ideal if for any $s \in \mathcal{S}$ and any $b \in \mathcal{B}$ we have $b * s \in \mathcal{S}$.*

Let $\mathcal{B}/\mathcal{S}$ denote the factor algebra in what follows.

THEOREM **4.17** *Let Π_j, $\mathcal{K}_j$, $j = 1,2$ be defined by (4.19). Then the following statements are true.*
(i) Π_j are rotationally invariant exact relations in $\mathrm{Sym}(\mathcal{T}_j)$*, $j = 1,2$.*
(ii) $\mathcal{K}_j$ are rotationally invariant Jordan ideals in Π_j, $j = 1,2$.
(iii) Let $\mathcal{F}_j = \Pi_j/\mathcal{K}_j$, $j = 1,2$. Then there is an $SO(3)$ Jordan isomorphism $\phi : \mathcal{F}_1 \to \mathcal{F}_2$.

PROOF: If $\{A_1, B_1\} \subset \Pi_1$ then there are $\{A_2, B_2\} \subset \mathrm{Sym}(\mathcal{T}_2)$ such that $\{[A_1, A_2], [B_1, B_2]\} \subset \Pi$. Then, since Π is an exact relation $[A_1, A_2] * [B_1, B_2] \in \Pi$. But then $A_1 * B_1 \in \Pi_1$. Let $R \in SO(3)$. Then $[R \cdot A_1, R \cdot A_2] = R \cdot [A_1, A_2] \in \Pi$. Thus $R \cdot A_1 \in \Pi_1$. Same argument works for Π_2.

Now if $A_1 \in \Pi_1$ and $K_1 \in \mathcal{K}_1$ then there is $A_2 \in \mathrm{Sym}(\mathcal{T}_2)$ such that $[A_1, A_2] \in \Pi$ then $[A_1 * K_1, 0] = [A_1, A_2] * [K_1, 0] \in \Pi$. Thus $A_1 * K_1 \in \mathcal{K}_1$. Let $R \in SO(3)$. Then $[R \cdot K_1, 0] = R \cdot [K_1, 0] \in \Pi$. Thus, $R \cdot K_1 \in \mathcal{K}_1$. Same argument works for $\mathcal{K}_2$.

Let $A_j \in \Pi_j$, then $\bar{A}_j$ denotes the equivalence class in $\mathcal{F}_j$ containing A_j, $j = 1,2$. For any $A_1 \in \Pi_1$, there is $A_2 \in \mathrm{Sym}(\mathcal{T}_2)$ such that $[A_1, A_2] \in \Pi$. Therefore, $A_2 \in \Pi_2$. Define $\phi(\bar{A}_1) = \bar{A}_2$. First we need to show that ϕ is a well-defined function mapping $\mathcal{F}_1$ into $\mathcal{F}_2$. Let $B_1 \in \bar{A}_1$. Then there is $B_2 \in \Pi_2$ such that $[B_1, B_2] \in \Pi$. Also, by our choice of B_1, $B_1 - A_1 \in \mathcal{K}_1$. Thus, $[B_1 - A_1, 0] \in \Pi$. So $[0, B_2 - A_2] = [B_1, B_2] - [A_1, A_2] - [B_1 - A_1, 0] \in \Pi$. Therefore, $\bar{B}_2 = \bar{A}_2$.

Let us show that the function ϕ is one-to-one. Indeed if $\phi(\bar{A}) = \bar{0}$, then there is $B \in \mathcal{K}_2$ such that $[A, B] \in \Pi$. But then $[A, 0] = [A, B] - [0, B] \in \Pi$. So, $\bar{A} = \bar{0}$. The function ϕ is also onto. If $\bar{B} \in \mathcal{F}_2$ then there is $A \in \Pi_1$ such that $[A, B] \in \Pi$. But then $\bar{B} = \phi(\bar{A})$.

Now we show that ϕ is a Jordan isomorphism between $\mathcal{F}_1$ and $\mathcal{F}_2$. Let $\{A_1, B_1\} \subset \Pi_1$ then there are $\{A_2, B_2\} \subset \mathrm{Sym}(\mathcal{T}_2)$ such that

$$\{[A_1, A_2], [B_1, B_2]\} \subset \Pi$$

. Then $[A_1 * B_1, A_2 * B_2] \in \Pi$. Thus

$$\phi(\overline{A_1 * B_1}) = \overline{A_2 * B_2}.$$

By construction of $\mathcal{F}_j$, $j = 1,2$ we have $\overline{A_j * B_j} = \bar{A}_j * \bar{B}_j$, $j = 1,2$. Then

$$\phi(\bar{A}_1 * \bar{B}_1) = \bar{A}_2 * \bar{B}_2.$$

But $\bar{A}_2 = \phi(\bar{A}_1)$ and $\bar{B}_2 = \phi(\bar{B}_1)$.

Finally, let us show that ϕ commutes with the $SO(3)$ action. Let $R \in SO(3)$ and let us *define* $R \cdot \bar{A} = \overline{R \cdot A}$ for any $A \in \Pi_j$, $j = 1, 2$. By rotational invariance of Π_j, $R \cdot A \in \Pi_j$ and our definition makes sense. If we choose any other representative $A' \in \bar{A}$ then $R \cdot A - R \cdot A' = R \cdot (A - A') \in \mathcal{K}_j$, since $A - A' \in \mathcal{K}_j$ and $\mathcal{K}_j$ is rotationally invariant. So, we have a well-defined $SO(3)$ action on $\mathcal{F}_j$. Now suppose $A \in \Pi_1$ then there is $B \in \Pi_2$ such that $[A, B] \in \Pi$. By rotational invariance of Π we have $[R \cdot A, R \cdot B] \in \Pi$. Thus we have $\phi(\bar{A}) = \bar{B}$ and $\phi(\overline{R \cdot A}) = \overline{R \cdot B}$. Then

$$\phi(R \cdot \bar{A}) = \phi(\overline{R \cdot A}) = \overline{R \cdot B} = R \cdot \bar{B} = R \cdot \phi(\bar{A}).$$

The theorem is proved. ∎

Remark 4.18 *If $\Pi_j = \mathrm{Sym}(\mathcal{T}_j)$, $j = 1, 2$ then $\mathcal{K}_j = \{0\}$ as Π_j are simple Jordan algebras, i.e. they do not have any non-trivial Jordan ideals. Thus $\mathrm{Sym}(\mathcal{T}_1)$ must be isomorphic to $\mathrm{Sym}(\mathcal{T}_2)$ in the sense of Theorem 4.17. But then all such isomorphisms ϕ are described by Theorem 4.9.*

We can use Theorem 4.17 to find exact relations involving volume fractions. Since we are not restricting ourselves to composites with finitely many phases we are seeking exact relations that involve $\langle f(L(\boldsymbol{x}))\rangle$ for a suitable function f. An example is the famous Hill's relation for elasticity (2.2). This example suggests our next construction. Let $\mathcal{T}_1 = \mathcal{T}$ with $\tilde{\Gamma}_1 = \tilde{\Gamma}$ and $\mathcal{T}_2 = \mathbb{R}^p$ with $\tilde{\Gamma}_2 = 0$. If we look for exact relations in $\mathcal{V}$ of the form $[L, f(L)]$, $L \in \mathbb{M}$, then the effective tensor corresponding to $[L(\boldsymbol{x}), f(L(\boldsymbol{x}))]$ will have the form $[L^*, \langle f(L(\boldsymbol{x}))\rangle]$, implying the relation $f(L^*) = \langle f(L(\boldsymbol{x}))\rangle$. See (6.7) for the explicit formula for the function f. In the W-variables this construction would correspond to a link Π, where Π_2 has the trivial Jordan structure: $A * B = 0$ for any $\{A, B\} \subset \Pi_2$ and an arbitrary $SO(3)$ action. Therefore, the ideal $\mathcal{K}_1$ in Π_1 should have the property that $\Pi_1 * \Pi_1 \subset \mathcal{K}_1$. Thus we have proved

THEOREM **4.19** *An exact relation $\Pi \subset Sym(\mathcal{T})$ can be sharpened by adding formulas of the form $f(L^*) = \langle f(L(\boldsymbol{x}))\rangle$ for some f if and only if $(\Pi\mathcal{A}\Pi)_{\mathrm{sym}} \neq \Pi$.*

Indeed, $\mathcal{K} = (\Pi\mathcal{A}\Pi)_{\mathrm{sym}}$ is a Jordan ideal and $\mathcal{F} = \Pi/\mathcal{K}$ has the trivial Jordan structure. Then we may think up an imaginary physical problem with subspaces $\mathcal{E} = \{0\}$ and $\mathcal{J} \oplus \mathcal{U} = \mathcal{H}$. We may choose $\mathcal{T}_2$ to be any linear space of sufficiently high dimension and with sufficiently rich $SO(3)$ structure so that we could pick a rotationally invariant subspace $\mathcal{K}_2 \subset \mathrm{Sym}(\mathcal{T}_2)$ such that $\mathcal{F} \cong \mathcal{F}_2 = \mathrm{Sym}(\mathcal{T}_2)/\mathcal{K}_2$ as $SO(3)$ modules, while we have already made sure that both spaces have the trivial Jordan structure. Thus we have constructed the objects $\Pi_1 = \Pi$, $\mathcal{K}_1 = \mathcal{K}$, Π_2 and $\mathcal{K}_2$ satisfying conditions of Theorem 4.17 together with ϕ effecting the isomorphism between $\mathcal{F}$ and $\mathcal{F}_2$. Since $\mathcal{K} \neq \Pi$, it follows that we have constructed a non-trivial link between our problem and an imaginary problem that has a very simple homogenization formula due to the choice of $\mathcal{E} = \{0\}$: $L^* = \langle L(\boldsymbol{x})\rangle$. In other words we have established a relation exemplified by the formula (2.2). Once a complete list of exact relations is computed in a given context, it is relatively easy to pick out all exact relations Π satisfying the requirements of Theorem 4.19.

5 How to solve $(\Pi\mathcal{A}\Pi)_{\rm sym}\subset\Pi$.

5.1 A general coupled problem.

So far we have developed a general theory of exact relations. But it is still not clear how we can get a complete list of exact relations even in a modestly sized problem. The naive approach that we had in our previous papers [19, 22] becomes infeasible starting with piezo-electricity. Here we present the next step in sophistication capable of dealing with relatively large problems. The method is based on the use of rotational invariance of polycrystalline exact relations, allowing a significant reduction in the computational size.

Consider the class of problems given by (1.4). The space $\mathcal{T}$ for conductivity will be denoted by N_1. This is the three-dimensional vector space with the standard action of the rotation group. The space $\mathcal{T}$ for elasticity is the space of three by three symmetric matrices, where $SO(3)$ acts by conjugation: $R\cdot A=RAR^T$. In this situation $\mathcal{T}$ can be represented as a direct sum of two irreducible subspaces: the hydrostatic part and the shear part. The hydrostatic part consists of all scalar multiples of the three by three identity matrix and will be denoted by N_0. The shear part consists of all symmetric trace-free matrices and is denoted by N_2. Thus, for elasticity, the six-dimensional space $\mathcal{T}$ of symmetric three by three matrices is split into the sum of a one-dimensional space N_0 and a five-dimensional space N_2. Finally, the one-dimensional space of temperature-like fields will be denoted by M_0. As representations M_0 and N_0 are indistinguishable, but they have different physical properties reflected in the Jordan multiplicative structure of exact relations.

For our most general problem

$$\mathcal{T}=\mathbb{R}^{n_0}\otimes M_0\oplus\mathbb{R}^{n_1}\otimes N_1\oplus\mathbb{R}_0^{n_2}\otimes N_0\oplus\mathbb{R}_2^{n_2}\otimes N_2, \tag{5.1}$$

where $\mathbb{R}_i^{n_2}$ is a copy of $\mathbb{R}^{n_2}$ attached to N_i, $i=0,2$. These two linear spaces have the same dimension but we will have to distinguish them in what follows. In the present context the local tensor $L\in\mathrm{Sym}(\mathcal{T})$ can be represented by a four by four block matrix.

For our next step we are going to use some representation theory of $SO(3)$, briefly summarized in our previous paper [22], in order to split $\mathrm{End}(\mathcal{T})$ into irreducibles. The irreducibles are the "unbreakable blocks" from which our exact relations Π will be built. Therefore, we can cut the size of our problem considerably by doing computations on the level of blocks rather than individual entries in the matrix representation of L. For example, for piezo-electricity L is a nine by nine symmetric matrix. But from our present point of view we will deal only with three by three matrices (here $n_0=0$, $n_1=n_2=1$).

For two linear spaces U and V we denote by $\mathrm{Hom}(U,V)$ the set of all linear maps from U to V, so $\mathrm{End}(U)=\mathrm{Hom}(U,U)$. We have a nice property of these functors: $\mathrm{Hom}(U_1\otimes U_2,V_1\otimes V_2)\cong\mathrm{Hom}(U_1,V_1)\otimes\mathrm{Hom}(U_2,V_2)$, where the tensor products are taken over $\mathbb{R}$. In particular $\mathrm{End}(U\otimes V)\cong\mathrm{End}(U)\otimes\mathrm{End}(V)$. We also need the "additive" property of the End functor:

$$\mathrm{End}(U\oplus V)\cong\mathrm{End}(U)\oplus\mathrm{End}(V)\oplus\mathrm{Hom}(U,V)\oplus\mathrm{Hom}(V,U).$$

Thus if $\mathcal{T}$ is given by (5.1) we see that $\mathrm{End}(\mathcal{T})$ can be written as a sixteen term expansion containing terms like $\mathrm{Hom}(\mathbb{R}^{n_i},\mathbb{R}^{n_j})\otimes\mathrm{Hom}(N_i,N_j)$, where the rotation group acts trivially on the first factor and "naturally" on the second one. We have already discussed how $SO(3)$ acts on the spaces N_i, so if $u \in N_i$ and $v \in N_j$ then $R \cdot u$ and $R \cdot v$ are assumed to be known. The "natural" action of $SO(3)$ on $\mathrm{Hom}(N_i, N_j)$ is given by the rule $R \cdot (Au) = (R \cdot A)(R \cdot u)$ for any $A \in \mathrm{Hom}(N_i, N_j)$ and any $R \in SO(3)$. It turns out that $\mathrm{Hom}(N_i, N_j)$ is not an irreducible representation of $SO(3)$, and that it can be split into a direct sum of irreducibles. The Clebsch-Gordon formula provides an answer:

$$\mathrm{Hom}(W_i, W_j) \cong \mathrm{Hom}(W_j, W_i) \cong W_{|j-i|} \oplus W_{|j-i|+1} \oplus \ldots \oplus W_{i+j-1} \oplus W_{i+j},$$

where W_n is a unique $2n + 1$ dimensional irreducible representation of $SO(3)$.

Our purpose is to encode an arbitrary rotationally invariant subspace of $\mathrm{Sym}(\mathcal{T}) \subset \mathrm{End}(\mathcal{T})$ and then "check" if it satisfies (3.21). In other words we will need to figure out how to multiply subrepresentations in $\mathrm{End}(\mathcal{T})$. The natural idea is to embed $\mathcal{T}$ given by (5.1) in $\mathcal{T}' \cong \mathbb{R}^N \otimes (W_0 \oplus W_1 \oplus W_2)$, $N = \max(n_0, n_1, n_2)$. Then there is an embedding map ϕ from $\mathrm{End}(\mathcal{T})$ into $\mathrm{End}(\mathbb{R}^N) \otimes \mathrm{End}(W_0 \oplus W_1 \oplus W_2)$. What is nice about it is that ϕ is an injective algebra homomorphism and commutes with the action of $SO(3)$. Thus, the problem reduces to figuring out the multiplication table in $\mathrm{End}(\mathcal{W})$, where $\mathcal{W} = W_0 \oplus W_1 \oplus W_2$. One approach is a brute force computation which is possible because $\mathcal{W}$ is just nine-dimensional. The other one is theoretical, yielding the multiplication rule for $\mathcal{W} = \bigoplus_{i=0}^{n} W_i$. We have decided in favor of the second approach, and describe it in Section 5.2.

Before we plunge into the details of the general case, we would like to consider the simplest interesting example of n coupled conductivity problems that avoids the technical difficulties present in more general situations. This case was considered by Milgrom in [41] and Milgrom and Shtrikman in [43].

Example 5.1

We have $\mathcal{T} \cong \mathbb{R}^n \otimes N_1$ and $\mathrm{End}(\mathcal{T}) \cong \mathrm{End}(\mathbb{R}^n) \otimes \mathrm{End}(N_1)$. According to the Clebsch-Gordon formula $\mathrm{End}(N_1) \cong W_0 \oplus W_1 \oplus W_2$. Then we need to compute $W_i W_j$ for $0 \le i, j \le 2$. Let ϕ be a linear map from $W_i \otimes W_j$ into $W_i W_j$ defined by the rule $\phi(a \otimes b) = ab$. Then ϕ is surjective and commutes with rotations. Therefore, $W_i W_j$ is the image of $\bigoplus_{k \in I(i,j)} W_k$ under ϕ, where $I(i, j)$ is the set of all integers between $|i - j|$ and $i + j$. Thus, $W_i W_j = \bigoplus_{k \in J} W_k$, where $J \subset I(i, j) \cap \{0, 1, 2\}$. It is not hard to check by hand (and it will follow from a more general theory developed in Section 5.2) that in fact $J = I'(i, j) = I(i, j) \cap \{0, 1, 2\}$. We remark that there are values $m > 1$ for which $J \neq I(i, j) \cap \{0, 1, \ldots, 2m\}$ in $\mathrm{End}(W_m)$.

If Π is a rotationally invariant subspace of $\mathrm{End}(\mathcal{T})$ then

$$\Pi = \mathcal{L}_0 \otimes W_0 \oplus \mathcal{L}_1 \otimes W_1 \oplus \mathcal{L}_2 \otimes W_2,$$

where $\mathcal{L}_0$ and $\mathcal{L}_2$ are subspaces of $\mathrm{Sym}(\mathbb{R}^n)$, while $\mathcal{L}_1 \subset \mathrm{Skew}(\mathbb{R}^n)$. The subspace $\mathcal{A}$ figuring in (3.21) is $\mathcal{A} = I_n \otimes W_2$, where I_n is a n by n identity matrix. A straightforward calculation using the formula

$$W_i W_j = \bigoplus_{k \in I'(i,j)} W_k, \quad I'(i, j) = I(i, j) \cap \{0, 1, 2\}$$

shows that (3.21) is equivalent to

$$\begin{aligned} &[(\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2)^2]_{\rm sym} && \subset \mathcal{L}_2, \\ &[(\mathcal{L}_1+\mathcal{L}_2)*(\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2)]_{\rm skew} && \subset \mathcal{L}_1, \\ &[(\mathcal{L}_0*\mathcal{L}_2)+(\mathcal{L}_1+\mathcal{L}_2)^2]_{\rm sym} && \subset \mathcal{L}_0, \end{aligned} \tag{5.2}$$

where $\mathcal{L}_i * \mathcal{L}_j = \mathcal{L}_i\mathcal{L}_j + \mathcal{L}_j\mathcal{L}_i$. It is remarkable that there is a nice characterization of all solutions of (5.2).

THEOREM **5.2** *Let $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ solve (5.2). Then $\mathcal{L}_0 = \mathcal{L}_2$ and there is an associative algebra $\mathcal{B} \subset End(\mathbb{R}^n)$ closed under transposition ($\mathcal{B}^T = \mathcal{B}$) such that $\mathcal{L}_1 = \mathcal{B}_{\rm skew}$ and $\mathcal{L}_2 = \mathcal{B}_{\rm sym}$.*

Milgrom and Shtrikman [43] (see also Milgrom [41]) have found an exact relation corresponding to $\mathcal{B} = \mathcal{D}_n$ the algebra of diagonal n by n matrices.

PROOF: Let $\mathcal{B} = \mathcal{L}_1 \oplus \mathcal{L}_2$, Then the subspaces $\mathcal{B}$ and $\mathcal{B}^2$ are closed under transposition. The first equation (5.2) says that $(\mathcal{B}^2)_{\rm sym} \subset \mathcal{L}_2 \subset \mathcal{B}$, while the second equation (5.2) says $(\mathcal{B}^2)_{\rm skew} \subset \mathcal{L}_1 \subset \mathcal{B}$. Therefore, $\mathcal{B}^2 = (\mathcal{B}^2)_{\rm sym} \oplus (\mathcal{B}^2)_{\rm skew} \subset \mathcal{B}$. Thus $\mathcal{B}$ is an associative subalgebra of $\mathrm{End}(\mathbb{R}^n)$. Since $\mathcal{B}^T = \mathcal{B}$, the algebra $\mathcal{B}$ is semisimple, and therefore, $\mathcal{B}^2 = \mathcal{B}$ (see [31] especially Proposition 4.7 of Chapter XVII). The third equation (5.2) says that $(\mathcal{B}^2)_{\rm sym} \subset \mathcal{L}_0$. Thus, $\mathcal{L}_2 \subset \mathcal{L}_0$. It remains to establish the reverse inclusion. From the first and second equations (5.2) we obtain respectively that $(\mathcal{L}_0 * \mathcal{B})_{\rm sym} \subset \mathcal{L}_2$ and $(\mathcal{L}_0 * \mathcal{B})_{\rm skew} \subset \mathcal{L}_1$. The two inclusions combine to

$$\mathcal{L}_0 * \mathcal{B} \subset \mathcal{B}. \tag{5.3}$$

This suggests our next step. Let

$$\mathcal{B}' = \{b \in \mathrm{End}(\mathbb{R}^n) : \ b\mathcal{B} \subset \mathcal{B} \text{ and } \mathcal{B}b \subset \mathcal{B}\}$$

be the largest associative subalgebra in $\mathrm{End}(\mathbb{R}^n)$ containing $\mathcal{B}$ as a two-sided ideal. The relation (5.3) can then be reformulated as $\mathcal{L}_0 \subset \mathcal{B}'$. It is easy to verify that $\mathcal{B}'$ is closed under transposition and, therefore, semisimple. Thus there exists a complement $\mathcal{C}$ to $\mathcal{B}$ in $\mathcal{B}'$ which is a two-sided ideal and such that $\mathcal{C} * \mathcal{B} = \{0\}$. In short

$$\mathcal{B}' = \mathcal{B} \oplus \mathcal{C}$$

as a direct sum of algebras. The complement $\mathcal{C}$ has a more explicit characterization:

$$\mathcal{C} = \{c \in \mathcal{B}' : \ c\mathcal{B} = \mathcal{B}c = \{0\}\}. \tag{5.4}$$

Indeed, let $\mathcal{C}'$ be equal to the right hand side of (5.4). Then, obviously, $\mathcal{C}'$ is a two-sided ideal in $\mathcal{B}'$ and $\mathcal{C} \subset \mathcal{C}'$. Also, $\mathcal{C}' \cap \mathcal{B} = \{0\}$. Therefore, $\mathcal{C}'$ is also a complement of $\mathcal{B}$ in $\mathcal{B}'$, and so $\mathcal{C}' = \mathcal{C}$. The characterization (5.4) is proved. In particular, (5.4) implies that the ideal $\mathcal{C}$ is also closed under transposition. Therefore, if $a \in \mathcal{L}_0$ and $a = b + c$ with $b \in \mathcal{B}$ and $c \in \mathcal{C}$, it follows that both b and c are symmetric matrices. From the first equation (5.2) we have $(\mathcal{L}_0^2)_{\rm sym} \subset \mathcal{L}_2$. Therefore, $a^2 = b^2 + c^2 \in \mathcal{B}$. Thus, $c^2 = 0$, implying $c = 0$ for a symmetric matrix c. So, we have proved that $\mathcal{L}_0 \subset \mathcal{L}_2$, which yields, together with the reverse inclusion, that $\mathcal{L}_0 = \mathcal{L}_2$. Now it is easy to verify that all equations (5.2) are satisfied. The theorem is proved. ∎

Corollary 5.3 *There are no non-trivial links between any number of uncoupled conductivity problems.*

PROOF: Let us look for exact relations Π in $\bigoplus_{i=1}^{n} \mathrm{Sym}(N_1) \subset \mathrm{Sym}(\mathbb{R}^n \otimes N_1)$. This means that the algebra $\mathcal{B}$ defined by Theorem 5.2 is a subalgebra of $\mathcal{D}_n$. To finish the proof we recall that any subalgebra of $\mathcal{D}_n$ is isomorphic to a direct sum of one-dimensional real algebras. If $\dim \mathcal{B} > 1$ then Π splits in the sense of Definition 4.15. If $\dim \mathcal{B} = 1$ then it corresponds to a trivial exact relation as in item 1 of the list in Section 2.2. ∎

Remark 5.4 *It easy to check that all solutions to (5.2) characterized by the theorem above satisfy conditions in Theorem 3.6 sufficient for stability under homogenization. In other words all rotationally invariant surfaces stable under lamination in the context of n coupled conductivities are also stable under homogenization.*

If $n = 2$ there are just six classes of algebras in $\mathrm{End}(\mathbb{R}^2)$ closed under transposition: $\mathcal{B}_0 = \{0\}$, $\mathcal{B}_1 = \{\lambda I : \lambda \in \mathbb{R}\}$, $\mathcal{B}_2(\boldsymbol{n}) = \{\lambda \boldsymbol{n} \otimes \boldsymbol{n} : \lambda \in \mathbb{R}\}$, for some unit vector $\boldsymbol{n} \in \mathbb{R}^2$, $\mathcal{B}_3(\boldsymbol{n}) = \{A \in \mathrm{Sym}(\mathbb{R}^2) : \boldsymbol{n}$ is an eigenvector of $A\}$, $\mathcal{B}_4 = \{\lambda R : \lambda \in \mathbb{R},\ R \in SO(2)\}$, $\mathcal{B}_5 = \mathrm{End}(\mathbb{R}^2)$. The four non-trivial algebras $\mathcal{B}_1$–$\mathcal{B}_4$ correspond to four non-trivial exact relations listed in Section 2.2.

5.2 A multiplication table.

In order to solve (3.21) we need to be able to multiply subrepresentations of $\mathrm{End}(\mathcal{W})$, where

$$\mathcal{W} = \bigoplus_{i=0}^{n} W_i. \tag{5.5}$$

For this purpose, we need to use some standard facts about the action of $SO(3)$ on an irreducible representation W_n (see for example [12]).

Let W_n be an irreducible real representation of $SO(3)$. This representation is of real type, i.e. $W_n \otimes \mathbb{C}$ is also irreducible. For that reason we use W_n for both a real vector space and for its complexification in a slight abuse of notation. The structure of the group action is completely determined by the induced action of its Lie algebra $so(3)$. The complexified Lie algebra $so(3,\mathbb{C}) = so(3) \otimes \mathbb{C}$ is spanned by three matrices

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & i & 1 \\ -i & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i & 1 \\ i & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

These matrices satisfy the following commutation relations

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = -H. \tag{5.6}$$

The complex linear space W_n has an orthogonal basis $\{v_{-n}, \ldots, v_n\}$ such that

$$H \cdot v_j = 2jv_j, \qquad X \cdot v_j = (n-j)v_{j+1}, \qquad Y \cdot v_j = -(n+j)v_{j-1}. \tag{5.7}$$

This basis is unique up to a single complex multiple applied to all basis vectors simultaneously. We will call it Cartan basis of weight vectors, and v_j will be called a vector of weight j. We will also use the normalized Cartan basis by choosing $\|v_n\| = \|v_{-n}\| = 1$. We will also require that $\bar{v}_j = (-1)^j v_{-j}$, where the bar denotes complex conjugation. In W_n there are only two choices for normalized Cartan basis. If $\{v_{-n}, \dots, v_n\}$ is one of them then $\{-v_{-n}, \dots, -v_n\}$ is the other. Under such a normalization we have

$$\|v_j\|^2 = 1/\binom{2n}{n+j}, \tag{5.8}$$

and v_0 is real. From now on we will fix $SO(3)$ invariant inner products $(\cdot, \cdot)_i$ on W_i's comprising $\mathcal{W}$ together with choices of normalized Cartan bases $\{v^i_k : |k| \le i\}$, $i = 0, \dots, n$.

We begin our analysis with an obvious observation that

$$\mathrm{End}(\mathcal{W}) = \bigoplus_{j,k=0}^{n} \mathrm{Hom}(W_k, W_j).$$

We use the fact that $\mathrm{Hom}(W_k, W_j) \cong W_j \otimes W_k$ via the following isomorphism.

$$(u \otimes v) \cdot x = u(x, \bar{v})_k,$$

for any $u \in W_j$ and $v, x \in W_k$. We will use the tensor product notation to denote elements of $\mathrm{Hom}(W_k, W_j)$. Thus, $\{v^j_A \otimes v^k_B : \ |A| \le j, \ |B| \le k\}$ forms a basis in $\mathrm{Hom}(W_k, W_j)$. Now consider the unique copy of $W_\alpha \subset \mathrm{Hom}(W_k, W_j)$. Its Cartan basis $\{e_C(j, k; \alpha) : |C| \le \alpha\}$ can be given as a linear combination of the basis vectors above:

$$e_C(j, k; \alpha) = \sum_A C^{j,k,\alpha}_{C,A} v^j_A \otimes v^k_{C-A}, \tag{5.9}$$

where the summation is over the admissible set of values of A. The coefficients $C^{j,k,\alpha}_{C,A}$ of this linear combination are multiples of the Clebsch-Gordon coefficients. We choose them to be

$$C^{j,k,\alpha}_{C,A} = \frac{i^{j+k-\alpha}}{\binom{2\alpha}{\alpha+C}} \sum_B (-1)^B \binom{j+k-\alpha}{B} \binom{j-k+\alpha+B}{j+A} \binom{2k-B}{k+C-A}.$$

Our choice of a multiple, that may depend on j, k and α, but not A or C, is fairly arbitrary, but it guarantees that $e_0(j, k; \alpha)$ are always real.

At this point we are ready to describe a convenient system of coordinates for irreducible subrepresentations sitting inside $\mathrm{End}(\mathcal{W})$. If $W_\alpha \subset \mathrm{End}(\mathcal{W})$ then its zero weight vector w_0 is a linear combination of zero weight vectors $e_0(j, k; \alpha)$:

$$w_0 = \sum_{j,k=0}^{n} x_{jk} e_0(j, k; \alpha).$$

The numbers x_{jk} are the homogeneous coordinates of $W_\alpha \subset \mathrm{End}(\mathcal{W})$. Conversely, let the matrix $X = (x_{jk})$ be given, then

$$e^\alpha_A(X) = \sum_{j,k=0}^{n} x_{jk} e_A(j, k; \alpha) \tag{5.10}$$

is a Cartan basis of weight vectors for W_α. We will use a rather convenient notation for W_α suggested by (5.10):

$$W_\alpha = X \otimes W_\alpha. \tag{5.11}$$

If instead of $\mathcal{W}$ given by (5.5) we take

$$\mathcal{W} \cong \bigoplus_{j=0}^{n} \left(\mathbb{R}^{n_j} \otimes W_j \right)$$

then the above discussion still holds without change, except x_{jk} are elements of $\mathrm{Hom}(\mathbb{R}^{n_k}, \mathbb{R}^{n_j})$.

Example 5.5

For our most general physical problem (5.1) we need to fix norms and normalized Cartan bases for M_0 and N_j, $j = 0, 1, 2$. The one-dimensional space M_0 is spanned by a unit vector $\overline{\omega}$. The normalized Cartan basis for N_1 is

$$f_1 = -\frac{1}{\sqrt{2}}(e_3 + ie_2), \qquad f_0 = \frac{1}{\sqrt{2}} e_1, \qquad f_{-1} = \frac{1}{\sqrt{2}}(e_3 - ie_2)$$

in terms of the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$. Introducing the notation ab for $1/2(a \otimes b + b \otimes a)$ for any two vectors a and b in $\mathbb{R}^3$, we have the following normalized Cartan bases $\{v_j : |j| \le 2\}$ for N_2 and $\hat{\omega}$ for N_0:

$$\begin{aligned} v_{-2} &= f_{-1}^2, & v_2 &= f_1^2, \\ v_{-1} &= f_0 f_{-1}, & v_1 &= f_0 f_1 \\ v_0 &= \frac{1}{3} f_1 f_{-1} + \tfrac{2}{3} f_0^2, & \hat{\omega} &= \tfrac{2}{\sqrt{3}}(f_0^2 - f_1 f_{-1}) = \frac{1}{\sqrt{3}} I, \end{aligned}$$

where I is the three by three identity matrix. These choices determine uniquely a system of homogeneous coordinates for rotationally invariant subspaces in $\mathrm{End}(\mathcal{T})$.

Using the tensor product notation (5.11) we can write the decomposition of $\mathrm{End}(\mathcal{T})$ and an arbitrary rotationally invariant subspace $\Pi \subset \mathrm{End}(\mathcal{T})$ into irreducibles as follows:

$$\mathrm{End}(\mathcal{T}) \cong \bigoplus_{i=0}^{4} \mathcal{V}_i \otimes W_i, \qquad \Pi = \bigoplus_{i=0}^{4} \mathcal{L}_i \otimes W_i, \tag{5.12}$$

where $\mathcal{L}_i \subset \mathcal{V}_i$ are arbitrary subspaces and $\mathcal{V}_i$ are the following subspaces of $\mathrm{End}(\mathbb{R}^{n_0+n_1+2n_2})$:

$$\begin{aligned} \mathcal{V}_0 &= \mathrm{End}(\mathbb{R}^{n_0} \oplus \mathbb{R}_0^{n_2}) \oplus \mathrm{End}(\mathbb{R}_2^{n_2}) \oplus \mathrm{End}(\mathbb{R}^{n_1}), \\ \mathcal{V}_1 &= \mathrm{End}(\mathbb{R}^{n_1} \oplus \mathbb{R}_2^{n_2}) \oplus \mathrm{Hom}(\mathbb{R}^{n_0} \oplus \mathbb{R}_0^{n_2}, \mathbb{R}^{n_1}) \oplus \mathrm{Hom}(\mathbb{R}^{n_1}, \mathbb{R}^{n_0} \oplus \mathbb{R}_0^{n_2}), \\ \mathcal{V}_2 &= \mathrm{End}(\mathbb{R}^{n_1} \oplus \mathbb{R}_2^{n_2}) \oplus \mathrm{Hom}(\mathbb{R}^{n_0} \oplus \mathbb{R}_0^{n_2}, \mathbb{R}_2^{n_2}) \oplus \mathrm{Hom}(\mathbb{R}_2^{n_2}, \mathbb{R}^{n_0} \oplus \mathbb{R}_0^{n_2}), \\ \mathcal{V}_3 &= \mathrm{End}(\mathbb{R}_2^{n_2}) \oplus \mathrm{Hom}(\mathbb{R}_2^{n_2}, \mathbb{R}^{n_1}) \oplus \mathrm{Hom}(\mathbb{R}^{n_1}, \mathbb{R}_2^{n_2}), \\ \mathcal{V}_4 &= \mathrm{End}(\mathbb{R}_2^{n_2}). \end{aligned}$$

In other words $\mathcal{V}_i$ is a direct sum of the blocks containing the number i in the diagram below:

	M_0	N_0	N_2	N_1
M_0	0	0	2	1
N_0	0	0	2	1
N_2	2	2	0, 1, 2, 3, 4	1, 2, 3
N_1	1	1	1, 2, 3	0, 1, 2

In order to be able to compute equation (3.21) for the case at hand (5.1) we need to have an explicit form of the generic symmetric, positive definite isotropic tensor L_0 and the rotationally invariant subspace $\mathcal{A}$ attached to L_0 that incorporates all the physics of the problem. We have

$$L_0 = \begin{pmatrix} \Lambda_{11} \otimes \omega_{11} & \Lambda_{12} \otimes \omega_{12} & 0 & 0 \\ \Lambda_{12}^T \otimes \omega_{12}^T & \Lambda_{22} \otimes \omega_{22} & 0 & 0 \\ 0 & 0 & \Lambda_{33} \otimes \omega_{33} & 0 \\ 0 & 0 & 0 & \Lambda_{44} \otimes \omega_{44} \end{pmatrix},$$

where Λ_{ij} are matrices of appropriate dimensions, symmetric if $i = j$, that make L_0 positive definite. The tensors ω_{ij} will always be implied but omitted from our notation of isotropic tensors. They are defined as follows. $\omega_{11} = \overline{\omega} \otimes \overline{\omega}$, $\omega_{12} = \hat{\omega} \otimes \overline{\omega}$, $\omega_{22} = \hat{\omega} \otimes \hat{\omega}$. $\omega_{33} = I_{N_2}$, and $\omega_{44} = I_{N_1}$.

The subspace $\mathcal{A}$ attached to L_0 is

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & A_{13} & 0 \\ 0 & 0 & A_{23} & 0 \\ A_{13}^T & A_{23}^T & A_{33} & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{3}} I_{n_1} \end{pmatrix} \otimes W_2 \oplus \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes W_4, \tag{5.13}$$

indicating that $\mathcal{A}$ is a direct sum of two irreducibles isomorphic to W_2 and W_4 respectively and whose homogeneous coordinates are given by the matrices above. In (5.13) A_{13}, A_{23}, A_{33} and G depend on Λ_{ij}. It is easier to give explicit formulae for A_{ij} and G in terms of L_{ij} block-components of $L_0^{1/2}$. Let

$$H = \Lambda_{33}^{1/2}(\Lambda_{22} + 2\Lambda_{33})^{-1}\Lambda_{33}^{1/2} = L_{33}[L_{12}^T L_{12} + L_{22}^2 + 2L_{33}^2]^{-1} L_{33}. \tag{5.14}$$

Then

$$A_{13} = L_{12} L_{33}^{-1} H, \qquad A_{23} = L_{22} L_{33}^{-1} H,$$

$$A_{33} = 2\sqrt{3}(I_{n_2} + 2H)/7, \qquad G = 3H - 2I_{n_2}.$$

Now we need to learn how to multiply the irreducibles in $\mathrm{End}(\mathcal{W})$. The theorem below describes the product $(X \otimes W_\alpha)(Y \otimes W_\beta)$ in terms of Racah (or 6-j) coefficients [11].

Theorem 5.6 *Let x_{jk} and y_{jk} be the homogeneous coordinates of $X \otimes W_\alpha$ and $Y \otimes W_\beta$ respectively. Then $(X \otimes W_\alpha)(Y \otimes W_\beta)$ contains no more than one copy of W_γ for each*

$|\alpha-\beta| \leq \gamma \leq \alpha+\beta$, *whose homogeneous coordinates are given by* z_{jk}*:*

$$z_{jm} = \sum_{k=0}^{n} x_{jk} y_{km} R^{\alpha\beta\gamma}_{jkm}, \tag{5.15}$$

where $R^{\alpha\beta\gamma}_{jkm}$ *are Racah coefficients. The irreducible* W_γ *does not appear if* $z_{jm} = 0$ *for all* j *and* m*.*

PROOF: The first part of the proof will develop a multiplication rule for "standard" irreducibles. These are the irreducibles contained in $\mathrm{Hom}(W_k, W_j)$. The Racah coefficients make their appearance there. Let us take $W_\alpha \subset \mathrm{Hom}(W_k, W_j)$ and a $W_\beta \subset \mathrm{Hom}(W_m, W_k)$ then $W_\alpha W_\beta \subset \mathrm{Hom}(W_m, W_j)$. As in Example 5.1 we observe that the linear map ϕ from $W_\alpha \otimes W_\beta$ into $W_\alpha W_\beta$ defined by the rule $\phi(a \otimes b) = ab$ is surjective and commutes with rotations. Therefore, $W_\alpha W_\beta$ contains at most a single copy of each irreducible W_γ, and if it does, then W_γ would have a Cartan basis $\{h_C : \; |C| \leq \gamma\}$:

$$h_C = \sum_A C^{\alpha,\beta,\gamma}_{C,A} e_A(j,k;\alpha) e_{C-A}(k,m;\beta) \tag{5.16}$$

according to (5.9). However W_γ appears only once in $\mathrm{Hom}(W_m, W_k)$. Therefore, it has another Cartan basis $\{e_C(k,m;\gamma) : \; |C| \leq \gamma\}$. As we have mentioned before, any two Cartan bases must be related by a complex multiple. This multiple is real in our case, since both bases have real zero weight vectors, and depends on j, k, m, α, β and γ. We call it $R^{\alpha\beta\gamma}_{jkm}$:

$$h_C = R^{\alpha\beta\gamma}_{jkm} e_C(k,m;\gamma). \tag{5.17}$$

Substituting (5.9) in (5.16) and using (5.8) we obtain

$$R^{\alpha\beta\gamma}_{jkm} C^{jm\gamma}_{C,B} = \sum_A \frac{(-1)^{A-B}}{\binom{2k}{B-A+k}} C^{\alpha\beta\gamma}_{C,A} C^{jk\alpha}_{A,B} C^{km\beta}_{C-A,B-A}. \tag{5.18}$$

Etingof and Sage independently observed that these numbers $R^{\alpha\beta\gamma}_{jkm}$ are in fact Racah or 6-j coefficients; for a proof, see [48]. However, for our purposes the formula (5.18) gives a definition and a practical way to evaluate these numbers, since we are dealing with irreducible representations of $SO(3)$ of weight no greater than four. (For large weight representations the formula (5.18) is no longer practical.)

Using the formulas (5.16) and (5.17) we easily derive the multiplication rule (5.15). Let $Z \otimes W_\gamma \subset (X \otimes W_\alpha)(Y \otimes W_\beta)$. Then a Cartan basis for $Z \otimes W_\gamma$ is given by

$$e^\gamma_C(Z) = \sum_A C^{\alpha\beta\gamma}_{C,A} e^\alpha_A(X) e^\beta_{C-A}(Y),$$

or using (5.10),

$$e^\gamma_C(Z) = \sum_{j,k,m=0}^{n} x_{jk} y_{km} \sum_A C^{\alpha\beta\gamma}_{C,A} e_A(j,k;\alpha) e_{C-A}(k,m;\beta).$$

Applying (5.16) and (5.17) we obtain

$$e_C^{\gamma}(Z) = \sum_{j,m=0}^{n} \left\{ \sum_{k=0}^{n} x_{jk} y_{km} R_{jkm}^{\alpha\beta\gamma} \right\} e_C(j,m;\gamma)$$

and (5.15) follows. ∎

5.3 Computing exact relations for coupled problems.

In this subsection we will focus exclusively on the situation where we have n_1 conductivity and n_2 elasticity problems all coupled together with n_0 uniform fields, so that the tensor space $\mathcal{T}$ is given by (5.1) and $\mathcal{A}$ is given by (5.13). First, we apply the covariance principle developed in Section 4.2 to simplify $\mathcal{A}$.

An arbitrary isotropic tensor $C \in \mathrm{End}(\mathcal{T})$ is given by

$$C = \begin{pmatrix} C_{11} & C_{12} & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 \\ 0 & 0 & C_{33} & 0 \\ 0 & 0 & 0 & C_{44} \end{pmatrix},$$

We consider only invertible C's of the above form. We can choose

$$\begin{aligned} &C_{11} = (L_0^{-1/2})_{11} = [L_{11} - L_{12} L_{22} L_{12}^T]^{-1},\ C_{12} = -C_{11} A_{13} A_{23}^{-1}, \\ &C_{21} = 0,\ C_{22} = (A_{23} C_{33}^T)^{-1},\ C_{33} = R(I + 2H)^{-1/2},\ C_{44} = I_{n_1}, \end{aligned} \tag{5.19}$$

where the orthogonal matrix R diagonalizes H, given by (5.14). Then

$$\mathcal{A}_0 = C\mathcal{A}C^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & I_{n_2} & \frac{2\sqrt{3}}{7} I_{n_2} & 0 \\ 0 & 0 & 0 & -\dfrac{1}{\sqrt{3}} I_{n_1} \end{pmatrix} \otimes W_2 \oplus \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & G' & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes W_4,$$

where $G' = (I_{n_2} + 2D_H)^{-1}(3D_H - 2I_{n_2})$ and $D_H = RHR^T$ is the diagonal form of H. We remark that G' is determined by the eigenvalues of H, or of $\Lambda_{33}\Lambda_{22}^{-1}$.

Our strategy is to find all subspaces Π_0 solving a simplified equation

$$(\Pi \mathcal{A}_0 \Pi)_{\mathrm{sym}} \subset \Pi. \tag{5.20}$$

There are two possible approaches. One is to try to answer the question using algebra, as was done for $n_2 = n_0 = 0$ and $n_1 = n$ in Example 5.1. The other was employed for finding subspaces corresponding to exact relations for thermo-piezo-electric composites. Unfortunately, the algebra needed for the former approach is not very well understood at present. Therefore, we proceed to describe the latter.

We have written a simple Maple program that computes $(\Pi_1\mathcal{A}\Pi_2)_{\rm sym}$ using Theorem 5.6. Then we take an irreducible $\Pi = X\otimes W_i$ and find the exact relation it generates by adjoining the new output of $(\Pi\mathcal{A}\Pi)_{\rm sym}$ to Π and applying the program again until we obtain an exact relation. This way we get a certain set of "minimal" exact relations generated by irreducibles in $\mathrm{Sym}(\mathcal{T})$. Then we combine two minimal exact relations Π_1 and Π_2 to see what $\Pi_1+\Pi_2$ generates. For that purpose it is sufficient to compute only $(\Pi_1\mathcal{A}\Pi_2)_{\rm sym}$, since Π_1 and Π_2 already satisfy (3.21). Thus we obtain all solutions of (3.21). Next we eliminate all intersections of exact relations. If we have Π_1 and Π_2, it is not necessary to keep $\Pi_1\cap\Pi_2$. Then we eliminate all physically trivial exact relations (the ones corresponding to uncoupled problems that split in the sense of Definition 4.15, etc.). The remaining relations are then sorted according to the physical context they really belong to. For example, when searching for exact relations for piezo-electricity, we necessarily pick up all elastic exact relations. After we have weeded and organized all exact relations we check if they satisfy the sufficient conditions. We modify our Maple program, so that it computes $\mathcal{L} = (\Pi\mathcal{A}\Pi)_{\rm skew}$ and then check, with our first Maple program if $\mathcal{L}$ satisfies the two conditions of Theorem 3.6. This plan works extremely well in the context of thermo-piezo-electricity, but might need to be combined with some algebraic analysis or even replaced by new tricks for coupled problems of larger size, as for example for two or more coupled elasticity problems.

Suppose now that (3.21) has been solved. The next step is to establish the stability under homogenization. Our observation is that it is enough to check stability under homogenization for a corresponding solution Π_0 of the simplified equation (5.20). The reason for this is that the Jordan isomorphism $\phi(X) = C^TXC$ has the property

$$\phi(X\mathcal{A}_0Y) = \phi(X)\mathcal{A}\phi(Y),$$

for any $\{X, Y\}\subset \mathrm{End}(\mathcal{T})$. Therefore, if a subspace Π_0 satisfies conditions of Theorem 3.6 then so does $\Pi = \phi(\Pi_0)$.

Another remark is that stability under homogenization is a property of an entire equivalence class under covariance transformation. In other words if the two exact relations passing through isotropic tensors $L_0^{(1)}$ and $L_0^{(2)}$ are related by a covariance transformation then they are simultaneously stable or unstable under homogenization. The statement follows from the remark below.

Remark 5.7 *Suppose $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ correspond to reference media $L_0^{(1)}$ and $L_0^{(2)}$, and suppose (4.8) holds for some isotropic C_0. Then there is $C\in\mathcal{C}$, defined in (4.9) such that*

$$\tilde{\Gamma}_1 = C\tilde{\Gamma}_2C^T. \tag{5.21}$$

PROOF: We observe that the property (3.3) implies that $\tilde{\Gamma}_i$, $i = 1, 2$ are weight vectors of weight 0. They generate rotationally invariant subspaces $\mathcal{A}_1$ and $\mathcal{A}_2$ each of which is a direct sum of two irreducibles isomorphic to W_2 and W_4. Therefore, we can split $\tilde{\Gamma}_i$, $i = 1, 2$, into the sum of orthogonal projections onto the irreducibles:

$$\tilde{\Gamma}_i = g_2^{(i)} + g_4^{(i)}, \quad i = 1, 2$$

If we further decompose $g_2^{(i)}$ and $g_4^{(i)}$ in a Cartan basis

$$g_2^{(i)} = \sum_{j,k=0}^{2} X_{jk}^{(i)} e_0(j,k;2), \quad g_4^{(i)} = \sum_{j,k=0}^{2} Y_{jk}^{(i)} e_0(j,k;4)$$

then we observe that both $X_{11}^{(i)}$ and $Y_{22}^{(i)}$ are negative definite for all $i = 1,2$. In fact $(X^{(i)}, Y^{(i)})$ are the homogeneous coordinates of $\mathcal{A}_i$, $i = 1,2$. Thus if $\mathcal{A}_1$ and $\mathcal{A}_2$ are related by a covariance transformation then so are $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$. ∎

A corollary of this is the following. If a subspace Π_1 satisfies (3.38) with $A_1(\boldsymbol{n})$ then $\Pi_2 = \phi(\Pi_1)$ satisfies (3.38) with $A_2(\boldsymbol{n})$. In other words, if ϕ preserves all surfaces closed under lamination it will also preserve those of them that are closed under homogenization. This gives us the possibility of establishing stability under homogenization of ϕ-images of exact relations.

We conclude this section with an example of 3D elasticity.

Example 5.8

$\mathcal{T} = N_0 \oplus N_2$ and $\mathrm{Sym}(\mathcal{T}) \cong W_0^2 \oplus W_2^2 \oplus W_4$. A generic isotropic tensor L_0 is characterized by the bulk and shear moduli κ and μ. In our system of notation

$$L_0 = \begin{pmatrix} 3\kappa & 0 \\ 0 & 2\mu \end{pmatrix}.$$

Then

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 1 & q \end{pmatrix} \otimes W_2 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes W_4,$$

where

$$q = \frac{\sqrt{2}}{7} \frac{3\kappa + 8\mu}{\sqrt{\kappa\mu}}.$$

Let us take a single irreducible subrepresentation in $\mathrm{Sym}(\mathcal{T})$ isomorphic to W_4:

$$\Pi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes W_4.$$

An easy calculation using Theorem 5.6 shows that

$$(\Pi\mathcal{A}\Pi)_{\mathrm{sym}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (W_0 \oplus W_2 \oplus W_4).$$

Thus if an exact relation Π contains a representation W_4 it must also contain the right hand side of the formula above. It is easy to check that the expression above is an exact relation. In fact it is a UFR given by $\mathrm{Ann}(N_0)$. It corresponds to the H-LCF exact relation described in Example 4.7.

Let

$$\Pi = \begin{pmatrix} 0 & a \\ a & 1 \end{pmatrix} \otimes W_2,$$

where $a \neq 0$. Then $(\Pi\mathcal{A}\Pi)_{\rm sym} = \mathrm{Sym}(\mathcal{T})$. If $a = 0$ then $(\Pi\mathcal{A}\Pi)_{\rm sym} =$ H-LCF. Now, if

$$\Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes W_2,$$

then

$$(\Pi\mathcal{A}\Pi)_{\rm sym} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes W_0 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes W_2. \tag{5.22}$$

Again it is easily checked that the last subspace is an exact relation. We will show that it corresponds to the "rank-one tensor plus a null-Lagrangian" exact relation, see [21, 22]. We will call it RPN.

Finally, we try

$$\Pi = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \otimes W_0,$$

where $a \neq 0$. Then

$$(\Pi\mathcal{A}\Pi)_{\rm sym} = \begin{pmatrix} 0 & a/q \\ a/q & 1 \end{pmatrix} \otimes W_2 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes W_4.$$

Thus, if an exact relation Π contains $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \otimes W_0$, then it must contain H-LCF *and* RPN. But the two exact relations add up to $\mathrm{Sym}(\mathcal{T})$. If $a = 0$ then

$$(\Pi\mathcal{A}\Pi)_{\rm sym} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes W_2 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes W_4.$$

Therefore, the minimal exact relation containing $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes W_0$ is H-LCF. The last possibility is

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes W_0. \tag{5.23}$$

We easily find that $(\Pi\mathcal{A}\Pi)_{\rm sym} = 0$ in this case. Thus Π is an exact relation. This is a UFR corresponding to $\mathrm{Ann}(N_2)$. This exact relation is due to Hill [25, 26], we will refer to it as HL. We have obtained a complete list of exact relations: H-LCF, RPN and HL. We observe that HL is a subset of RPN.

Now we address stability under homogenization. From our general theory the UFR are always stable under homogenization. This leaves only RPN. We find that for $\Pi=\mathrm{RPN}$

$$\mathcal{L}=(\Pi\mathcal{A}\Pi)_{\mathrm{skew}}=\begin{pmatrix} & 0 & 1\\ -1 & 0\end{pmatrix}\otimes W_2.$$

Then

$$(\mathcal{L}\mathcal{A}\Pi)_{\mathrm{sym}}=\mathrm{RPN}$$

and

$$(\mathcal{L}\mathcal{A}\mathcal{L})_{\mathrm{sym}}=\mathrm{RPN}.$$

So the conditions of Theorem 3.6 are satisfied and RPN is stable under homogenization.

This detailed information about exact relations for 3D elasticity allows us to describe all possible links between the two uncoupled elasticity problems in the sense of Section 4.3. We need to identify all pairs $(\Pi,\mathcal{K})$ of Jordan algebras and their Jordan ideals according to Theorem 4.17. This is very easy to do. The Jordan algebra $\mathrm{Sym}(\mathcal{T})$ is simple, so the only ideal $\mathcal{K}\neq\mathrm{Sym}(\mathcal{T})$ is $\mathcal{K}=\{0\}$. The same $\mathcal{K}$ is the only possibility for $\Pi=\mathrm{HL}$ or $\Pi=$H-LCF. A simple calculation shows that $((\mathrm{HL})\mathcal{A}(\mathrm{RPN}))_{\mathrm{sym}}=0$; therefore, HL is an ideal in RPN. This gives us five possibilities listed in the table below

Π	$\mathcal{K}$	$\mathcal{F}=\Pi/\mathcal{K}$
$\mathrm{Sym}(\mathcal{T})$	$\{0\}$	$W_0^2\oplus W_2^2\oplus W_4$
H-LCF	$\{0\}$	$W_0\oplus W_2\oplus W_4$
HL	$\{0\}$	W_0
RPN	$\{0\}$	$W_0\oplus W_2$
RPN	HL	W_2

In the third column we have listed the isomorphism classes of $\mathcal{F}$'s as representations of $SO(3)$. We see that each entry in the third column is different. Therefore, the only possibility is $\Pi_1=\Pi_2$ and $\mathcal{K}_1=\mathcal{K}_2$. Schur's lemma in group representation theory can be used to describe all linear maps $\phi:\mathcal{F}\to\mathcal{F}$ that commute with the action of the group. In our case ϕ is just a linear combination of projections onto the irreducible subrepresentations of $\mathcal{F}$. But, according to Theorem 4.17, the map ϕ has to preserve the Jordan multiplication in $\mathcal{F}$. This places a severe constraints on ϕ. Examining all five cases closely we easily show that ϕ can only be a Milgrom-Shtrikman covariance transformation. This results in a single non-trivial link: $\Pi_1=\Pi_2=$RPN and $\mathcal{K}_1=\mathcal{K}_2=\mathrm{HL}$.

All the exact relations above did not involve volume fractions. We can find out which exact relations may involve volume fractions by applying Theorem 4.19 to our situation. We need to find an exact relation Π such that $\mathcal{K}=(\Pi\mathcal{A}\Pi)_{\mathrm{sym}}\neq\Pi$. We easily identify the item: $\Pi=\mathrm{HL}$ and $\mathcal{K}=\{0\}$.

6 How to convert Π into $\mathbb{M}$.

Having found all subspaces Π satisfying (3.21) and having verified that they satisfy all conditions of Theorem 3.6 we need to return to the physical variables L and present our exact relations in the beautiful form of Section 2. The naive answer to the title question is to try to invert the equation (3.16):

$$L = L_0 - L_0^{1/2}[\mathcal{I} + K\Gamma']^{-1}KL_0^{1/2}, \quad K \in \Pi.$$

However the computation of $[\mathcal{I}+K\Gamma']^{-1}K$ may be complicated because Γ' is not particularly simple in most cases. It turns out that it is possible to replace Γ' by a much simpler tensor.

Proposition 6.1 *Let $M \in Sym(\mathcal{T})$ be such that $K(\overline{\Gamma} - M)K \in \Pi$ for all $K \in \Pi$, where $\overline{\Gamma}$ is the isotropic part of Γ'. Then*

$$\mathbb{M} = \{L = L_0 - L_0^{1/2}[\mathcal{I} + KM]^{-1}KL_0^{1/2}, \quad K \in \Pi\}. \tag{6.1}$$

We remark that $M = \overline{\Gamma}$ satisfies conditions of the proposition. In our examples from Section 2 we were able to find even simpler choices for M. In particular, when an exact relation $\mathbb{M}$ happens to be an affine subspace, then $M = 0$ is the best choice. In all other cases $M = m(I \otimes I)$ was the choice we actually used for suitable value of m. We employed a simple Maple program that finds the best M for a given subspace Π.

PROOF: Suppose M satisfies conditions of the proposition. Then $K(\Gamma' - M)K \in \Pi$ for all $K \in \Pi$. Indeed

$$K(\Gamma' - M)K = K(\Gamma' - \overline{\Gamma} + \overline{\Gamma} - M)K = K\tilde{\Gamma}K + K(\overline{\Gamma} - M)K \in \Pi$$

by Theorem 4.1. As we have done in the proof of Theorem 3.5 we observe that (3.21) and (4.1) imply that

$$[\mathcal{I} + K(\Gamma' - M)]^{-1}K = K - K(\Gamma' - M)K + \ldots + (-1)^n K((\Gamma' - M)K)^n + \ldots \in \Pi$$

for any $K \in \Pi$. Therefore $f(K) = [\mathcal{I}+K(\Gamma'-M)]^{-1}K$ maps Π onto Π. Now, let us compute the function $f(W(L))$ that maps $\mathbb{M}$ into Π.

$$f(W(L)) = [S(L) - \Gamma' + \Gamma' - M]^{-1} = [S(L) - M]^{-1}.$$

Solving $K = f(W(L))$ for L we obtain the statement of the proposition. ∎

Remark 6.2 *A simple corollary of Theorem 5.2 is that $(\Pi^2)_{\text{sym}} = \Pi$ for any exact relation Π for n coupled conductivity problems. In view of Proposition 6.1 we conclude that $M = 0$, works and all exact relations have the form $L = L_0 - L_0^{1/2}KL_0^{1/2}$, $K \in \Pi$.*

We can simplify (6.1) further by doing a covariance transformation the same way we have simplified (3.21) (reducing it to (5.20)).

Proposition 6.3 *Let C be given by 5.19. Then*

$$\overline{\Gamma}_0 = C\overline{\Gamma}C^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & D_H^{-1} + 2I_{n_2} & 0 & 0 \\ 0 & 0 & \frac{2}{5}(I_{n_2} + 2D_H)^{-1}(I_{n_2} + D_H) & 0 \\ 0 & 0 & 0 & \frac{2}{3}I_{n_1} \end{pmatrix}.$$

Suppose that M is such that $K(\overline{\Gamma}_0 - M)K \in \Pi_0$ for any $K \in \Pi_0$. Then $\mathbb{M} = W^{\rm inv}(\Pi)$ is given by

$$\mathbb{M} = \{L = L_0 - \tilde{L}^T[\mathcal{I} + KM]^{-1}K\tilde{L}, \ \ K \in \Pi_0\}, \tag{6.2}$$

where

$$\tilde{L} = CL_0^{1/2} = \begin{pmatrix} I_{n_0} & 0 & 0 & 0 \\ C_{22}L_{12}^T & C_{22}L_{22} & 0 & 0 \\ 0 & 0 & C_{33}L_{33} & 0 \\ 0 & 0 & 0 & L_{44} \end{pmatrix}.$$

Proof: The formulas for Γ', $\tilde{\Gamma}$, $\overline{\Gamma}$ and $\mathcal{A}$ were obtained by a straightforward but tedious calculation. An elementary calculation shows that $K(\overline{\Gamma}_0 - M)K \in \Pi_0$ if and only if $\phi(K)(\overline{\Gamma} - M')\phi(K) \in \Pi = \phi(\Pi_0)$, where M' is defined by $M = CM'C^T$. Thus, if M satisfies the above condition, then we have, according to Proposition 6.1

$$L = L_0 - L_0^{1/2}[\mathcal{I} + KM']^{-1}KL_0^{1/2}, \ \ K \in \Pi.$$

An elementary calculation gives (6.2). ∎

Now we show how we use Proposition 6.1 for 3D elasticity, where we have already computed all solutions Π. The Proposition 6.3 was used to compute exact relations for thermo-piezo-electric composites.

Example 6.4

The key for converting Π into $\mathbb{M}$ is having a nice representation of Π as a subspace of $\mathrm{Sym}(\mathcal{T})$. In all of our examples it was fairly easy to do. In the case of RPN for 3D elasticity (see (5.22)) we have

$$\mathrm{RPN} = \{I \otimes B + B \otimes I : B \in \mathcal{T} = N_0 \oplus N_2\},$$

where I is the three by three identity matrix (it is a basis vector for N_0). We are looking for an exact relation $\mathbb{M}$ corresponding to RPN and passing through the isotropic tensor L_0.

We can use Proposition 6.1 with

$$M = \frac{\kappa}{3\kappa + 4\mu} I \otimes I \tag{6.3}$$

to find it. Our choice of M is a better one than $\overline{\Gamma}$ (in fact M is a hydrostatic part of $\overline{\Gamma}$). An explicit calculation according to formula (6.1) shows that

$$\begin{aligned} L &= L_0 - L_0^{1/2}\left\{\mathcal{I} + \frac{\kappa}{3\kappa + 4\mu}[(\mathbf{Tr}B)I + 3B] \otimes I\right\}^{-1}(I \otimes B + B \otimes I)L_0^{1/2} \\ &= 2\mu(\mathcal{I} - I \otimes I) + \frac{[\sqrt{6\kappa\mu}(3B - (\mathbf{Tr}B)I) - (3\kappa + 4\mu)I]^{\otimes 2}}{3(3\kappa + 4\mu + 6\kappa\mathbf{Tr}B)}, \end{aligned}$$

where $\mathcal{I}$ is the identity operator on $\mathcal{T}$ (as opposed to I which is an element of $\mathcal{T}$) and $X^{\otimes 2}$ stands for $X \otimes X$. A necessary condition for L to be positive definite is that

$$3\kappa + 4\mu + 6\kappa \mathbf{Tr} B > 0. \tag{6.4}$$

Let

$$A = [\sqrt{6\kappa\mu}(3B - (\mathbf{Tr}B)I) - (3\kappa + 4\mu)I]/\sqrt{3(3\kappa + 4\mu + 6\kappa\mathbf{Tr}B)}, \tag{6.5}$$

then the associated surface $\mathbb{M}$, will consists all positive definite elasticity tensors L expressible in the form

$$L = 2\mu T + A \otimes A.$$

The positive definiteness of L is equivalent to $(\mathbf{Tr}A)^2 - 2\mathbf{Tr}(A^2) > 4\mu$, or

$$2\mathbf{Tr}B + 3\mathbf{Tr}(B - \frac{1}{3}(\mathbf{Tr}B)I)^2 < \frac{3\kappa + 4\mu}{4\mu}. \tag{6.6}$$

We conclude this section with a discussion of the question of incorporating the volume fraction information in an exact relation formula. Theorem 4.19 tells us that an exact relation Π admits a volume fraction sharpening if and only if $(\Pi\mathcal{A}\Pi)_{\text{sym}} \neq \Pi$.

THEOREM **6.5** *Let $\mathcal{K} = (\Pi\mathcal{A}\Pi)_{\text{sym}} \neq \Pi$ and let $\mathcal{N}$ be an orthogonal complement of $\mathcal{K}$ in Π. Then we have*

$$\mathcal{P}_{\mathcal{N}} W_M(L^*) = \mathcal{P}_{\mathcal{N}} \langle W_M(L(\boldsymbol{x}))\rangle, \tag{6.7}$$

where $P_{\mathcal{N}}$ denotes the orthogonal projection onto $\mathcal{N}$ and M satisfies conditions of Proposition 6.1.

PROOF: Suppose $[L, f(L)]$ is a link between the two uncoupled problems. The second one corresponding to $\mathcal{A} = 0$. Then we can choose $M' = [M, 0]$ satisfying conditions of Proposition 6.1 and $\Pi' = [W_M(L), \mathcal{I} - L_0^{-1/2} f(L) L_0^{-1/2}]$ is the subspace corresponding to the link. Thus

$$\mathcal{I} - L_0^{-1/2} f(L) L_0^{-1/2} = \mathcal{P}_{\mathcal{N}}(W_M(L)).$$

Therefore, the volume fraction relation $f(L^*) = \langle f(L(\boldsymbol{x}))\rangle$ from Theorem 4.19 can be written as (6.7). ∎

Example 6.6

For Hill's exact relation (5.23) we have

$$L(\boldsymbol{x}) = 2\mu_0 I_{N_2} + \kappa(\boldsymbol{x}) I \otimes I.$$

An easy calculation gives for $W_M(L(\boldsymbol{x}))$ with M as in (6.3)

$$W_M(L(\boldsymbol{x})) = c\frac{\kappa_0 - \kappa(\boldsymbol{x})}{4\mu_0 + 3\kappa(\boldsymbol{x})}(I \otimes I),$$

where $c = (3\kappa_0 + 4\mu_0)/(3\kappa_0)$. The formula (6.7) then becomes

$$\langle \frac{\kappa_0 - \kappa(\boldsymbol{x})}{4\mu_0 + 3\kappa(\boldsymbol{x})}\rangle = \frac{\kappa_0 - \kappa^*}{4\mu_0 + 3\kappa^*},$$

which is equivalent to (2.2).

A The maps W_n are analytic diffeomorphisms.

Theorem **A.1** *Let Γ be an orthogonal projector onto a subspace $\mathcal{E}$ of $\mathcal{T}$. The map*

$$W(L) = [(I-L)^{-1} - \Gamma]^{-1}$$

defined on a dense open subset of Sym$^+(\mathcal{T})$ can be extended to all of Sym$^+(\mathcal{T})$ by continuity. Moreover, the extended map is an analytic diffeomorphism between Sym$^+(\mathcal{T})$ and its image under W.

Proof: Observe, that on a dense subset, where all the inverses in the definition of W exist, we can rewrite the formula for W as follows:

$$W(L) = [I-(I-L)\Gamma]^{-1}(I-L) = (I-L)[I-\Gamma(I-L)]^{-1}. \tag{A.8}$$

This formula extends W to all of $\mathrm{Sym}^+(\mathcal{T})$. Indeed, the operator $A = I-(I-L)\Gamma$ is always invertible. In order to see this we write $A = \Gamma^{\perp} + L\Gamma$, where $\Gamma^{\perp}$ is the orthogonal projection onto $\mathcal{E}^{\perp}$. Suppose that there is $t \in \mathcal{T}$ such that $At = 0$. Writing t as the sum of a vector $e \in \mathcal{E}$ and a vector $e' \in \mathcal{E}^{\perp}$ we get $0 = At = Le + e'$. Taking the inner product with e we obtain that $(Le, e) = 0$, which implies that $e = 0$. Thus $e' = 0$ as well. So, the operator A is invertible and W can be extended by continuity to all of $\mathrm{Sym}^+(\mathcal{T})$ via (A.8).

Now we will show that W is injective, i.e. W maps distinct operators into distinct operators. Suppose that $W(L_1) = W(L_2)$. Then

$$[I-(I-L_2)\Gamma](I-L_1) = (I-L_2)[I-\Gamma(I-L_1)].$$

Eliminating parentheses we obtain $L_1 = L_2$. The inverse map W^{inv} is given by

$$W^{\mathrm{inv}}(K) = I - [I+K\Gamma]^{-1}K \tag{A.9}$$

Indeed, if K is in the image of $\mathrm{Sym}^+(\mathcal{T})$ under W, then $K = A^{-1}(I-L)$, where $A = \Gamma' + L\Gamma$ for some $L \in \mathrm{Sym}^+(\mathcal{T})$. An easy calculation shows that $I + K\Gamma = A^{-1}$, and therefore, invertible. Thus,

$$W^{\mathrm{inv}}(K) = I - A(A^{-1}(I-L)) = L.$$

Since both formulas (A.8) and (A.9) define analytic maps, we may conclude that W is an analytic diffeomorphism. ∎

If in the theorem above we set $\Gamma = \Gamma'(\boldsymbol{n})$ and replace L by $L_0^{-1/2}LL_0^{-1/2}$, where the reference medium L_0 is assumed to be symmetric and positive definite, then we conclude that the functions $W_{\mathbf{n}}$ are analytic diffeomorphism between $\mathrm{Sym}^+(\mathcal{T})$ and $W_{\mathbf{n}}(\mathrm{Sym}^+(\mathcal{T}))$ for every unit vector $\boldsymbol{n}$. Even though we do not know a precise description of the image of $W_{\mathbf{n}}$ we may observe that $W_{\mathbf{n}}(\mathrm{Sym}^+(\mathcal{T}))$ is open by the theorem above and convex by Theorem 3.1.

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References

[1] M. Avellaneda, A. V. Cherkaev, L. V. Gibiansky, G. W. Milton, and M. Rudelson. A complete characterization of the possible bulk and shear moduli of planar polycrystals. *J. Mech. Phys. Solids*, 44(7):1179–1218, 1996.

[2] G. E. Backus. Long-wave elastic anisotropy produced by horizontal layering. *J. Geophys. Res.*, 67:4427–4440, 1962.

[3] B. Y. Balagurov. Isomorphism of certain problems of percolation theory. *Zh. Eksper. Teoret. Fiz.*, 85(2):568–584, 1983.

[4] Y. Benveniste. Exact results in the micromechanics of fibrous piezoelectric composites exhibiting pyroelectricity. *Proc. Roy. Soc. Lond. A*, 441:59–81, 1993.

[5] Y. Benveniste. Exact connections between polycrystal and crystal properties in two-dimensional polycrystalline aggregates. *Proc. Roy. Soc. Lond. A*, 447:1–22, 1994.

[6] Y. Benveniste. Correspondence relations among equivalent classes of heterogeneous piezoelectric solids under anti-plane mechanical and in-plane electrical fields. *J. Mech. Phys. Solids*, 43(4):553–571, 1995.

[7] Y. Benveniste and G. J. Dvorak. On correspondence between mechanical and thermal effects in two-phase composites. In G. Weng et al., editors, *Micromechanics and Inhomogeneity*, pages 65–80, Berlin, 1990. Springer-Verlag.

[8] Y. Benveniste and G. J. Dvorak. Uniform fields and universal relations in piezoelectric composites. *J. Mech. Phys. Solids*, 40(6):1295–1312, 1992.

[9] V. L. Berdichevsky. *Variational principles of continuum mechanics.* (Russian) Nauka, Moscow, 1983.

[10] D. J. Bergman and D. Stroud. Physical properties of macroscopically inhomogeneous media. *Solid State Phys.*, 46:147–269, 1992.

[11] L. C. Biedenharn and J. D. Louck. *Angular momentum in quantum physics*, volume 8 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981. Theory and application, With a foreword by Peter A. Carruthers.

[12] T. Bröcker and T. tom Dieck. *Representations of Compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.

[13] H. B. Callen. *Thermodynamics*. John Wiley & Sons, Inc., New York, 1960.

[14] J. L. Cribb. Shrinkage and thermal expansion of a two-phase material. *Nature, Lond.*, 220:576–577, 1968.

[15] M. L. Dunn. Exact relations between the thermoelectroelastic moduli of heterogeneous materials. *Proc. Roy. Soc. London Ser. A*, 441(1913):549–557, 1993.

[16] G. J. Dvorak. On uniform fields in heterogeneous media. *Proc. Roy. Soc. London Ser. A*, 431(1881):89–110, 1990.

[17] G. J. Dvorak. On some exact results in thermoplasticity of composite materials. *J. Thermal Stresses*, 15(2):211–228, 1992. Sixty-fifth Birthday of Bruno A. Boley Symposium, Part 2 (Atlanta, GA, 1991).

[18] A. M. Dykhne. Conductivity of a two-dimensional two-phase system. *Sov. Phys. JETP*, 32:63–65, 1971.

[19] Y. Grabovsky. Exact relations for effective tensors of polycrystals. I: Necessary conditions. *Arch. Ration. Mech. Anal.*, 143(4):309–330, 1998.

[20] Y. Grabovsky and G. W. Milton. Exact relations for composites: Towards a complete solution. *Doc. Math. J. DMV*, Extra Volume ICM III:623–632, 1998.

[21] Y. Grabovsky and G. W. Milton. Rank one plus a null-lagrangian is an inherited property of two-dimensional compliance tensors under homogenization. *Proc. Roy. Soc. Edinburgh, Series A*, 128:283–299, 1998.

[22] Y. Grabovsky and D. S. Sage. Exact relations for effective tensors of polycrystals. II: Applications to elasticity and piezoelectricity. *Arch. Ration. Mech. Anal.*, 143(4):331–356, 1998.

[23] Z. Hashin. Thermal expansion of polycrystalline aggregates: I. Exact results. *J. Mech. Phys. Solids*, 32:149–157, 1984.

[24] J. Helsing, G. W. Milton, and A. B. Movchan. Duality relations, correspondences and numerical results for planar elastic composites. *J. Mech. Phys. Solids*, 45(4):565–590, 1997.

[25] R. Hill. Elastic properties of reinforced solids: Some theoretical principles. *J. Mech. Phys. Solids*, 11:357–372, 1963.

[26] R. Hill. Theory of mechanical properties of fibre-strengthened materials: I. Elastic behaviour. *J. Mech. Phys. Solids*, 12:199–212, 1964.

[27] N. Jacobson. *Structure and representations of Jordan algebras*. American Mathematical Society, Providence, R.I., 1968. American Mathematical Society Colloquium Publications, Vol. XXXIX.

[28] J. B. Keller. A theorem on the conductivity of a composite medium. *J. Math. Phys.*, 5:548–549, 1964.

[29] R. S. Lakes. Cellular solid structures with unbounded thermal expansion. *J. Mater. Sci. Lett.*, 15:475–477, 1996.

[30] R. S. Lakes. Dense solid microstructures with unbounded thermal expansion. *J. Mech. Behav. Mts.*, 7:85–92, 1996.

[31] S. Lang. *Algebra.* Addison-Wesley Publishing Co., Reading, Mass., third edition, 1993.

[32] N. Laws. On thermostatics of composite materials. *J. Mech. Phys. Solids*, 21:9–17, 1973.

[33] V. M. Levin. Thermal expansion coefficients of heterogeneous materials. *MTT*, 2(1):88–94, 1967.

[34] K. A. Lurie and A. V. Cherkaev. Exact estimates of conductivity of composites formed by two isotropically conducting media taken in prescribed proportion. *Proc. Royal Soc. Edinburgh*, 99A:71–87, 1984.

[35] K. A. Lurie and A. V. Cherkaev. G-closure of some particular sets of admissible material characteristics for the problem of bending of thin plates. *J. Opt. Th. Appl.*, 42:305–316, 1984.

[36] K. A. Lurie and A. V. Cherkaev. Effective characteristics of composite materials and the optimal design of structural elements. *Uspekhi Mekhaniki (Advances in Mechanics)*, 9:3–81, 1986.

[37] K. A. Lurie and A. V. Cherkaev. Exact estimates of a binary mixture of isotropic components. *Proc. Royal Soc. Edinburgh*, 104A:21–38, 1986.

[38] K. A. Lurie and A. V. Cherkaev. Effective characteristics of composite materials and the optimal design of structural elements. In A. Cherkaev and R. Kohn, editors, *Topics in the mathematical modeling of composite materials*, volume 31 of *Progress in nonlinear differential equations and their applications*, pages 175–258. Birkhäuser, Boston, 1997.

[39] K. A. Lurie, A. V. Cherkaev, and A. V. Fedorov. On the existence of solutions to some problems of optimal design for bars and plates. *J. Optim. Th. Appl.*, 42(2):247–281, 1984.

[40] K. S. Mendelson. A theorem on the conductivity of two-dimensional heterogeneous medium. *J. Appl. Phys.*, 46:4740–4741, 1975.

[41] M. Milgrom. Some more exact results concerning multifield moduli of two-phase composites. *J. Mech. Phys. Solids*, 45(3):399–404, 1997.

[42] M. Milgrom and S. Shtrikman. Linear responce of polycrystals to coupled fields: Exact relations among the coefficients. *Phys. Rev. B*, 40(9):5991–5994, 1989.

[43] M. Milgrom and S. Shtrikman. Linear responce of two-phase composites with cross moduli: Exact universal relations. *Phys. Rev. A*, 40(3):1568–1575, 1989.

[44] G. W. Milton. On characterizing the set of possible effective tensors of composites: the variational method and the translation method. *Comm. Pure Appl. Math.*, 43:63–125, 1990.

[45] G. W. Milton. Composites: A myriad of microstructure independent relations. In T. Tatsumi, E. Watanabe, and T. Kambe, editors, *Theoretical and applied mechanics (Proc. of the XIX International Congress of Theoretical and Applied mechanics, Kyoto, 1996)*, pages 443–459. Elsevier, Amsterdam, 1997.

[46] G. W. Milton. *The theory of composites*, volume 6 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, 2002.

[47] B. W. Rosen and Z. Hashin. Effective thermal expansion coefficients and specific heats of composite materials. *Int. J. Engng. Sci*, 8:157–173, 1970.

[48] D. S. Sage. Racah coefficients, subrepresentation semirings, and composite materials. *Adv. in Appl. Math.*, 34(2):335–357, 2005.

[49] K. Schulgasser. Thermal expansion of polycrystalline aggregates with texture. *J. Mech. Phys. Solids*, 35(1):34–42, 1987.

[50] O. Sigmund and S. Torquato. Design of materials with extreme thermal expansion coefficients. *Appl. Phys. Lett.*, 69:3203–3205, 1996.

[51] J. P. Straley. Thermoelectric properties of inhomogeneous materials. *J. Phys. D*, 14:2101–2105, 1981.

[52] L. Tartar. Estimation de coefficients homogénéisés. In *Computing methods in applied sciences and engineering (Proc. Third Internat. Sympos., Versailles, 1977), I,*, volume 704 of *Lecture Notes in Mathematics*, pages 364–373. Springer-Verlag, Berlin, 1979.

[53] L. Tartar. Estimation fines des coefficients homogénéisés. In P. Kree, editor, *Ennio de Giorgi's Colloquium*, pages 168–187, London, 1985. Pitman.

[54] V. V. Zhikov. Estimates for the homogenized matrix and the homogenized tensor. *Russian Math Surveys*, 46(3):65–136, 1991.