Erratum to "Direct approach to the problem of strong local minima in Calculus of Variations", Calc. Var., 29 (2007): 59–83.

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October 2, 2007

The proof of Theorem 5 does not prove the statement in Theorem 5. The problem is that the cut-off functions $\theta_k^r(\boldsymbol{x})$ that vanish on $\partial\Omega$ cannot correctly recover the function $\mathcal{I}(\boldsymbol{x}_0, \boldsymbol{x}_0)$ at the points $\boldsymbol{x}_0 \in \partial\Omega_1$, where the measure $\widetilde{\pi}$ has a non-zero mass. This problem is easily fixed by using cut-off functions $\theta_k^r \in C_0^\infty(B(\boldsymbol{x}_0, r))$ that do not vanish on $\partial\Omega$, if $\boldsymbol{x}_0 \in \partial\Omega$. We require that $\theta_k^r(\boldsymbol{x}) \to \chi_{B(\boldsymbol{x}_0,r)}(\boldsymbol{x})$, as $k \to \infty$ for all $\boldsymbol{x} \in \mathbb{R}^d$. This change corrects the proof of Theorem 5 but makes the arguments in Step 4 in the proof of Theorem 2 (p. 82) invalid, since $\theta_k^r(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x})$ no longer has to vanish on $\partial B_\Omega(\boldsymbol{x}_0, r) \cap \partial\Omega$. This problem is fixed by augmenting the statement of the Decomposition Lemma (Lemma 1). We claim that it is possible to modify sequences $\boldsymbol{z}_n$ and $\boldsymbol{v}_n$ is such a way that they vanish on $\partial\Omega_1$, while satisfying all the other properties required by Lemma 1. Using the modified functions in the consequent analysis leads now to the functions $\theta_k^r(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x})$ that do vanish on $\partial B_\Omega(\boldsymbol{x}_0, r)$.

THEOREM **1** *Suppose the sequence of functions $\boldsymbol{\psi}_n \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ that vanish on $\partial\Omega_1$ (relatively open subset of a C^1 boundary $\partial\Omega$ of Ω) is bounded in $W^{1,2}(\Omega; \mathbb{R}^m)$. Suppose $\alpha_n \to 0$, as $n \to \infty$ is a sequence of positive numbers such that $\alpha_n \boldsymbol{\psi}_n \to \boldsymbol{0}$ in $W^{1,\infty}$ weak-*. Suppose $\boldsymbol{z}_n$ and $\boldsymbol{v}_n$ are as in Lemma 1, i.e.*

(a) $\boldsymbol{\psi}_n(\boldsymbol{x}) = \boldsymbol{z}_n(\boldsymbol{x}) + \boldsymbol{v}_n(\boldsymbol{x})$;

(b) For all $\boldsymbol{x} \in \Omega \setminus R_n$ *we have* $\boldsymbol{z}_n(\boldsymbol{x}) = \boldsymbol{\psi}_n(\boldsymbol{x})$ *and* $\nabla \boldsymbol{z}_n(\boldsymbol{x}) = \nabla \boldsymbol{\psi}_n(\boldsymbol{x})$;

(c) The sequence $\{|\nabla \boldsymbol{z}_n|^2\}$ *is equiintegrable;*

(d) $\boldsymbol{v}_n \rightharpoonup 0$ *weakly in* $W^{1,2}(\Omega; \mathbb{R}^m)$;

(e) $|R_n| \to 0$ *as* $n \to \infty$;

(f) $\alpha_n \nabla \boldsymbol{z}_n$ *is bounded in* L^∞;

Then there exists modified versions $\widetilde{\boldsymbol{z}}_n$ and $\widetilde{\boldsymbol{v}}_n$ of $\boldsymbol{z}_n$ and $\boldsymbol{v}_n$, respectively, such that they satisfy all the properties (a)–(f) and additionally the following two properties *(g)* $\alpha_n \widetilde{\boldsymbol{z}}_n \to \mathbf{0}$, *as* $n \to \infty$ *uniformly in* $\boldsymbol{x} \in \Omega$;

(h) $\widetilde{\boldsymbol{z}}_n(\boldsymbol{x}) = \widetilde{\boldsymbol{v}}_n(\boldsymbol{x}) = \boldsymbol{\psi}_n(\boldsymbol{x}) = \mathbf{0}$ *for all* $\boldsymbol{x} \in \partial\Omega_1$.

PROOF: **Step 1.** Let $\boldsymbol{w}_n(\boldsymbol{x}) = (w_n^{(1)}(\boldsymbol{x}), \ldots, w_n^{(m)}(\boldsymbol{x}))$ be defined by

$$w_n^{(i)}(\boldsymbol{x}) = \min\{z_n^{(i)}(\boldsymbol{x}), \|\boldsymbol{\psi}_n\|_{L^\infty(\Omega\setminus R_n)}\}, \quad i = 1, \ldots, m.$$

Then $\boldsymbol{w}_n(\boldsymbol{x})$ is Lipschitz continuous with $|\nabla \boldsymbol{w}_n(\boldsymbol{x})| \le |\nabla \boldsymbol{z}_n(\boldsymbol{x})|$ for a.e. $\boldsymbol{x} \in \Omega$, and $\boldsymbol{w}_n(\boldsymbol{x}) = \boldsymbol{\psi}_n(\boldsymbol{x})$ for almost all $\boldsymbol{x} \in \Omega \setminus R_n$. Similarly, let $\boldsymbol{u}_n(\boldsymbol{x}) = (u_n^{(1)}(\boldsymbol{x}), \ldots, u_n^{(m)}(\boldsymbol{x}))$ be defined by

$$u_n^{(i)}(\boldsymbol{x}) = \max\{w_n^{(i)}(\boldsymbol{x}), -\|\boldsymbol{\psi}_n\|_{L^\infty(\Omega\setminus R_n)}\}, \quad i = 1, \ldots, m.$$

Then $\boldsymbol{u}_n(\boldsymbol{x})$ is Lipschitz continuous with $|\nabla \boldsymbol{u}_n(\boldsymbol{x})| \le |\nabla \boldsymbol{z}_n(\boldsymbol{x})|$ for a.e. $\boldsymbol{x} \in \Omega$, and $\boldsymbol{u}_n(\boldsymbol{x}) = \boldsymbol{\psi}_n(\boldsymbol{x})$ for almost all $\boldsymbol{x} \in \Omega \setminus R_n$. Therefore $|\nabla \boldsymbol{u}_n|^2$ is also equiintegrable and

$$\alpha_n \|\boldsymbol{u}_n\|_\infty \le \alpha_n \|\boldsymbol{\psi}_n\|_\infty \to 0, \text{ as } n \to \infty.$$

Step 2. Let $\boldsymbol{\psi}_0$ be a $W^{1,2}$-weak limit of (a subsequence of) $\boldsymbol{\psi}_n$. Then there exists a sequence $\widehat{\boldsymbol{\psi}}_n \in C^1(\overline{\Omega}; \mathbb{R}^m)$ such that $\widehat{\boldsymbol{\psi}}_n$ vanishes on $\partial\Omega_1$, converges to $\boldsymbol{\psi}_0$ in the $W^{1,2}$ norm and additionally satisfies

$$\lim_{n\to\infty} \alpha_n \|\widehat{\boldsymbol{\psi}}_n\|_{1,\infty} = 0.$$

It follows from $|R_n| \to 0$, as $n \to \infty$ that $\boldsymbol{\psi}_n - \boldsymbol{u}_n \rightharpoonup \mathbf{0}$ weakly in $W^{1,2}$. Thus, by Rellich's lemma,

$$\lim_{n\to\infty} \|\boldsymbol{u}_n - \widehat{\boldsymbol{\psi}}_n\|_2 = 0.$$

Let $\eta_n(\boldsymbol{x})$ be a Lipschitz cut-off function such $0 \le \eta_n(\boldsymbol{x}) \le 1$ and

$$\eta_n(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in \partial\Omega, \\ 0, & \operatorname{dist}(\boldsymbol{x}, \partial\Omega) \ge \delta_n. \end{cases} \tag{1}$$

It is possible to do so, while ensuring that

$$\|\nabla \eta_n\|_\infty \le \frac{C}{\delta_n}, \tag{2}$$

where C depends only on Ω and $\delta_n \to 0$, as $n \to \infty$ so slowly that

$$\lim_{n\to\infty} \frac{\alpha_n \|\boldsymbol{u}_n\|_\infty}{\delta_n} = \lim_{n\to\infty} \frac{\alpha_n \|\widehat{\boldsymbol{\psi}}_n\|_{1,\infty}}{\delta_n} = \lim_{n\to\infty} \frac{\|\boldsymbol{u}_n - \widehat{\boldsymbol{\psi}}_n\|_2}{\delta_n} = 0.$$

Step 3. Let

$$\widetilde{\boldsymbol{z}}_n = (1 - \eta_n)\boldsymbol{u}_n + \eta_n \widehat{\boldsymbol{\psi}}_n.$$

It is obvious that $\widetilde{\boldsymbol{z}}_n(\boldsymbol{x})$ vanishes on $\partial\Omega_1$ and that $\alpha_n \widetilde{\boldsymbol{z}}_n(\boldsymbol{x}) \to 0$, as $n \to \infty$ uniformly in $\boldsymbol{x} \in \Omega$. Defining $\widetilde{\boldsymbol{v}}_n = \boldsymbol{\psi}_n - \widetilde{\boldsymbol{z}}_n$, proves (a), (g) and (h). By definition $\widetilde{\boldsymbol{z}}_n(\boldsymbol{x}) = \boldsymbol{u}_n(\boldsymbol{x})$ for all

$\boldsymbol{x}$, such that $\operatorname{dist}(\boldsymbol{x}, \partial\Omega) \geq \delta_n$. Hence, (b) and (e) are also established. The property (d) is a consequence of (b), (c) and (e). Let us now establish (f).

$$\nabla \widetilde{\boldsymbol{z}}_n = (1 - \eta_n)\nabla \boldsymbol{u}_n + \eta_n \nabla \widehat{\boldsymbol{\psi}}_n + (\widehat{\boldsymbol{\psi}}_n - \boldsymbol{u}_n) \otimes \nabla \eta_n. \tag{3}$$

Hence,

$$\alpha_n \|\nabla \widetilde{\boldsymbol{z}}_n\|_\infty \leq \alpha_n \|\nabla \boldsymbol{z}_n\|_\infty + \alpha_n \|\widehat{\boldsymbol{\psi}}_n\|_{1,\infty} + \frac{\alpha_n \|\widehat{\boldsymbol{\psi}}_n\|_{1,\infty}}{\delta_n} + \frac{\alpha_n \|\boldsymbol{u}_n\|_\infty}{\delta_n}.$$

The property (f) is proved. To prove property (c), we observe that

$$\|\nabla \widetilde{\boldsymbol{z}}_n - \nabla \boldsymbol{u}_n\|_2 \leq \|\eta_n(\nabla \boldsymbol{u}_n - \nabla \widehat{\boldsymbol{\psi}}_n)\|_2 + \frac{\|\boldsymbol{u}_n - \widehat{\boldsymbol{\psi}}_n\|_2}{\delta_n} \to 0, \text{ as } n \to \infty$$

because $|\nabla \boldsymbol{u}_n - \nabla \widehat{\boldsymbol{\psi}}_n|^2$ is equiintegrable and η_n is bounded and supported on a set of small measure. Hence, (c) follows from the equiintegrability of $|\nabla \boldsymbol{u}_n|^2$. ∎

Another correction has to be made to the statements of Theorems 5 and 6. Theorem 5 has to apply to points in $\Omega \cup \partial\Omega_1$, while Theorem 6 to points in $\overline{\partial\Omega_2}$. Finally, no additional smoothness assumptions are required of $\partial\Omega_1$. We only need $\partial\Omega_1$ to be relatively open and coincide with the interior of its closure. If this condition is not satisfied we should just redefine $\partial\Omega_1$ as the interior of the closure of the original $\partial\Omega_1$. The set $\partial\Omega_2$ is then defined as $\partial\Omega \setminus \overline{\partial\Omega_1}$.