Scaling instability in buckling of axially compressed cylindrical shells

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Abstract

In this paper we continue the development of mathematically rigorous theory of "near-flip" buckling of slender bodies of arbitrary geometry, based on hyperelasticity. In order to showcase the capabilities of this theory we apply it to buckling of axially compressed circular cylindrical shells. The theory confirms the classical formula for the buckling load, whereby the perfect structure buckles at the stress that scales as the first power of shell's thickness. However, in the case of imperfections of load, the theory predicts scaling instability of the buckling stress. Depending on the type of load imperfections, buckling may occur at stresses that scale as thickness to the power 1.5 or 1.25, corresponding to the lower and upper ends, respectively, of the historically accumulated experimental data.

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1 Introduction

1.1 Near-flip buckling

The term buckling, understood broadly, refers to a failure of structural stability. As such it encompasses a plethora of disparate phenomena from classical buckling of the Euler column to plastic failure. In this paper we study one specific notion of buckling that we refer to, following [16], as "near-flip" buckling of slender structures in order to distinguish it from other meanings of the word used in the literature.

Buckling of the Euler column is a typical example of "near-flip" buckling. To explain the term we recall D'Alembert's objection to the famous Euler formula for the critical stress of the compressed column. D'Alembert has observed that such a column, to which compressive dead loads are applied at both ends, would loose stability at zero stress, since an infinitesimal rotation of the column would cause it to flip. Of course, this objection does not invalidate Euler's results, since in his work [7] Euler addressed this concern explicitly: "If the column be so constituted that it cannot slip, nothing else need be feared from the weight P, if it be not excessively great, except the bending of the column…". Mathematically, this suggests that Euler meant more complicated boundary conditions than pure dead loads applied at both ends of the column. However, D'Alembert's observation is an important one. It was shown in [16] that the same mechanism that was responsible for the flip instability is still at play in the buckling of Euler's column. In fact, buckling can be viewed as a "parametrized flip", whereby displacements undergone by small portions of the column are superpositions of a rigid motion (flip) and a small elastic deformation. Such configurations can have lower energy than the unbuckled ones, since rigid motions do not contribute to the total energy of the structure. This property of elastic energy is referred to as "objectivity" or "frame indifference". Its validity for all elastic structures implies that the flip mechanism described above applies to buckling of *all* slender[1] bodies, such as plates and shells, provided the prebuckled configuration satisfies certain conditions that we are going to discuss now.

Our use of the term "near-flip buckling" is limited to structures whose *prebuckled* state falls in the linearly elastic regime. For example, the stress in a compressed Euler column in its prebuckled state could be found from equations of linear elasticity with high degree of accuracy. At a critical value of the compressive stress this seemingly "linearly elastic" configuration looses its stability. It is crucial for the applicability of our theory that the value

[1]In this paper we will use the term "slender" in a mathematically precise sense defined in [16] (see also Definition 2.5 below).

of the critical stress goes to zero as the column's slenderness parameter (ratio of its diameter to its length) goes to zero. In general, we define "near-flip" buckling as a failure of stability of such linearly elastic configurations of slender bodies, whereby the critical value of the loading parameter (such as compressive stress for a column) goes to zero, as the slenderness parameter goes to zero (see Definition 2.2).

The general theory of near-flip buckling, developed in [16], detects when the prebuckled state (the trivial branch) stops being a weak local minimizer of the fully non-linear energy functional in the context of 3D hyperelasticity[2]. The main feature of the theory is the *constitutive linearization principle* that simplifies the second variation of the energy by replacing local material response with the linear elastic constitutive law, while retaining all the geometric non-linearity that is necessary to capture the leading order asymptotics of the buckling load. The idea is that at the initial stages of buckling (and before) the absolute magnitude of the stress at every point lies well into the linear elastic regime for the material that the structure is made of. The non-linearity of the structure's response to applied loads is of *purely geometric nature* [39, 9]. It is important to emphasize that constitutive linearization is not an assumption, but a theorem that guarantees that the constitutively linearized second variation captures the leading order asymptotics of the buckling load and the buckling mode correctly.

In this paper we prove the constitutive linearization theorem under weaker assumptions on the trivial branch than in [16]. This strengthening of the theory is necessary for analyzing buckling of axially compressed cylindrical shells, which fail to satisfy conditions for constitutive linearization in [16]. The passage from the second variation of the original non-linear energy to its constitutively linearized version suggests an equivalence relation on the set of functionals, called *B-equivalence* in [16]. This notion was the main simplification tool in [15], where the first completely rigorous derivation of the classical formula for the critical strain and an explicit description of a very large space of buckling modes, described in terms of the Koiter circle [25], were obtained by the application of the general theory of near-flip buckling. The strengthened constitutive linearization theorem and simple and effective criteria for B-equivalence established in this paper (see Theorem 2.11) have served as the theoretical foundation of the analysis in [15].

1.2 Imperfection sensitivity

The classical shell theory supplies the following formula for the critical stress [30, 37] (see also [38]):

$$\sigma_{\rm cr}=\frac{Eh}{\sqrt{3(1-\nu^2)}}, \tag{1.1}$$

where E and ν are the Young modulus and the Poisson ratio, respectively, and $h=t/R$ is the ratio of the wall thickness to the radius of the cylinder. A large body of experimental results indicates that the theoretical value of the buckling load is about 4 to 5 times higher than the one observed in experiments. This is generally attributed to the sensitivity of the

[2] While hyperelasticity is hardly the "ultimate" theory of elasticity, it is sufficiently general to capture buckling of slender structures.

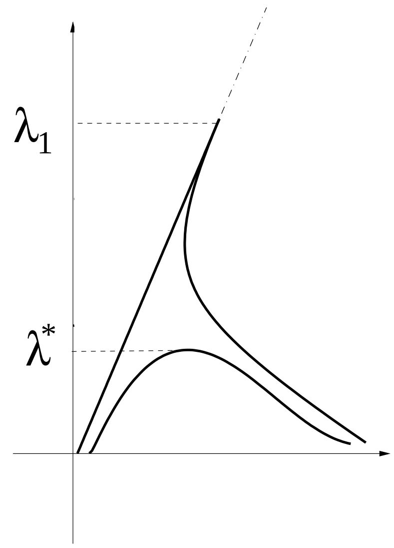

Figure 1: Imperfection sensitivity caused by a sub-critical bifurcation point

buckling load to imperfections of load and shape [1, 36, 41, 13, 42]. The mechanism of such high imperfection sensitivity is believed to be the sub-critical nature of the bifurcation [4] in equations of shell theory, such as von-Kármán-Donnell equations, for example [23, 24, 29, 21]. The gist of such an explanation is captured in Fig. 1. Generically, imperfections eliminate sharp bifurcation transitions [3]. However, the abrupt appearance of the dimple-shaped buckle, accompanied by an audible click and a sharp drop in load in our soda can buckling experiments [17] suggests that the imperfections might not eliminate bifurcations. Instead, they may cause the system to follow a slightly different "trivial branch" that also undergoes near-flip buckling at its own critical value of the loading parameter. In fact, in Section 4.1 and in Appendix A we exhibit families of trivial branches parametrized by a load imperfection parameter and show that they all undergo a near-flip buckling instability. In this case our theory can accurately predict the critical load and the corresponding buckling mode for the imperfect structure.

A more serious problem with the standard explanation of imperfection sensitivity was pointed out in [43, 6]. In the traditional interpretation, imperfections do not affect the scaling of the critical load with respect to the slenderness parameter h. For example, Koiter in his dissertation [25] gives a formula for λ^* (see Fig. 1):

$$\lambda^* \approx \lambda_1(1 - c\sqrt{\epsilon}), \tag{1.2}$$

where ϵ describes the magnitude of (shape) imperfections. An analysis in [43] of the large body of experimental data in [2] showed that in practice the critical stress σ_{cr} scales like h^α for some $\alpha \in [1.3, 1.5]$ in direct contradiction with (1.1), and even (1.2). A similar empirical scaling law, but with exponent $\alpha = 1.4$ was found in [39], based on the experimental data available at the time. The experiments of Calladine and Barber [5] and Mandal and Calladine [31] give a more consistent $\alpha = 1.5$ exponent.

An attempt to explain the empirical $h^{3/2}$ scaling was made in [6, 43]. The idea was to find a relation between the vertical displacement of the top edge of the shell and the elastic energy stored in the buckle (dimple) that was observed to form in experiments. To solve the problem a relation was found between the depth of a round bend in an inextensible line (or a strip) and the corresponding displacement of its endpoint. The elastic energy stored in the bend was taken to be the energy stored in the inverted spherical cap of the same depth as the bend in the line (strip). The compressive force necessary to sustain the dimple was then computed as the derivative of this energy with respect to the vertical displacement.

Relatively recently, a different approach has been proposed in [20] to explain high sensitivity of the buckling load to imperfections. It is based on the examination of the global energy landscape of the perfectly circular infinitely long cylindrical shell and discovery of different equilibria and mountain passes between them. Imperfections of load capable of overcoming energy barriers (mountain pass heights) between different equilibria, especially those that feature lateral loading, were shown to have a dramatic effect on the shell conformation. Single dimple solutions were obtained as the mountain pass states. Our approach also examines the energy functional of the perfect cylinder, but explores only the local energy landscape in the vicinity of the trivial branches corresponding to various loading imperfection schemes.

Recent years have also seen significant progress in the mathematically rigorous analysis of dimensionally reduced theories of plates and shells based on Γ-convergence [10, 32, 11, 27, 28]. In this approach, one must postulate the scaling of energy and the forces[3] a priori, whereby different scaling assumptions lead to different dimensionally reduced plate and shell equations. These analyses prove that the tacit assumptions of validity of specific shell theories must be justified before conclusions about the elastic behavior of such shells can be regarded as rigorous. In the case of buckling of circular cylindrical shells it is the scaling of energy that needs to be determined, and hence, these theories cannot be applied directly. By contrast, the theory of near-flip buckling has no need for such a priori assumptions, since it pursues a less ambitious goal of identifying the leading order asymptotics of the buckling load and buckling mode of a specific structure without determining the equations governing its global post-buckling behavior.

In this paper we offer a mathematically rigorous and mechanically rational explanation of the $h^{3/2}$ scaling. We show that imperfections in loading can cause the shell deformation to follow a slightly altered trivial branch that becomes unstable at a critical load that no longer scale as h, as in (1.1). We call this new phenomenon *scaling instability*. We demonstrate that imperfections of loading that are modeled by adding pressure terms lead to the $h^{3/2}$ scaling law, justifying the von Mises-Southwell formula [33, 34, 35, 40]. If imperfections of load are permitted to alter only the boundary conditions at the top of the cylinder, they may change the scaling of σ_{cr} only to $h^{5/4}$, as in [12]. It is important to emphasise that the power law $h^{3/2}$ arises in our theory as the scaling of the universal lower bound on safe loads given by the Korn constant [16, 14] of the axially loaded cylindrical shell. These results in the framework of the general theory of near-flip buckling illustrate the mechanism of scaling instability in buckling loads of slender bodies. This is discussed in detail in Section 5.

This paper is organized as follows. In Section 2 we recall the general theory of buckling of slender bodies from [16]. We prove the strengthened version of the constitutive linearization

[3] Alternatively, scaling of the forces and additional information on how the forces are applied, sufficient to determine the scaling of the energy, may be given.

theorem, applicable to axially compressed cylindrical shells, and derive the effective criteria for B-equivalence. Section 3 applies the theory to the problem of buckling of perfect circular cylindrical shells. In Section 4 we consider imperfections of load by adding a pressure component and a torsional component at the top edge of the shell. We demonstrate that such perturbation in loading causes scaling instability in the critical value of the applied strain. In Section 5 we discuss the mechanism of scaling instability in buckling loads of slender structures, as seen through the theory of near-flip buckling.

2 Near-flip buckling of slender structures

In this section we review and extend the general theory of buckling developed in [16], so that it can be applied to the buckling of axially compressed cylindrical shells. The theory provides a recipe for computing the asymptotics of buckling loads of slender structures, as the slenderness parameter goes to zero.

We say that the elastic configuration $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega \subset \mathbb{R}^3$ is stable if it is a weak local minimizer of the energy, i.e. if it has the lowest energy among all configurations that are sufficiently close to $\boldsymbol{y}(\boldsymbol{x})$ in the C^1 norm and satisfy the imposed boundary conditions. We assume that the energy functional is given by

$$\mathcal{E}(\boldsymbol{y}) = \int_\Omega W(\nabla \boldsymbol{y}) d\boldsymbol{x} - \int_{\partial\Omega} \boldsymbol{y} \cdot \boldsymbol{t}(\boldsymbol{x}) dS(\boldsymbol{x}),$$

where $W(\boldsymbol{F})$ is the energy density function of the body and $\boldsymbol{t}(\boldsymbol{x})$ is the vector of dead load tractions. The energy density function $W(\boldsymbol{F}) \in C^3(\mathbb{R}^{3\times 3})$ is assumed to satisfy the four fundamental properties:

(P1) Absence of prestress: $W_{\boldsymbol{F}}(\boldsymbol{I}) = \boldsymbol{0}$;

(P2) Frame indifference: $W(\boldsymbol{R}\boldsymbol{F}) = W(\boldsymbol{F})$ for every $\boldsymbol{R} \in SO(3)$;

(P3) Local stability of the undeformed state $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{x}$: $\langle \mathsf{L}_0 \boldsymbol{\xi}, \boldsymbol{\xi} \rangle \geq 0$ for any $\boldsymbol{\xi} \in \mathbb{R}^{3\times 3}$, where $\mathsf{L}_0 = W_{\boldsymbol{F}\boldsymbol{F}}(\boldsymbol{I})$ is the linearly elastic tensor of material properties;

(P4) Non-degeneracy: $\langle \mathsf{L}_0 \boldsymbol{\xi}, \boldsymbol{\xi} \rangle = 0$ if and only if $\boldsymbol{\xi}^T = -\boldsymbol{\xi}$.

Here, and elsewhere in this paper we use the notation $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \mathrm{Tr}\,(\boldsymbol{A}\boldsymbol{B}^T)$ for the Frobenius inner product on the space of 3×3 matrices, and the first and second gradients of $W(\boldsymbol{F})$ with respect to $\boldsymbol{F}$ are denoted $W_{\boldsymbol{F}}$ and $W_{\boldsymbol{F}\boldsymbol{F}}$, respectively:

$$W_{\boldsymbol{F}} = \frac{\partial W}{\partial F_{i\alpha}}, \qquad W_{\boldsymbol{F}\boldsymbol{F}} = \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}}.$$

We remark that properties (P3) and (P4) of L_0 imply a uniform lower bound

$$\langle \mathsf{L}_0 \boldsymbol{\xi}, \boldsymbol{\xi} \rangle \geq \alpha_{\mathsf{L}_0} |\boldsymbol{\xi}_{\mathrm{sym}}|^2, \qquad \boldsymbol{\xi}_{\mathrm{sym}} = \frac{1}{2}(\boldsymbol{\xi} + \boldsymbol{\xi}^T) \tag{2.1}$$

for some $\alpha_{\mathsf{L}_0} > 0$.

2.1 The trivial branch

Consider a sequence of progressively slender[4] domains Ω_h parametrized by a dimensionless parameter h. For example, for circular cylindrical shells, h is the ratio of cylinder wall thickness to the cylinder radius (assuming we keep the ratio of cylinder height to its radius constant). We consider a loading program parametrized by the loading parameter λ describing the magnitude of the applied tractions $\boldsymbol{t}(\boldsymbol{x};h,\lambda)=\lambda\boldsymbol{t}^h(\boldsymbol{x})+O(\lambda^2)$, as $\lambda\to 0$, or as a measure of the prescribed strain entering through the displacement boundary conditions. Here and below $O(\lambda^\alpha)$ is understood *uniformly* in $\boldsymbol{x}\in\Omega_h$ and $h\in[0,h_0]$. Let $\boldsymbol{y}(\boldsymbol{x};h,\lambda)$ be a family of Lipschitz equilibria of

$$\mathcal{E}(\boldsymbol{y};h,\lambda)=\int_{\Omega_h}W(\nabla\boldsymbol{y})d\boldsymbol{x}-\int_{\partial\Omega_h}\boldsymbol{y}(\boldsymbol{x})\cdot\boldsymbol{t}(\boldsymbol{x};h,\lambda)dS(\boldsymbol{x}) \tag{2.2}$$

defined on $W^{1,\infty}(\overline{\Omega}_h;\mathbb{R}^3)\times[0,h_0]\times[0,\lambda_0]$ for some $h_0>0$ and $\lambda_0>0$. The general theory places no direct restrictions on the type of boundary conditions we can have. To describe a wide class of ones, we restrict $\boldsymbol{y}$ to an affine subspace of $W^{1,\infty}(\Omega_h;\mathbb{R}^3)$, given by

$$\boldsymbol{y}\in\overline{\boldsymbol{y}}(\boldsymbol{x};h,\lambda)+V_h^\circ, \tag{2.3}$$

where V_h° is a linear subspace of $W^{1,\infty}(\Omega_h;\mathbb{R}^3)$ that contains $W_0^{1,\infty}(\Omega_h;\mathbb{R}^3)$ and does not depend on the loading parameter λ. The given function $\overline{\boldsymbol{y}}(\boldsymbol{x};h,\lambda)\in W^{1,\infty}(\Omega_h;\mathbb{R}^3)$ describes the "displacement part" of the boundary conditions, while the vector $\boldsymbol{t}(\boldsymbol{x};h,\lambda)$ describes the traction part. The use of a fairly general subspace V_h° permits us to describe boundary conditions in which desired components of displacements and tractions, or their linear combinations are prescribed on the boundary. An example of such a description of the boundary conditions for the cylindrical shell will be given in Section 3.1.

Definition 2.1. *We call the family of Lipschitz equilibria $\boldsymbol{y}(\boldsymbol{x};h,\lambda)$ of $\mathcal{E}(\boldsymbol{y};h,\lambda)$ a **regular trivial branch** if there exist $h_0>0$ and $\lambda_0>0$, so that for every $h\in[0,h_0]$ and $\lambda\in[0,\lambda_0]$*

(i) $\boldsymbol{y}(\boldsymbol{x};h,0)=\boldsymbol{x}$.

(ii) There exist a family of Lipschitz functions $\boldsymbol{u}^h(\boldsymbol{x})$, independent of λ, such that

$$\left\|\frac{\partial(\nabla\boldsymbol{y})}{\partial\lambda}(\boldsymbol{x};h,\lambda)-\nabla\boldsymbol{u}^h(\boldsymbol{x})\right\|_{L^\infty(\Omega_h)}\le C\lambda, \tag{2.4}$$

where the constant C is independent of h and λ.

Several remarks are in order. We note that neither uniqueness nor stability of the trivial branch are assumed. We also note that uniformity of the estimate (2.4) in h is the main factor determining the applicability of this theory to buckling of specific structures. We also observe that integrating estimate (2.4) in λ and using property (i) we obtain the estimate

$$\|\nabla\boldsymbol{y}(\boldsymbol{x};h,\lambda)-\boldsymbol{I}-\lambda\nabla\boldsymbol{u}^h(\boldsymbol{x})\|_{L^\infty(\Omega_h)}\le C\lambda^2, \tag{2.5}$$

[4] The notion of slenderness, introduced in [16] is recalled in Definition 2.5.

where the constant C is independent of h and λ.

The equilibrium equations and the boundary conditions satisfied by the trivial branch $\boldsymbol{y}(\boldsymbol{x}; h, \lambda)$ can be written explicitly in the weak form:

$$\int_{\Omega_h} \langle W_{\boldsymbol{F}}(\nabla \boldsymbol{y}(\boldsymbol{x}; h, \lambda)), \nabla \boldsymbol{\phi} \rangle d\boldsymbol{x} - \int_{\partial \Omega_h} \boldsymbol{\phi} \cdot \boldsymbol{t}(\boldsymbol{x}; h, \lambda) dS = 0, \qquad \forall \boldsymbol{\phi} \in V_h^{\circ}, \tag{2.6}$$

Differentiating (2.6) in λ at $\lambda = 0$, which is allowed due to (2.4), we obtain

$$\int_{\Omega_h} \langle \mathsf{L}_0 \nabla \boldsymbol{u}^h(\boldsymbol{x}), \nabla \boldsymbol{\phi} \rangle d\boldsymbol{x} - \int_{\partial \Omega_h} \boldsymbol{\phi} \cdot \boldsymbol{t}^h(\boldsymbol{x}) dS = 0, \qquad \forall \boldsymbol{\phi} \in V_h^{\circ}. \tag{2.7}$$

In other words, the function $\boldsymbol{u}^h$ can be computed by solving equations of *linear elasticity*.

The loss of stability occurs for some $\lambda = \lambda^*(h)$ when it becomes energetically more advantageous to activate bending modes rather than store more compressive stress. This exchange of stability is detected by the change in sign of the second variation in the energy functional $\mathcal{E}(\boldsymbol{y})$,

$$\delta^2 \mathcal{E}(\boldsymbol{\phi}; h, \lambda) = \int_{\Omega_h} \langle W_{\boldsymbol{F}\boldsymbol{F}}(\nabla \boldsymbol{y}(\boldsymbol{x}; h, \lambda)) \nabla \boldsymbol{\phi}, \nabla \boldsymbol{\phi} \rangle d\boldsymbol{x}, \qquad \boldsymbol{\phi} \in V_h, \tag{2.8}$$

as λ crosses its critical value $\lambda^*(h)$. Here $V_h = \overline{V_h^{\circ}}$ is the closure of V_h° in $W^{1,2}(\Omega_h; \mathbb{R}^3)$. The passage from V_h° to V_h is not essential, but is simplifies language and notation in what follows.

2.2 Buckling load and buckling mode

We now assume that a regular trivial branch $\boldsymbol{y}(\boldsymbol{x}; h, \lambda)$ is fixed. Using the second variation criterion for stability we define the buckling load as

$$\lambda^*(h) = \inf\{\lambda > 0 : \delta^2 \mathcal{E}(\boldsymbol{\phi}; h, \lambda) < 0 \text{ for some } \boldsymbol{\phi} \in V_h\}. \tag{2.9}$$

Definition 2.2. *We say that a regular trivial branch undergoes a* ***near-flip buckling*** *if $\lambda^*(h) > 0$ for all $h \in (0, h_0)$, for some $h_0 > 0$, and $\lambda^*(h) \to 0$, as $h \to 0$.*

We refer to [16] for an extensive discussion of why this terminology is appropriate.

The buckling mode is generally understood as the variation $\boldsymbol{\phi}_h^* \in V_h \setminus \{0\}$, such that $\delta^2 \mathcal{E}(\boldsymbol{\phi}_h^*; h, \lambda^*(h)) = 0$[5]. However, if we are only interested in the *asymptotics* of the critical load, as $h \to 0$, then we should not distinguish between $\lambda^*(h)$ and another function $\lambda(h)$, as long as $\lambda(h)/\lambda^*(h) \to 1$, as $h \to 0$. If we replace $\lambda^*(h)$ with $\lambda_\epsilon(h) = \lambda^*(h)(1 + \epsilon)$, then, using a formal Taylor expansion, we estimate

$$\delta^2 \mathcal{E}(\boldsymbol{\phi}_h^*; h, \lambda^*(h)(1 + \epsilon)) \approx \lambda^*(h) \epsilon \frac{\partial(\delta^2 \mathcal{E})}{\partial \lambda}(\boldsymbol{\phi}_h^*; h, \lambda^*(h)).$$

[5]The question of existence of the buckling mode $\boldsymbol{\phi}_h^*$ is irrelevant here, since the goal of this discussion is to motivate our somewhat unusual definition of a buckling mode below, which makes no existence assumptions.

This means that for the purposes of asymptotics we should not distinguish differences in values of second variation that are infinitesimal, compared to

$$\lambda^*(h)\frac{\partial(\delta^2\mathcal{E})}{\partial\lambda}(\boldsymbol{\phi}_h^*; h, \lambda^*(h)).$$

In keeping with these observations, we redefine the notion of the buckling load and buckling mode, under the assumption that the body undergoes a near-flip buckling in the sense of Definition 2.2.

Definition 2.3. *We say that $\lambda(h) \to 0$, as $h \to 0$ is a buckling load if*

$$\lim_{h\to 0}\frac{\lambda(h)}{\lambda^*(h)} = 1. \tag{2.10}$$

A ***buckling mode*** *is a family of variations $\boldsymbol{\phi}_h \in V_h \setminus \{0\}$, such that*

$$\lim_{h\to 0}\frac{\delta^2\mathcal{E}(\boldsymbol{\phi}_h; h, \lambda^*(h))}{\lambda^*(h)\dfrac{\partial(\delta^2\mathcal{E})}{\partial\lambda}(\boldsymbol{\phi}_h; h, \lambda^*(h))} = 0. \tag{2.11}$$

The most important insight in [16] is that the stress at or below the critical value of the loading parameter $\lambda^*(h)$ is well inside the linearly elastic regime. Hence, the local material response can be linearized, and we can replace the original second variation (2.8) by a simpler *constitutively linearized* second variation:

$$\delta^2\mathcal{E}_{cl}(\boldsymbol{\phi}; h, \lambda) = \int_{\Omega_h}\{\langle \mathsf{L}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle + \lambda\langle \boldsymbol{\sigma}_h, \nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi}\rangle\}d\boldsymbol{x}, \qquad \boldsymbol{\phi} \in V_h, \tag{2.12}$$

where $e(\boldsymbol{\phi}) = \frac{1}{2}(\nabla\boldsymbol{\phi} + (\nabla\boldsymbol{\phi})^T)$ and

$$\boldsymbol{\sigma}_h(\boldsymbol{x}) = \mathsf{L}_0 e(\boldsymbol{u}^h(\boldsymbol{x})) \tag{2.13}$$

are the linear elastic strain and stress, respectively.

The goal of this section is to determine under what conditions the original second variation (2.8) can be replaced with the constitutively linearized one. In order to do that we will have to show that the constitutively linearized buckling load, coming from examining the sign of the constitutively linearized second variation (2.12), has the same asymptotics as $\lambda^*(h)$. We will also need to show that the buckling modes determined by $\delta^2\mathcal{E}$ and $\delta^2\mathcal{E}_{cl}$ are the same.

The sign of $\delta^2\mathcal{E}_{cl}$ is determined by the competition between the two terms in (2.12). The first term

$$\mathfrak{S}_h(\boldsymbol{\phi}) = \int_{\Omega_h}\langle \mathsf{L}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle d\boldsymbol{x} \tag{2.14}$$

is always non-negative due to the assumption (P3), and hence, we interpret it as the measure of stability of the trivial branch. The second term, or rather

$$\mathfrak{C}_h(\boldsymbol{\phi}) = \frac{\partial(\delta^2\mathcal{E}_{cl})}{\partial\lambda}(\boldsymbol{\phi}; h, \lambda) = \int_{\Omega_h}\langle \boldsymbol{\sigma}_h, \nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi}\rangle d\boldsymbol{x} \tag{2.15}$$

measures the destabilizing influence of compressive stresses. Note that, if the stress tensor $\boldsymbol{\sigma}_h$ is negative definite (i.e. compressive) then $\mathfrak{C}_h(\boldsymbol{\phi})$ is negative[6]. Thus,

$$\mathcal{A}_h = \{\boldsymbol{\phi} \in V_h : \mathfrak{C}_h(\boldsymbol{\phi}) < 0\} \tag{2.16}$$

is the set of all potentially destabilizing variations for (2.12). If $\mathcal{A}_h = \emptyset$, then the structure might not be susceptible to a near-flip buckling. For example, it will not undergo a near-flip buckling, if there exists a constant $c > 0$, such that $\mathfrak{C}_h(\boldsymbol{\phi}) \geq c\|\nabla\boldsymbol{\phi}\|^2$. This can be seen from the error estimate (2.26), for example. Here and elsewhere in this paper $\|\cdot\|$ always denotes the L^2-norm on Ω_h.

Henceforth we will work under the assumption that the applied loading has a compressive nature, i.e. $\mathcal{A}_h \neq \emptyset$ for all $h \in (0, h_0)$ for some $h_0 > 0$. The constitutively linearized version of the critical load can now be computed my minimizing the Rayleigh quotient

$$\mathfrak{R}(h, \boldsymbol{\phi}) = -\frac{\int_{\Omega_h} \langle \mathsf{L}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle d\boldsymbol{x}}{\int_{\Omega_h} \langle \boldsymbol{\sigma}_h, \nabla\boldsymbol{\phi}^T \nabla\boldsymbol{\phi}\rangle d\boldsymbol{x}} = -\frac{\mathfrak{S}_h(\boldsymbol{\phi})}{\mathfrak{C}_h(\boldsymbol{\phi})}, \tag{2.17}$$

and the constitutively linearized buckling mode is defined by (2.11), in which $\delta^2\mathcal{E}$ is replaced with $\delta^2\mathcal{E}_{cl}$. We summarize this in the following definition.

Definition 2.4. *The* ***constitutively linearized buckling load*** $\lambda_{\rm cl}(h)$ *is defined by*

$$\lambda_{\rm cl}(h) = \inf_{\boldsymbol{\phi} \in \mathcal{A}_h} \mathfrak{R}(h, \boldsymbol{\phi}). \tag{2.18}$$

We say that the family of variations $\{\boldsymbol{\phi}_h \in \mathcal{A}_h : h \in (0, h_0)\}$ *is a* ***constitutively linearized buckling mode*** *if*

$$\lim_{h \to 0} \frac{\mathfrak{R}(h, \boldsymbol{\phi}_h)}{\lambda_{\rm cl}(h)} = 1. \tag{2.19}$$

In order to formulate conditions under which the buckling load and the buckling mode can be determined from the constitutively linearized second variation, we recall the definition of the Korn constant

$$K(V_h) = \inf_{\boldsymbol{\phi} \in V_h} \frac{\|e(\boldsymbol{\phi})\|^2}{\|\nabla\boldsymbol{\phi}\|^2}. \tag{2.20}$$

Definition 2.5. *We say that the body*[7] Ω_h *is* ***slender*** *if*

$$\lim_{h \to 0} K(V_h) = 0. \tag{2.21}$$

We remark that this notion of slenderness, introduced in [16], is not purely geometric, but depends on the type of loading described by the subspace V_h. On the one hand, a thin rod or a plate in the hard device will not be regarded as slender, since their Korn constant

[6]This is a consequence of a classical result in matrix algebra, due to I. Schur, that even though the product of two positive definite matrices does not have to be positive definite, its trace is always positive.

[7]Of course, it is the family of bodies Ω_h that may or may not be "slender" according to our definition. We abuse the terminology for the sake of euphony.

is 1/2, regardless of their geometric slenderness. On the other hand, a geometrically non-slender body, such as a ball or a cube will not be slender under our definition, for any set of boundary conditions that excludes all rigid body motions. In fact, any admissible continuous family of rotations will cause the Korn constant to be identically zero, since the gradient of the infinitesimal generator of such a family will be a skew-symmetric matrix.

We are now ready to state and prove the constitutive linearization theorem. (The reader, who wishes to skip the proof, can go to Section 2.3.)

THEOREM **2.6** (Asymptotics of the critical load). *Suppose that the body is slender in the sense of Definition 2.5. Assume that the constitutively linearized critical load $\lambda_{\rm cl}(h)$, defined in (2.18) satisfies $\lambda_{\rm cl}(h) > 0$ for all sufficiently small h and*

$$\lim_{h\to 0}\frac{\lambda_{\rm cl}(h)^2}{K(V_h)} = 0. \tag{2.22}$$

Then $\lambda_{\rm cl}(h)$ is the buckling load and any constitutively linearized buckling mode $\boldsymbol{\phi}_h$ is a buckling mode in the sense of Definition 2.3.

The theorem is proved by means of the basic estimate, which is a simple modification of the estimates in [16] used in the derivation of the formula for $\delta^2\mathcal{E}_{cl}(\boldsymbol{\phi};h,\lambda)$:

LEMMA **2.7.** *Suppose $\boldsymbol{y}(\boldsymbol{x};h,\lambda)$ is a regular trivial branch in the sense of Definition 2.1 and $W(\boldsymbol{F})$ has the properties (P1)–(P4). Then*

$$\left|\delta^2\mathcal{E}(\boldsymbol{\phi};h,\lambda)-\delta^2\mathcal{E}_{cl}(\boldsymbol{\phi};h,\lambda)\right|\le C\left(\frac{\lambda}{\sqrt{K(V_h)}}+\frac{\lambda^2}{K(V_h)}\right)\mathfrak{S}_h(\boldsymbol{\phi}) \tag{2.23}$$

and

$$\left|\frac{\partial(\delta^2\mathcal{E})}{\partial\lambda}(\boldsymbol{\phi};h,\lambda)-\mathfrak{C}_h(\boldsymbol{\phi})\right|\le C\left(\frac{1}{\sqrt{K(V_h)}}+\frac{\lambda}{K(V_h)}\right)\mathfrak{S}_h(\boldsymbol{\phi}), \tag{2.24}$$

where the constant C is independent of h, λ and $\boldsymbol{\phi}$.

Proof. According to the frame indifference property (P2), $W(\boldsymbol{F})=\widehat{W}(\boldsymbol{F}^T\boldsymbol{F})$. Differentiating this formula twice we obtain

$$\langle W_{\boldsymbol{FF}}(\boldsymbol{F})\boldsymbol{\xi},\boldsymbol{\xi}\rangle = 4\langle\widehat{W}_{\boldsymbol{CC}}(\boldsymbol{C})(\boldsymbol{F}^T\boldsymbol{\xi}),\boldsymbol{F}^T\boldsymbol{\xi}\rangle+2\langle\widehat{W}_{\boldsymbol{C}}(\boldsymbol{C}),\boldsymbol{\xi}^T\boldsymbol{\xi}\rangle,\qquad \boldsymbol{C}=\boldsymbol{F}^T\boldsymbol{F}. \tag{2.25}$$

We can estimate

$$|\langle\widehat{W}_{\boldsymbol{CC}}(\boldsymbol{C})(\boldsymbol{F}^T\boldsymbol{\xi}),\boldsymbol{F}^T\boldsymbol{\xi}\rangle-\langle\widehat{W}_{\boldsymbol{CC}}(\boldsymbol{I})\boldsymbol{\xi},\boldsymbol{\xi}\rangle|\le|\langle\widehat{W}_{\boldsymbol{CC}}(\boldsymbol{C})(\boldsymbol{F}^T-\boldsymbol{I})\boldsymbol{\xi},(\boldsymbol{F}^T-\boldsymbol{I})\boldsymbol{\xi}\rangle|+$$
$$|\langle(\widehat{W}_{\boldsymbol{CC}}(\boldsymbol{C})-\widehat{W}_{\boldsymbol{CC}}(\boldsymbol{I}))\boldsymbol{\xi},\boldsymbol{\xi}\rangle|+2|\langle\widehat{W}_{\boldsymbol{CC}}(\boldsymbol{C})\boldsymbol{\xi},(\boldsymbol{F}^T-\boldsymbol{I})\boldsymbol{\xi}\rangle|.$$

When $\boldsymbol{F}$ is uniformly bounded we obtain

$$|\langle\widehat{W}_{\boldsymbol{CC}}(\boldsymbol{C})(\boldsymbol{F}^T\boldsymbol{\xi}),\boldsymbol{F}^T\boldsymbol{\xi}\rangle-\langle\widehat{W}_{\boldsymbol{CC}}(\boldsymbol{I})\boldsymbol{\xi},\boldsymbol{\xi}\rangle|\le C\left(|\boldsymbol{F}-\boldsymbol{I}|^2|\boldsymbol{\xi}|^2+|\boldsymbol{C}-\boldsymbol{I}||\boldsymbol{\xi}_{\rm sym}|^2+|\boldsymbol{F}-\boldsymbol{I}||\boldsymbol{\xi}_{\rm sym}||\boldsymbol{\xi}|\right).$$

Similarly,

$$|\langle \widehat{W}_{\boldsymbol{C}}(\boldsymbol{C}) - \widehat{W}_{\boldsymbol{CC}}(\boldsymbol{I})(\boldsymbol{C}-\boldsymbol{I}), \boldsymbol{\xi}^T\boldsymbol{\xi}\rangle| \le C|\boldsymbol{C}-\boldsymbol{I}|^2|\boldsymbol{\xi}|^2.$$

When $\boldsymbol{F} = \nabla\boldsymbol{y}(\boldsymbol{x}; h, \lambda)$ and $\boldsymbol{\xi} = \nabla\boldsymbol{\phi}$ we obtain, taking into account (2.4) and (2.5), that

$$|\boldsymbol{C} - \boldsymbol{I} - 2\lambda e(\boldsymbol{u}^h)| \le C\lambda^2, \qquad |\boldsymbol{F}-\boldsymbol{I}| \le C\lambda, \qquad |\boldsymbol{C}-\boldsymbol{I}| \le C\lambda.$$

Observing that $4\widehat{W}_{\boldsymbol{CC}}(\boldsymbol{I}) = W_{\boldsymbol{FF}}(\boldsymbol{I}) = \mathsf{L}_0$ we obtain the estimate

$$|\langle W_{\boldsymbol{FF}}(\boldsymbol{F})\boldsymbol{\xi}, \boldsymbol{\xi}\rangle - \langle \mathsf{L}_0\boldsymbol{\xi}_{\rm sym}, \boldsymbol{\xi}_{\rm sym}\rangle - \lambda\langle \boldsymbol{\sigma}_h, \boldsymbol{\xi}^T\boldsymbol{\xi}\rangle| \le C(\lambda|\boldsymbol{\xi}_{\rm sym}||\boldsymbol{\xi}| + \lambda^2|\boldsymbol{\xi}|^2). \tag{2.26}$$

Integrating over Ω_h as using the coercivity (2.1) of L_0 we obtain the estimate (2.23).

In order to prove the estimate (2.24) we substitute $\boldsymbol{F} = \nabla\boldsymbol{y}(\boldsymbol{x}; h, \lambda)$ and $\boldsymbol{\xi} = \nabla\boldsymbol{\phi}$ into (2.25) and differentiate in λ, obtaining

$$\frac{\partial\langle W_{\boldsymbol{FF}}(\boldsymbol{F})\boldsymbol{\xi}, \boldsymbol{\xi}\rangle}{\partial\lambda} = 4\langle(\widehat{W}_{\boldsymbol{CCC}}(\boldsymbol{C})\dot{\boldsymbol{C}})(\boldsymbol{F}^T\boldsymbol{\xi}), \boldsymbol{F}^T\boldsymbol{\xi}\rangle + 8\langle \widehat{W}_{\boldsymbol{CC}}(\boldsymbol{C})(\boldsymbol{F}^T\boldsymbol{\xi}), \dot{\boldsymbol{F}}^T\boldsymbol{\xi}\rangle + 2\langle \widehat{W}_{\boldsymbol{CC}}(\boldsymbol{C})\dot{\boldsymbol{C}}, \boldsymbol{\xi}^T\boldsymbol{\xi}\rangle,$$

where $\dot{\boldsymbol{C}}$ and $\dot{\boldsymbol{F}}$ denote differentiation with respect to λ. Using the uniform boundedness of $\dot{\boldsymbol{C}}$, which is a corollary of (2.4), we estimate

$$|\langle(\widehat{W}_{\boldsymbol{CCC}}(\boldsymbol{C})\dot{\boldsymbol{C}})(\boldsymbol{F}^T\boldsymbol{\xi}), \boldsymbol{F}^T\boldsymbol{\xi}\rangle| \le C(|\boldsymbol{\xi}_{\rm sym}|^2 + \lambda|\boldsymbol{\xi}||\boldsymbol{\xi}_{\rm sym}|)$$

and

$$|\langle \widehat{W}_{\boldsymbol{CC}}(\boldsymbol{C})(\boldsymbol{F}^T\boldsymbol{\xi}), \dot{\boldsymbol{F}}^T\boldsymbol{\xi}\rangle| \le C(|\boldsymbol{\xi}||\boldsymbol{\xi}_{\rm sym}| + \lambda|\boldsymbol{\xi}|^2).$$

We also estimate, using $|\boldsymbol{C}-\boldsymbol{I}| \le C\lambda$ and $|\dot{\boldsymbol{C}} - 2e(\boldsymbol{u}^h)| \le C\lambda$, that are consequences of (2.4):

$$|2\langle \widehat{W}_{\boldsymbol{CC}}(\boldsymbol{C})\dot{\boldsymbol{C}}, \boldsymbol{\xi}^T\boldsymbol{\xi}\rangle - \langle \boldsymbol{\sigma}_h, \boldsymbol{\xi}^T\boldsymbol{\xi}\rangle| \le C\lambda|\boldsymbol{\xi}|^2.$$

□

Proof of Theorem 2.6. By definition of $\lambda_{\rm cl}(h)$, for any $\epsilon > 0$ and any $h \in (0, h_0)$ there exists $\boldsymbol{\phi}_h \in \mathcal{A}_h$ such that

$$\mathfrak{S}_h(\boldsymbol{\phi}_h) + \lambda_{\rm cl}(h)(1+\epsilon)\mathfrak{C}_h(\boldsymbol{\phi}_h) < 0. \tag{2.27}$$

Thus,

$$\delta^2\mathcal{E}_{cl}(\boldsymbol{\phi}_h; h, \lambda_{\rm cl}(h)(1+2\epsilon)) \le -\frac{\epsilon\mathfrak{S}_h(\boldsymbol{\phi}_h)}{1+\epsilon}.$$

The estimate (2.23) gives the upper bound on the second variation:

$$\delta^2\mathcal{E}(\boldsymbol{\phi}_h; h, \lambda_{\rm cl}(h)(1+2\epsilon)) \le \left(-\frac{\epsilon}{(1+\epsilon)} + C\left(\frac{\lambda_{\rm cl}(h)}{\sqrt{K(V_h)}} + \frac{\lambda_{\rm cl}(h)^2}{K(V_h)}\right)\right)\mathfrak{S}_h(\boldsymbol{\phi}_h).$$

Therefore, due to (2.22), for sufficiently small h, we have $\delta^2\mathcal{E}(\boldsymbol{\phi}_h; h, \lambda_{\rm cl}(h)(1+2\epsilon)) < 0$, and hence $\lambda^*(h) \le \lambda_{\rm cl}(h)(1+2\epsilon)$. We conclude that

$$\varlimsup_{h\to 0}\frac{\lambda^*(h)}{\lambda_{\rm cl}(h)} \le 1. \tag{2.28}$$

To prove the opposite inequality we observe that by definition of $\lambda_{\rm cl}(h)$ we have

$$\mathfrak{S}_h(\boldsymbol{\phi}) + \lambda_{\rm cl}(h)\mathfrak{C}_h(\boldsymbol{\phi}) \geq 0$$

for any $\boldsymbol{\phi} \in V_h$. Therefore, for any $\epsilon > 0$ and any $0 < \lambda \leq \lambda_{\rm cl}(h)(1-\epsilon)$ we have

$$\delta^2\mathcal{E}_{cl}(\boldsymbol{\phi}; h, \lambda) \geq \epsilon\mathfrak{S}_h(\boldsymbol{\phi}).$$

The estimate (2.23) now gives the lower bound on the second variation:

$$\delta^2\mathcal{E}(\boldsymbol{\phi}; h, \lambda) \geq \left(\epsilon - C\left(\frac{\lambda_{\rm cl}(h)}{\sqrt{K(V_h)}} + \frac{\lambda_{\rm cl}(h)^2}{K(V_h)}\right)\right)\mathfrak{S}_h(\boldsymbol{\phi}).$$

Thus for all sufficiently small h and all $\boldsymbol{\phi} \in V_h \setminus \{0\}$ we have $\delta^2\mathcal{E}(\boldsymbol{\phi}; h, \lambda) > 0$ for all $0 < \lambda \leq \lambda_{\rm cl}(h)(1-\epsilon)$, which means that $\lambda^*(h) \geq \lambda_{\rm cl}(h)(1-\epsilon)$. This implies

$$\varliminf_{h\to 0} \frac{\lambda^*(h)}{\lambda_{\rm cl}(h)} \geq 1. \tag{2.29}$$

Combining (2.28) and (2.29) we conclude that $\lambda_{\rm cl}(h)$ is the buckling load.

Assume now that $\boldsymbol{\phi}_h$ is a constitutively linearized buckling mode, i.e. (2.19) holds. Set $\lambda = \lambda^*(h)$ and $\boldsymbol{\phi} = \boldsymbol{\phi}_h$ in the inequality (2.23). Then, dividing both sides of the inequality by $-\lambda^*(h)\mathfrak{C}_h(\boldsymbol{\phi}_h) > 0$ we obtain

$$\left|\frac{\delta^2\mathcal{E}(\boldsymbol{\phi}_h; h, \lambda^*(h))}{-\lambda^*(h)\mathfrak{C}_h(\boldsymbol{\phi}_h)} - \left(\frac{\mathfrak{R}(h, \boldsymbol{\phi}_h)}{\lambda^*(h)} - 1\right)\right| \leq C\left(\frac{\lambda^*(h)}{\sqrt{K(V_h)}} + \frac{(\lambda^*(h))^2}{K(V_h)}\right)\frac{\mathfrak{R}(h, \boldsymbol{\phi}_h)}{\lambda^*(h)}.$$

Since we have proved that $\lambda_{\rm cl}(h)$ is the buckling load we conclude that

$$\lim_{h\to 0} \frac{\delta^2\mathcal{E}(\boldsymbol{\phi}_h; h, \lambda^*(h))}{\lambda^*(h)\mathfrak{C}_h(\boldsymbol{\phi}_h)} = 0.$$

Similarly, setting $\lambda = \lambda^*(h)$ and $\boldsymbol{\phi} = \boldsymbol{\phi}_h$ in the inequality (2.24) and dividing both sides of the inequality by $-\mathfrak{C}_h(\boldsymbol{\phi}_h) > 0$ we obtain

$$\left|\frac{\dfrac{\partial(\delta^2\mathcal{E})}{\partial\lambda}(\boldsymbol{\phi}_h; h, \lambda^*(h))}{-\mathfrak{C}_h(\boldsymbol{\phi}_h)} + 1\right| \leq C\left(\frac{\lambda^*(h)}{\sqrt{K(V_h)}} + \frac{(\lambda^*(h))^2}{K(V_h)}\right)\frac{\mathfrak{R}(h, \boldsymbol{\phi}_h)}{\lambda^*(h)}.$$

We conclude that

$$\lim_{h\to 0} \frac{\dfrac{\partial(\delta^2\mathcal{E})}{\partial\lambda}(\boldsymbol{\phi}_h; h, \lambda^*(h))}{\mathfrak{C}_h(\boldsymbol{\phi}_h)} = 1.$$

It follows now that $\boldsymbol{\phi}_h$ satisfies (2.11), and the theorem is proved. $\square$

2.3 B-equivalence

In Section 2.2 we showed that the asymptotics of the critical load and buckling mode can be captured by the Rayleigh quotient $\mathfrak{R}(h, \boldsymbol{\phi})$. Even though such a characterization of buckling represents a significant simplification, compared to the original characterization based on the second variation of a fully non-linear energy functional, further simplifications may be necessary in order to obtain an explicit analytic expression for the buckling load. We envision two ways in which the required analysis can be simplified. One, is the simplification of the functional $\mathfrak{R}(h, \boldsymbol{\phi})$. The other, is replacing the space of all admissible functions $\mathcal{A}_h$ with a smaller space $\mathcal{B}_h$. For example, we may want to justify a restriction to a specific ansatz, like the Kirchhoff ansatz in buckling of rods and plates. In order to formalize our simplification procedure we make the following definitions.

Definition 2.8. *Assume that $J(h, \boldsymbol{\phi})$ is a functional defined on $\mathcal{B}_h \subset \mathcal{A}_h$. We say that the pair $(\mathcal{B}_h, J(h, \boldsymbol{\phi}))$* ***characterizes buckling*** *if the following three conditions are satisfied*

(a) *Characterization of the buckling load: If*

$$\lambda(h) = \inf_{\boldsymbol{\phi} \in \mathcal{B}_h} J(h, \boldsymbol{\phi}),$$

then $\lambda(h)$ is a buckling load in the sense of Definition 2.3.

(b) *Characterization of the buckling mode: If $\boldsymbol{\phi}_h \in \mathcal{B}_h$ is a buckling mode in the sense of Definition 2.3, then*

$$\lim_{h \to 0} \frac{J(h, \boldsymbol{\phi}_h)}{\lambda(h)} = 1. \tag{2.30}$$

(c) *Faithful representation of the buckling mode: If $\boldsymbol{\phi}_h \in \mathcal{B}_h$ satisfies (2.30) then it is a buckling mode.*

Definition 2.9. *Two pairs $(\mathcal{B}_h, J_1(h, \boldsymbol{\phi}))$ and $(\mathcal{C}_h, J_2(h, \boldsymbol{\phi}))$ are called* ***B-equivalent*** *if the pair $(\mathcal{B}_h, J_1(h, \boldsymbol{\phi}))$ characterizes buckling if and only if $(\mathcal{C}_h, J_2(h, \boldsymbol{\phi}))$ does.*

The notion of B-equivalence of *functionals* $(\mathcal{B}_h, J(h, \boldsymbol{\phi}))$ is an extension of the notion of B-equivalence, introduced in [16], now encompassing buckling modes in addition to buckling loads.

Let us first address a question of restricting the space of functions $\mathcal{B}_h$ to an "ansatz" $\mathcal{C}_h$. The answer is fairly obvious. The space $\mathcal{B}_h$ can be replaced by an "ansatz" $\mathcal{C}_h$ if the ansatz contains the buckling mode. We state this observation as a lemma.

LEMMA **2.10.** *Suppose the pair $(\mathcal{B}_h, J(h, \boldsymbol{\phi}))$ characterizes buckling. Let $\mathcal{C}_h \subset \mathcal{B}_h$ be such that it contains a buckling mode. Then the pair $(\mathcal{C}_h, J(h, \boldsymbol{\phi}))$ characterizes buckling.*

Proof. Let

$$\lambda(h) = \inf_{\boldsymbol{\phi} \in \mathcal{B}_h} J(h, \boldsymbol{\phi}), \qquad \widetilde{\lambda}(h) = \inf_{\boldsymbol{\phi} \in \mathcal{C}_h} J(h, \boldsymbol{\phi}).$$

Then, clearly, $\widetilde{\lambda}(h) \geq \lambda(h)$. By assumption there exists a buckling mode $\boldsymbol{\phi}_h \in \mathcal{C}_h \subset \mathcal{B}_h$. Therefore,

$$\overline{\lim_{h\to 0}} \frac{\widetilde{\lambda}(h)}{\lambda(h)} \leq \lim_{h\to 0} \frac{J(h,\boldsymbol{\phi}_h)}{\lambda(h)} = 1,$$

since the pair $(\mathcal{B}_h, J(h,\boldsymbol{\phi}))$ characterizes buckling. Hence

$$\lim_{h\to 0} \frac{\widetilde{\lambda}(h)}{\lambda(h)} = 1, \tag{2.31}$$

and part (a) of Definition 2.8 is established.

If $\boldsymbol{\phi}_h \in \mathcal{C}_h \subset \mathcal{B}_h$ is a buckling mode then

$$\lim_{h\to 0} \frac{J(h,\boldsymbol{\phi}_h)}{\lambda(h)} = 1,$$

since the pair $(\mathcal{B}_h, J(h,\boldsymbol{\phi}))$ characterizes buckling. Part (b) now follows from (2.31).

Finally, if $\boldsymbol{\phi}_h \in \mathcal{C}_h$ satisfies

$$\lim_{h\to 0} \frac{J(h,\boldsymbol{\phi}_h)}{\widetilde{\lambda}(h)} = 1,$$

then, $\boldsymbol{\phi}_h \in \mathcal{B}_h$ and by (2.31) we also have

$$\lim_{h\to 0} \frac{J(h,\boldsymbol{\phi}_h)}{\lambda(h)} = 1.$$

Therefore, $\boldsymbol{\phi}_h$ is a buckling mode. The Lemma is proved now. □

Our key tool for simplification of the functionals $J(h,\boldsymbol{\phi})$ characterizing buckling is the following theorem.

THEOREM **2.11** (B-equivalence)**.** *Suppose that $\lambda(h)$ is a buckling load in the sense of Definition 2.3. If either*

$$\lim_{h\to 0} \lambda(h) \sup_{\boldsymbol{\phi}\in\mathcal{B}_h} \left| \frac{1}{J_1(h,\boldsymbol{\phi})} - \frac{1}{J_2(h,\boldsymbol{\phi})} \right| = 0, \tag{2.32}$$

or

$$\lim_{h\to 0} \frac{1}{\lambda(h)} \sup_{\boldsymbol{\phi}\in\mathcal{B}_h} |J_1(h,\boldsymbol{\phi}) - J_2(h,\boldsymbol{\phi})| = 0, \tag{2.33}$$

then the pairs $(\mathcal{B}_h, J_1(h,\boldsymbol{\phi}))$ and $(\mathcal{B}_h, J_2(h,\boldsymbol{\phi}))$ are B-equivalent in the sense of Definition 2.9.

Proof. Let us introduce the following notation:

$$\lambda_i(h) = \inf_{\boldsymbol{\phi}\in\mathcal{B}_h} J_i(h,\boldsymbol{\phi}), \quad i = 1, 2.$$

$$\delta_{\pm}(h) = \lambda(h)^{\pm 1} \sup_{\boldsymbol{\phi}\in\mathcal{B}_h} \left| J_1(h,\boldsymbol{\phi})^{\pm 1} - J_2(h,\boldsymbol{\phi})^{\pm 1} \right|.$$

Then
$$\left|\frac{\lambda(h)}{\lambda_1(h)}-\frac{\lambda(h)}{\lambda_2(h)}\right|=\lambda(h)\left|\sup_{\boldsymbol{\phi}\in\mathcal{B}_h}\frac{1}{J_1(h,\boldsymbol{\phi})}-\sup_{\boldsymbol{\phi}\in\mathcal{B}_h}\frac{1}{J_2(h,\boldsymbol{\phi})}\right|\le\delta_-(h)$$
and
$$\frac{|\lambda_1(h)-\lambda_2(h)|}{\lambda(h)}=\frac{1}{\lambda(h)}\left|\inf_{\boldsymbol{\phi}\in\mathcal{B}_h}J_1(h,\boldsymbol{\phi})-\inf_{\boldsymbol{\phi}\in\mathcal{B}_h}J_2(h,\boldsymbol{\phi})\right|\le\delta_+(h).$$
Assume that $(\mathcal{B}_h, J_1(h,\boldsymbol{\phi}))$ characterizes buckling. Then we have just proved that if either $\delta_+(h)\to 0$ or $\delta_-(h)\to 0$, as $h\to 0$, then $\lambda_2(h)/\lambda(h)\to 1$, as $h\to 0$, and condition (a) in Definition 2.8 is proved for $J_2(h,\boldsymbol{\phi})$.

Observe that by parts (b) and (c) of Definition 2.8 $\boldsymbol{\phi}_h\in\mathcal{B}_h$ is the buckling mode if and only if
$$\lim_{h\to 0}\frac{J_1(h,\boldsymbol{\phi}_h)}{\lambda_1(h)}=1.$$
This is equivalent to
$$\lim_{h\to 0}\frac{\lambda(h)}{J_1(h,\boldsymbol{\phi}_h)}=1.$$
Therefore,
$$\lim_{h\to 0}\frac{J_2(h,\boldsymbol{\phi}_h)}{\lambda(h)}=1,$$
since either
$$\left|\frac{\lambda(h)}{J_1(h,\boldsymbol{\phi}_h)}-\frac{\lambda(h)}{J_2(h,\boldsymbol{\phi}_h)}\right|\le\delta_-(h)$$
or
$$\frac{|J_1(h,\boldsymbol{\phi}_h)-J_2(h,\boldsymbol{\phi}_h)|}{\lambda(h)}\le\delta_+(h).$$
Thus, in view of part (a), $\boldsymbol{\phi}_h$ is a buckling mode if and only if
$$\lim_{h\to 0}\frac{J_2(h,\boldsymbol{\phi}_h)}{\lambda_2(h)}=1.$$
□

As an application of Theorem 2.11 we show that we can simplify the Rayleigh quotient $\mathfrak{R}(h,\boldsymbol{\phi})$ further.

Theorem **2.12.** *Suppose that the critical load $\lambda_{\rm cl}(h)$ satisfies (2.22). Let*
$$\mathfrak{R}^*(h,\boldsymbol{\phi})=-\frac{\int_{\Omega_h}\langle\mathsf{L}_0e(\boldsymbol{\phi}),e(\boldsymbol{\phi})\rangle d\boldsymbol{x}}{\frac{1}{4}\int_{\Omega_h}\langle\widetilde{\boldsymbol{\sigma}}_h\nabla\times\boldsymbol{\phi},\nabla\times\boldsymbol{\phi}\rangle d\boldsymbol{x}}=-\frac{\mathfrak{S}_h(\boldsymbol{\phi})}{\mathfrak{C}_h^*(\boldsymbol{\phi})},$$
where
$$\widetilde{\boldsymbol{\sigma}}_h=(\operatorname{Tr}\boldsymbol{\sigma}_h)\boldsymbol{I}-\boldsymbol{\sigma}_h \tag{2.34}$$
is the compression tensor. Then $(\mathcal{A}_h,\mathfrak{R}(h,\boldsymbol{\phi}))$ and $(\mathcal{A}_h,\mathfrak{R}^*(h,\boldsymbol{\phi}))$ are B-equivalent.

Proof. For $\boldsymbol{a}\in\mathbb{R}^3$, let $\pi(\boldsymbol{a})$ denote a 3×3 antisymmetric matrix defined by the cross-product map:

$$\pi(\boldsymbol{a})\boldsymbol{u}=\boldsymbol{a}\times\boldsymbol{u}.$$

Then $\nabla\boldsymbol{\phi}-(\nabla\boldsymbol{\phi})^T=\pi(\nabla\times\boldsymbol{\phi})$. We observe that replacing $\nabla\boldsymbol{\phi}$ with $e(\boldsymbol{\phi})+\pi(\nabla\times\boldsymbol{\phi})/2$ in $\mathfrak{C}_h(\boldsymbol{\phi})$ we obtain

$$\mathfrak{C}_h(\boldsymbol{\phi})=\int_{\Omega_h}\langle\boldsymbol{\sigma}_h, e(\boldsymbol{\phi})^2+e(\boldsymbol{\phi})\pi(\nabla\times\boldsymbol{\phi})\rangle d\boldsymbol{x}+\mathfrak{C}_h^*(\boldsymbol{\phi}).$$

It follows that for every $\boldsymbol{\phi}\in V_h$

$$|\mathfrak{C}_h(\boldsymbol{\phi})-\mathfrak{C}_h^*(\boldsymbol{\phi})|\le\|\boldsymbol{\sigma}_h\|_\infty(\|e(\boldsymbol{\phi})\|^2+2\|e(\boldsymbol{\phi})\|\|\nabla\boldsymbol{\phi}\|)\le\|\boldsymbol{\sigma}_h\|_\infty\|e(\boldsymbol{\phi})\|^2\left(1+\frac{2}{\sqrt{K(V_h)}}\right).$$

Recalling that, due to (2.1),

$$\mathfrak{S}_h(\boldsymbol{\phi})\ge\alpha_{\mathsf{L}_0}\|e(\boldsymbol{\phi})\|^2$$

we obtain

$$\lambda_{\rm cl}(h)\left|\frac{1}{\mathfrak{R}(h,\boldsymbol{\phi})}-\frac{1}{\mathfrak{R}^*(h,\boldsymbol{\phi}))}\right|\le\frac{\|\boldsymbol{\sigma}_h\|_\infty}{\alpha_{\mathsf{L}_0}}\left(\lambda_{\rm cl}(h)+\frac{2\lambda_{\rm cl}(h)}{\sqrt{K(V_h)}}\right).$$

Thus (2.22) implies that the sufficient condition (2.32) for B-equivalence is satisfied. The theorem is proved. □

Theorem 2.12 shows that in order to quantify the destabilizing effect of stress in a pre-buckled state, it is not necessary to deal with a quadratic form $\boldsymbol{H}\mapsto\langle\boldsymbol{\sigma}_h,\boldsymbol{H}^T\boldsymbol{H}\rangle$ on the 9-dimensional space of 3×3 matrices. The same information is also contained in a quadratic form on $\mathbb{R}^3$, given by the symmetric 3×3 matrix $\widetilde{\boldsymbol{\sigma}}_h$. For example, the set of destabilizing variations $\mathcal{A}_h$ could be empty, even if $\boldsymbol{\sigma}_h$ has negative eigenvalues. By contrast, $\mathcal{A}_h$ is non-empty if and only if , the compression tensor has a negative eigenvalue. Similarly, $\mathcal{A}_h=V_h$, if and only if $\widetilde{\boldsymbol{\sigma}}_h$ is negative definite, which does not require all of the principal stresses to be negative. [8] In that respect the compression tensor $\widetilde{\boldsymbol{\sigma}}_h$ is a much better descriptor of compressiveness than the original stress tensor $\boldsymbol{\sigma}_h$. The tensor $\widetilde{\boldsymbol{\sigma}}_h$ is a natural quantity and has been appearing regularly, albeit implicitly, in analyses of stability, e.g. [19, 18, 8].

Another consequence of Theorem 2.12 is a dramatic difference between buckling in two and three dimensions. It comes from the fact that $\nabla\times\boldsymbol{\phi}$ is a *scalar* in 2D, and similar calculations show that the functional $\mathfrak{R}(h,\boldsymbol{\phi})$ can be replaced in 2D by

$$\mathfrak{R}^{2D}(h,\boldsymbol{\phi})=-\frac{\int_{\Omega_h}\langle\mathsf{L}_0 e(\boldsymbol{\phi}),e(\boldsymbol{\phi})\rangle d\boldsymbol{x}}{\frac{1}{2}\int_{\Omega_h}(\mathrm{Tr}\,\boldsymbol{\sigma}_h(\boldsymbol{x}))|\nabla\boldsymbol{\phi}|^2 d\boldsymbol{x}},$$

or, in the case of a homogeneous trivial branch, by

$$\mathfrak{R}^{2D}_{\rm hom}(h,\boldsymbol{\phi})=-\frac{2\int_{\Omega_h}\langle\mathsf{L}_0 e(\boldsymbol{\phi}),e(\boldsymbol{\phi})\rangle d\boldsymbol{x}}{\mathfrak{c}\|\nabla\boldsymbol{\phi}\|^2},$$

[8] The former happens when $\sigma_1+\sigma_2<0$, where $\sigma_1\le\sigma_2\le\sigma_3$ are the principal stresses, the latter when $\sigma_2+\sigma_3<0$.

where

$$\mathfrak{c} = \lim_{h\to 0} \operatorname{Tr} \boldsymbol{\sigma}_h.$$

Therefore, we can formulate the *2D alternative*. Generically, either $\mathfrak{c} > 0$, in which case the structure will not buckle (in the sense of near-flip buckling), or $\mathfrak{c} < 0$, in which case we have a general formula for the critical load [16]:

$$\lambda_{\rm cl}(h) = -\frac{2K_{\mathsf{L}_0}(V_h)}{\mathfrak{c}},$$

where

$$K_{\mathsf{L}_0}(V_h) = \inf_{\boldsymbol{\phi}\in V_h} \frac{\int_{\Omega_h} \langle \mathsf{L}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle d\boldsymbol{x}}{\|\nabla\boldsymbol{\phi}\|^2}. \tag{2.35}$$

By contrast, the situation in 3D is much more nuanced, even in the case of a homogeneous trivial branch, since the asymptotics of the critical load depends on a *tensor* $\widetilde{\boldsymbol{\sigma}}_h$, given by (2.34), instead of a scalar $\operatorname{Tr} \boldsymbol{\sigma}_h$.

3 Axially compressed circular cylindrical shells

In this section we apply the theory of near-flip buckling developed in Section 2 to the buckling of circular cylindrical shells under axial compression. In the remainder of the paper we assume that the energy density $W(\boldsymbol{F})$ is isotropic.

3.1 The trivial branch

Consider the circular cylindrical shell given in cylindrical coordinates (r, θ, z) as follows:

$$\mathcal{C}_h = I_h \times \mathbb{T} \times [0, L], \qquad I_h = [1 - h/2, 1 + h/2],$$

where $\mathbb{T}$ is a 1-dimensional torus (circle) describing 2π-periodicity in θ. We impose the the following boundary conditions on the deformation $\boldsymbol{y} : \mathcal{C}_h \to \mathbb{R}^3$

$$y_\theta(r, \theta, 0) = y_z(r, \theta, 0) = y_\theta(r, \theta, L) = 0, \quad y_z(r, \theta, L) = (1 - \lambda)L, \tag{3.1}$$

where y_r, y_θ and y_z are coordinates of the vector field $\boldsymbol{y}$ in the standard moving orthonormal frame $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$ associated with cylindrical coordinates. The loading is parametrized by the compressive strain $\lambda > 0$ in the axial direction. According to (3.1) the the base of the shell is fixed in the vertical and circumferential directions, but is allowed to undergo radial Poisson expansion, when compressed. At the top rim the loading device is controlling only the vertical component of the displacement.

To use our theory of buckling we need to describe the applied loading in the form (2.2), (2.3). This is done by defining

$$\overline{\boldsymbol{y}}(\boldsymbol{x}; h, \lambda) = (1 - \lambda)z\boldsymbol{e}_z, \qquad \boldsymbol{t}(\boldsymbol{x}; h, \lambda) = \boldsymbol{0}. \tag{3.2}$$

In that case

$$V_h = \{\boldsymbol{\phi} \in W^{1,2}(\mathcal{C}_h;\mathbb{R}^3) : \phi_\theta(r,\theta,0) = \phi_z(r,\theta,0) = \phi_\theta(r,\theta,L) = \phi_z(r,\theta,L) = 0\}. \qquad (3.3)$$

In our notation the dependence on the fixed ratio L of length to the radius will be consistently suppressed, while the essential dependence on $h \to 0$ will be emphasized. Let us show that (under a certain non-degeneracy condition) there exists a homogeneous trivial branch $\boldsymbol{y}(\boldsymbol{x}; h, \lambda)$, given in cylindrical coordinates by

$$y_r = (a(\lambda) + 1)r, \qquad y_\theta = 0, \qquad y_z = (1 - \lambda)z, \qquad (3.4)$$

where the function $a(\lambda)$, satisfying $a(0) = 0$, is uniquely determined by the energy density function $W(\boldsymbol{F})$ through the natural boundary conditions at the two lateral boundaries of the shell

$$\boldsymbol{P}(\nabla \boldsymbol{y})\boldsymbol{e}_r = \boldsymbol{0}, \quad r = 1 \pm \frac{h}{2}. \qquad (3.5)$$

Here $\boldsymbol{P}(\boldsymbol{F}) = W_{\boldsymbol{F}}(\boldsymbol{F})$, the gradient of W with respect to $\boldsymbol{F}$, denotes the Piola-Kirchhoff stress tensor.

LEMMA **3.1.** *Assume that $W(\boldsymbol{F})$ is three times continuously differentiable in a neighborhood of $\boldsymbol{F} = \boldsymbol{I}$, satisfies properties (P1)–(P3) and that $W(\boldsymbol{F})$ is isotropic, i.e.*

$$W(\boldsymbol{F}\boldsymbol{R}) = W(\boldsymbol{F}), \qquad \forall \boldsymbol{R} \in SO(3). \qquad (3.6)$$

Then there exists a unique function $a(\lambda)$, of class C^2 on a neighborhood of 0, such that $a(0) = 0$ and the natural boundary conditions (3.5) are satisfied by the trivial branch (3.4).

This lemma is a particular case of Lemma 4.1, corresponding to $p = 0$. Lemma 3.1 implies that the fundamental regularity assumption (2.4) is satisfied, since the trivial branch is smooth in λ and does not depend on h explicitly. It is straightforward to compute, using (3.4) and (3.5), that

$$\boldsymbol{\sigma}_h = -E\boldsymbol{e}_z \otimes \boldsymbol{e}_z, \qquad (3.7)$$

where E is the Young's modulus.

3.2 Scaling of the critical load

THEOREM **3.2.** *Suppose that $\boldsymbol{\sigma}_h$ is given by (3.7). Then there exist constants $c > 0$ and $C > 0$ depending only on L and the elastic moduli, such that*

$$ch \le \lambda_{\text{cl}}(h) \le Ch. \qquad (3.8)$$

Proof. Observe that

$$\mathfrak{C}_h(\boldsymbol{\phi}) = \int_{\mathcal{C}_h} \langle \boldsymbol{\sigma}_h, \nabla \boldsymbol{\phi}^T \nabla \boldsymbol{\phi} \rangle d\boldsymbol{x} = -E(\|\phi_{r,z}\|^2 + \|\phi_{\theta,z}\|^2 + \|\phi_{z,z}\|^2),$$

and there exist constants $\alpha > 0$ and $\beta > 0$ (depending only on the elastic moduli) such that

$$\alpha\|e(\boldsymbol{\phi})\|^2 \le \mathfrak{S}_h(\boldsymbol{\phi}) \le \beta\|e(\boldsymbol{\phi})\|^2.$$

Thus, in order to compute the scaling of $\lambda_{\rm cl}(h)$, given by (2.18) and verify conditions of Theorem 2.6 we need to estimate the Korn constant $K(V_h)$, as well as the norms of gradient components $\|\phi_{r,z}\|^2$, $\|\phi_{\theta,z}\|^2$ and $\|\phi_{z,z}\|^2$ in terms of $\|e(\boldsymbol{\phi})\|$. This was accomplished in [14]. The desired estimates are stated in the following theorem.

THEOREM **3.3** (Korn-type inequalities)**.** *There exist constants $C, c > 0$ depending only on L such that*

$$ch^{3/2} \le K(V_h) \le Ch^{3/2}. \tag{3.9}$$

$$\|\phi_{\theta,z}\|^2 \le \frac{C}{\sqrt{h}}\|e(\boldsymbol{\phi})\|^2, \tag{3.10}$$

$$\|\phi_{r,z}\|^2 \le \frac{C}{h}\|e(\boldsymbol{\phi})\|^2. \tag{3.11}$$

Moreover, the powers of h in the inequalities (3.9)–(3.11) are optimal, achieved simultaneously *by the ansatz*

$$\begin{cases} \phi_r^h(r,\theta,z) = & -W_{,\eta\eta}\left(\frac{\theta}{\sqrt[4]{h}}, z\right) \\ \phi_\theta^h(r,\theta,z) = & r\sqrt[4]{h}W_{,\eta}\left(\frac{\theta}{\sqrt[4]{h}}, z\right) + \frac{r-1}{\sqrt[4]{h}}W_{,\eta\eta\eta}\left(\frac{\theta}{\sqrt[4]{h}}, z\right), \\ \phi_z^h(r,\theta,z) = & (r-1)W_{,\eta\eta z}\left(\frac{\theta}{\sqrt[4]{h}}, z\right) - \sqrt{h}W_{,z}\left(\frac{\theta}{\sqrt[4]{h}}, z\right), \end{cases} \tag{3.12}$$

for any smooth compactly supported function $W(\eta, z)$ on $\mathbb{R} \times (0, L)$, with the understanding that the function $\boldsymbol{\phi}^h(\theta, z)$ is extended 2π-periodically in θ from $[0, 2\pi)$ to $\mathbb{R}$.

Adding inequalities (3.10), (3.11) and an obvious inequality $\|\phi_{z,z}\|^2 \le \|e(\boldsymbol{\phi})\|^2$ we obtain

$$-\mathfrak{C}_h(\boldsymbol{\phi}) \le \frac{C}{h}\mathfrak{S}_h(\boldsymbol{\phi}). \tag{3.13}$$

The power of h in (3.13) is optimal, achieved by the ansatz (3.12). Hence, the variational definition (2.18) of $\lambda_{\rm cl}(h)$ implies (3.8). □

We observe that the upper bound in (3.8) implies that condition (2.22) in Theorem 2.6 is satisfied. This proves that $\lambda_{\rm cl}(h)$ is the buckling load in the sense of Definition 2.3. If the Hooke's law tensor L_0 is isotropic, then, as we have shown in [15], the exact asymptotics of $\lambda_{\rm cl}$, given by (2.18), is

$$\lim_{h\to 0}\frac{\lambda_{\rm cl}(h)}{h} = \frac{1}{\sqrt{3(1-\nu^2)}},$$

which together with (3.7) agrees with the classical formula (1.1) for the critical stress.

4 Imperfections of load and scaling instability

Let us now assume that the cylinder remains perfectly circular, however, some of the boundary conditions (3.1), (3.2) are perturbed. We also continue working under the assumption that $W(\boldsymbol{F})$ is isotropic. The goal is to show that there are imperfections of load that can lead to two distinct scalings of the critical strain $h^{3/2}$ and $h^{5/4}$. This is a "proof of concept" result, exhibiting the workings of scaling instability mechanism, rather than an explanation of particular experimental data. We do not pretend that the boundary conditions in our loading programs represent an actual experimental setup. However, we do believe that the scaling instability mechanism is still at work under real experimental conditions.

4.1 Scaling $\lambda^*(h) \sim h^{3/2}$

Let us consider the loading program where in addition to increasing the compressive strain λ we also simultaneously increase pressure. In other words, the perturbed loading is now given by (3.1) and

$$\overline{\boldsymbol{y}}(\boldsymbol{x}; h, \lambda) = (1-\lambda) z \boldsymbol{e}_z, \qquad \boldsymbol{t}(\boldsymbol{x}; h, \lambda) = -\lambda p \boldsymbol{n}, \tag{4.1}$$

where $p > 0$ and $\boldsymbol{n}$ is the outward unit normal on $\partial \mathcal{C}_h$. Since we study the loading for very small values of λ, the applied pressure λp can be regarded as a load imperfection, even for values of p of order 1. It is important to note that the space V_h has remained the same, since only the value of the traction vector was modified.

Let us show that such a loading is still consistent with a homogeneous trivial branch of the form (3.4) for an isotropic material.

LEMMA **4.1.** *Assume that $W(\boldsymbol{F})$ is three times continuously differentiable in a neighborhood of $\boldsymbol{F} = \boldsymbol{I}$, satisfies properties (P1)–(P3) and is isotropic, i.e. satisfies (3.6). Then for every $p_0 > 0$ there exists $\delta > 0$ and a unique function $a(\lambda, p)$, of class $C^2((-\delta, \delta) \times (-p_0, p_0))$, such that $a(0, p) = 0$ and the natural boundary conditions*

$$\boldsymbol{P}(\nabla \boldsymbol{y}) \boldsymbol{e}_r = -\lambda p \boldsymbol{e}_r, \quad r = 1 \pm \frac{h}{2},$$

are satisfied by the trivial branch (3.4).

Proof. By (P2) $W(\boldsymbol{F}) = \hat{W}(\boldsymbol{F}^T \boldsymbol{F})$. The function $\hat{W}(\boldsymbol{C})$ is three times continuously differentiable in a neighborhood of $\boldsymbol{C} = \boldsymbol{I}$. Thus,

$$\boldsymbol{P}(\boldsymbol{F}) = W_{\boldsymbol{F}}(\boldsymbol{F}) = 2\boldsymbol{F} \hat{W}_{\boldsymbol{C}}(\boldsymbol{F}^T \boldsymbol{F}).$$

The isotropy of $W(\boldsymbol{F})$ implies that $\hat{W}(\boldsymbol{R}\boldsymbol{C}\boldsymbol{R}^T) = \hat{W}(\boldsymbol{C})$ for all $\boldsymbol{R} \in SO(3)$. Differentiating this relation in $\boldsymbol{R}$ at $\boldsymbol{R} = \boldsymbol{I}$ we conclude that $\hat{W}_{\boldsymbol{C}}(\boldsymbol{C})$ must commute with $\boldsymbol{C}$. In particular, this implies that the matrix $\hat{W}_{\boldsymbol{C}}(\boldsymbol{C})$ must be diagonal, whenever $\boldsymbol{C}$ is diagonal. We compute that in cylindrical coordinates

$$\boldsymbol{F} = \nabla \boldsymbol{y} = \begin{bmatrix} 1+a & 0 & 0 \\ 0 & 1+a & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}, \quad \boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = \begin{bmatrix} (1+a)^2 & 0 & 0 \\ 0 & (1+a)^2 & 0 \\ 0 & 0 & (1-\lambda)^2 \end{bmatrix}.$$

Hence, $\boldsymbol{P}(\boldsymbol{F})$ is diagonal, and boundary conditions (3.5) reduce to a single scalar equation

$$\hat{W}_{\boldsymbol{C}}((1+a)^2(\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta) + (1-\lambda)^2 \boldsymbol{e}_z \otimes \boldsymbol{e}_z)\boldsymbol{e}_r \cdot \boldsymbol{e}_r = -\frac{\lambda p}{2(1+a)}. \tag{4.2}$$

Condition (P1) implies that $(\lambda, p, a) = (0, p, 0)$ is a solution. The conclusion of the lemma is guaranteed by the implicit function theorem, provided

$$\mathsf{L}_0(\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta)\boldsymbol{e}_r \cdot \boldsymbol{e}_r \neq 0. \tag{4.3}$$

When L_0 is isotropic, the non-degeneracy condition (4.3) becomes

$$\frac{E}{(1+\nu)(1-2\nu)} \neq 0.$$

This condition is always satisfied because the left-hand side is positive due to (P3)–(P4). □

In order to find the critical load (asymptotically) we need to compute

$$\boldsymbol{\sigma}_h = \mathsf{L}_0 \nabla \boldsymbol{u}^h = \mathsf{L}_0 \left. \frac{\partial \boldsymbol{F}}{\partial \lambda} \right|_{\lambda=0}.$$

Differentiating (4.2) in λ at $\lambda = 0$ to find $\partial a / \partial \lambda$ we compute

$$\boldsymbol{\sigma}_h = -p\boldsymbol{I} - (E - 2p(1-\nu))\boldsymbol{e}_z \otimes \boldsymbol{e}_z. \tag{4.4}$$

THEOREM **4.2.** *The pair $(V_h, J(h, \boldsymbol{\phi}))$ characterizes buckling, where*

$$J(h, \boldsymbol{\phi}) = \frac{\int_{\Omega_h} \langle \mathsf{L}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi}) \rangle d\boldsymbol{x}}{p \|\nabla \boldsymbol{\phi}\|^2}.$$

Proof. We compute

$$\left| \frac{1}{\mathfrak{R}(h, \boldsymbol{\phi})} - \frac{1}{J(h, \boldsymbol{\phi})} \right| = \frac{|E - 2p(1-\nu)| \|(\nabla \boldsymbol{\phi})\boldsymbol{e}_z\|^2}{\int_{\Omega_h} \langle \mathsf{L}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi}) \rangle d\boldsymbol{x}}.$$

Thus, according to Theorem 3.3

$$\sup_{\boldsymbol{\phi} \in V_h} \left| \frac{1}{\mathfrak{R}(h, \boldsymbol{\phi})} - \frac{1}{J(h, \boldsymbol{\phi})} \right| \leq \frac{|E - 2p(1-\nu)|}{\alpha_{\mathsf{L}_0}} \sup_{\boldsymbol{\phi} \in V_h} \frac{\|(\nabla \boldsymbol{\phi})\boldsymbol{e}_z\|^2}{\|e(\boldsymbol{\phi})\|^2} \leq \frac{C}{h}.$$

Denoting $\lambda(h) = \inf_{\boldsymbol{\phi} \in V_h} J(h, \boldsymbol{\phi})$, we obtain

$$\left| \frac{\lambda(h)}{\lambda_{\mathrm{cl}}(h)} - 1 \right| = \lambda(h) \left| \sup_{\boldsymbol{\phi} \in V_h} \frac{1}{\mathfrak{R}(h, \boldsymbol{\phi})} - \sup_{\boldsymbol{\phi} \in V_h} \frac{1}{J(h, \boldsymbol{\phi})} \right| \leq \frac{\lambda(h) C}{h}.$$

Applying Theorem 3.3 once again we conclude that $\lambda(h) \leq C h^{3/2}$, and therefore,

$$\lim_{h \to 0} \frac{\lambda(h)}{\lambda_{\mathrm{cl}}(h)} = 1.$$

Thus, condition (2.22) holds, and hence, both $\lambda_{\rm cl}(h)$ and $\lambda(h)$ are buckling loads. The proof is concluded by the application of Theorem 2.11, since

$$\lim_{h\to 0}\lambda(h)\sup_{\boldsymbol{\phi}\in V_h}\left|\frac{1}{\mathfrak{R}(h,\boldsymbol{\phi})}-\frac{1}{J(h,\boldsymbol{\phi})}\right|=0.$$

□

As a corollary of Theorem 4.2 we obtain that

$$\lambda^*(h)\sim\frac{K_{\mathsf{L}_0}(V_h)}{p}\sim h^{3/2}, \tag{4.5}$$

where $K_{\mathsf{L}_0}(V_h)$ is defined in (2.35). This corresponds to the von Mises-Southwell formula [33, 34, 35, 40].

We remark that adding pressure component to the load can be regarded as a load imperfection. According to (4.5), the critical stress and pressure are

$$\sigma_{\rm cr}\sim E\lambda^*(h)\sim\frac{E^2h^{3/2}}{p},\qquad P_{\rm cr}=\lambda_{\rm cr}p\sim Eh^{3/2},$$

respectively. Hence, the condition for interpreting pressure as a load imperfection is $p\ll E$. Since E ranges between 1 and 100 GPa for typical solids, the values of p on the order of KPa would qualify as small, compared to E.

4.2 Scaling $\lambda^*(h)\sim h^{5/4}$

Let us examine a more subtle imperfection of load that occurs if we require to keep shell's lateral boundary stress-free. We also assume that boundary conditions (3.1) hold, except that $y_\theta(r,\theta,L)$ is no longer required to be zero. We will work under the assumption that the new loading results in a regular trivial branch in the sense of Definition 2.1. While we cannot prove that such a trivial branch exists for all energy density functions $W(\boldsymbol{F})$, satisfying (P1)–(P4), we can exhibit one explicitly for a neo-Hookean material, which is done in Appendix A. However, it is important to emphasize that once the existence of a regular trivial branch is established, the theory expresses the asymptotics of the buckling load entirely in terms of linear elastic trivial branch.

Let us show that regardless of the specific functional form of $y_\theta(r,\theta,L)$ the stress $\boldsymbol{\sigma}^h$ that enters the constitutively linearized second variation is almost completely determined in an asymptotic sense. For this calculation we make an additional assumption that the family of Lipschitz functions $\boldsymbol{u}^h$ from Definition 2.1 depends regularly on r and h. This assumption can be verified directly for our explicit trivial in Appendix A. Hence, we have the relations

$$\boldsymbol{u}^h(r,\theta,z)\approx\widetilde{\boldsymbol{u}}^h(r,\theta,z)=\boldsymbol{u}^0(\theta,z)+(r-1)\boldsymbol{u}^1(\theta,z)+\frac{(r-1)^2}{2}\boldsymbol{u}^2(\theta,z), \tag{4.6}$$

that are understood in the following sense:

$$\lim_{h\to 0}\boldsymbol{u}^h=\lim_{h\to 0}\widetilde{\boldsymbol{u}}^h=\boldsymbol{u}^0,\qquad \lim_{h\to 0}\nabla\boldsymbol{u}^h=\lim_{h\to 0}\nabla\widetilde{\boldsymbol{u}}^h=\lim_{h\to 0}\nabla(\boldsymbol{u}^0+(r-1)\boldsymbol{u}^1),$$

$$\lim_{h\to 0} \nabla\nabla \boldsymbol{u}^h = \lim_{h\to 0} \nabla\nabla \widetilde{\boldsymbol{u}}^h,$$

where the first two limits are understood in the a.e. sense, while the last limit is understood in the sense of distributions.

THEOREM **4.3.** *Suppose that $\boldsymbol{u}^h(r,\theta,z)$ from Definition 2.1 depends regularly on r and h. Suppose further that*

(i) $\nabla\cdot(\mathsf{L}_0 e(\boldsymbol{u}^h)) = \boldsymbol{0}$,

(ii) $\boldsymbol{\sigma}_h \boldsymbol{e}_r = \boldsymbol{0}$ *at* $r = 1 \pm h/2$, *where* $\boldsymbol{\sigma}_h = \mathsf{L}_0 e(\boldsymbol{u}^h)$,

(iii) $u_z^h(r,\theta,0) = u_\theta^h(r,\theta,0) = 0$.

Then there exist two constants s and t, such that

$$u_r^0 + (r-1)u_r^1 = -\frac{t\nu}{E}r, \quad u_\theta^0 + (r-1)u_\theta^1 = \frac{2(1+\nu)s}{E}rz, \quad u_z^0 + (r-1)u_z^1 = \frac{t}{E}z, \tag{4.7}$$

and consequently

$$\boldsymbol{\sigma}^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & s \\ 0 & s & t \end{bmatrix}. \tag{4.8}$$

Proof. By the assumptions of regularity (4.6) and by condition (i) we have

$$\lim_{h\to 0} \nabla\cdot\boldsymbol{\sigma}_h = \lim_{h\to 0} \nabla\cdot(\boldsymbol{\sigma}^0(\theta,z) + (r-1)\boldsymbol{\sigma}^1(\theta,z)) = 0, \tag{4.9}$$

where

$$\boldsymbol{\sigma}^0 = \lim_{h\to 0} \mathsf{L}_0 e(\boldsymbol{u}^0 + (r-1)\boldsymbol{u}^1), \qquad \boldsymbol{\sigma}^1 = \lim_{h\to 0} \mathsf{L}_0 e\left(\boldsymbol{u}^1 + \frac{r-1}{2}\boldsymbol{u}^2\right). \tag{4.10}$$

Passing to the limit as $h \to 0$ in (4.9), we obtain

$$\begin{cases} \sigma_{rr}^1 + \sigma_{r\theta,\theta}^0 + \sigma_{rr}^0 - \sigma_{\theta\theta}^0 + \sigma_{rz,z}^0 = 0, \\ \sigma_{r\theta}^1 + \sigma_{\theta\theta,\theta}^0 + 2\sigma_{r\theta}^0 + \sigma_{\theta z,z}^0 = 0, \\ \sigma_{rz}^1 + \sigma_{\theta z,\theta}^0 + \sigma_{rz}^0 + \sigma_{zz,z}^0 = 0. \end{cases} \tag{4.11}$$

The traction-free boundary conditions $\boldsymbol{\sigma}_h \boldsymbol{e}_r = \boldsymbol{0}$ at $r = 1 \pm h/2$ imply that

$$\boldsymbol{\sigma}^0(\theta,z)\boldsymbol{e}_r = \boldsymbol{\sigma}^1(\theta,z)\boldsymbol{e}_r = \boldsymbol{0}$$

for all $(\theta,z) \in \mathbb{T}\times(0,L)$. Substituting these equations into (4.11) we obtain

$$\sigma_{\theta\theta}^0 = 0, \quad \sigma_{\theta z,z}^0 = 0, \qquad \sigma_{\theta z,\theta}^0 + \sigma_{zz,z}^0 = 0.$$

Solving these equations we obtain

$$\boldsymbol{\sigma}^0(\theta,z) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & s(\theta) \\ 0 & s(\theta) & t(\theta) - zs'(\theta) \end{bmatrix} \tag{4.12}$$

for some functions $s(\theta)$ and $t(\theta)$. The first equation in (4.10) can now be written as

$$\begin{cases} u_\theta^1 = u_\theta^0 - u_{r,\theta}^0, \quad u_z^1 = -u_{r,z}^0, \quad u_{\theta,z}^0 + u_{z,\theta}^0 = \frac{2(1+\nu)}{E}s(\theta), \\ u_r^1 = -\frac{\nu}{1-\nu}(u_r^0 + u_{\theta,\theta}^0 + u_{z,z}^0), \quad u_r^0 + u_{\theta,\theta}^0 + \frac{\nu}{1-\nu}(u_r^1 + u_{z,z}^0) = 0, \\ u_{z,z}^0 + \frac{\nu}{1-\nu}(u_r^1 + u_r^0 + u_{\theta,\theta}^0) = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}(t(\theta) - zs'(\theta)). \end{cases} \tag{4.13}$$

Solving these equations subject to the conditions

$$u_z^0(\theta,0) = u_\theta^0(\theta,0) = u_z^1(\theta,0) = u_\theta^1(\theta,0) = 0$$

we conclude that the functions $s(\theta)$ and $t(\theta)$ have to be constant[9] and that the formulas (4.7) hold. The formula (4.8) follows from (4.12). □

We conclude that imperfections of load give rise to a family of trivial branches characterized by the parameter s in (4.7)–(4.8). The perfect axial compression described in Section 3.1 corresponds to $s = 0$, while $s \neq 0$ introduces a torsional component to the stress.

We show now that trivial branches characterized by a non-zero parameter s in (4.7)–(4.8) become unstable at the critical load that scales as $h^{5/4}$, establishing scaling instability of the critical load (1.1) as a function of h. It is an important feature of constitutive linearization, that in order to compute the asymptotics of the buckling load and the buckling mode we do not need to know the non-linear trivial branch explicitly. (We only need to know that it exists and is regular in the sense of Definition 2.1.) The desired asymptotics is given by Theorem 2.6 in terms of the solution $\boldsymbol{u}^h$ of the equations of *linear elasticity*[10]. In order to obtain $\boldsymbol{\sigma}^0$ of the form (4.8) we observe that

$$u_r^h = \nu r, \qquad u_\theta^h = srz, \qquad u_z^h = -z, \tag{4.14}$$

solves

$$\begin{cases} \nabla \cdot (\mathsf{L}_0 e(\boldsymbol{u}^h)) = \boldsymbol{0}, & \text{in } \mathcal{C}_h, \\ \boldsymbol{\sigma}_h \boldsymbol{e}_r = \boldsymbol{0}, & r = 1 \pm \frac{h}{2}, \\ u_z^h = u_\theta^h = 0, & z = 0, \end{cases} \tag{4.15}$$

resulting in

$$\boldsymbol{\sigma}_h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{Esr}{2(\nu+1)} \\ 0 & \dfrac{Esr}{2(\nu+1)} & -E \end{bmatrix}, \qquad \boldsymbol{\sigma}^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{Es}{2(\nu+1)} \\ 0 & \dfrac{Es}{2(\nu+1)} & -E \end{bmatrix}. \tag{4.16}$$

[9] We note that s =constant is a consequence of $u_z^1(\theta,0) = 0$, while t =constant is a consequence of $u_\theta^1(\theta,0) = 0$.

[10] We emphasize that we are studying stability of the non-linearly elastic trivial branch in the context of fully non-linear hyperelasticity. Linear elastic equations supply the leading order asymptotics of the fully non-linear critical load.

In this explicit solution the imperfections of load are described by a single torsion parameter s. This specific representation of $\boldsymbol{\sigma}^0$ is, nevertheless, generic for arbitrary regular imperfections of load at $z = L$, according to Theorem 4.3. Similarly to Theorem 3.2, the formulas (4.16) determine the scaling of the critical load with h, which, for every fixed $s \neq 0$, is *different* from (3.8).

THEOREM **4.4.** *Suppose that* $\boldsymbol{\sigma}^0$ *is given by (4.16). Then there are positive constants* c *and* C*, depending only on* L *and the elastic moduli, such that*

$$\frac{c}{|s|} \leq \lim_{h\to 0} \frac{\lambda_{\rm cl}(h)}{h^{5/4}} \leq \frac{C}{|s|}. \tag{4.17}$$

Proof. Let us first obtain an upper bound on the critical load. This is done by using the test functions (3.12) in the estimate

$$\lambda_{\rm cl}(h) \leq \mathfrak{R}(h, \boldsymbol{\phi}^h).$$

We have

$$\mathfrak{S}_h(\boldsymbol{\phi}^h) \leq C\|e(\boldsymbol{\phi}^h)\|^2 \leq Ch^2,$$

$$-\mathfrak{C}_h(\boldsymbol{\phi}^h) = E\|(\nabla\boldsymbol{\phi}^h)\boldsymbol{e}_z\|^2 - \frac{sE}{\nu+1}((\nabla\boldsymbol{\phi}^h)\boldsymbol{e}_z, (\nabla\boldsymbol{\phi}^h)\boldsymbol{e}_\theta),$$

where (f, g) denotes the inner product in $L^2(\mathcal{C}_h)$. Extracting the leading order term in $\mathfrak{C}_h(\boldsymbol{\phi}^h)$ we rewrite it as

$$-\mathfrak{C}_h(\boldsymbol{\phi}^h) = -\frac{sE}{\nu+1}(r(\nabla\boldsymbol{\phi}^h)\boldsymbol{e}_z, (\nabla\boldsymbol{\phi}^h)\boldsymbol{e}_\theta) + \rho_h,$$

where, according to Theorem 3.3,

$$|\rho_h| \leq \frac{C}{h}\|e(\boldsymbol{\phi}^h)\|^2 \leq Ch.$$

Using the explicit formulas (3.12) for $\boldsymbol{\phi}^h$ we compute

$$\lim_{h\to 0} h^{-3/4}(r(\nabla\boldsymbol{\phi}^h)_{rz}, (\nabla\boldsymbol{\phi}^h)_{r\theta}) = \int_0^{2\pi}\int_0^L W_{,\eta\eta\eta}(\eta, z)W_{,\eta\eta z}(\eta, z)d\eta dz. \tag{4.18}$$

Hence, in order to prove the upper bound in (4.17) we only need to exhibit a fixed compactly supported function $W(\eta, z)$, such that the right-hand side in (4.18) is a non-zero number, whose sign is opposite to $\mathrm{sign}(s)$. This is done by choosing two arbitrary non-zero compactly supported functions $\phi(\eta)$ and $\psi(z)$ and setting

$$W(\eta, z) = \phi(\eta)\psi'(z) \pm \phi'(\eta)\psi(z).$$

Then

$$W_{,\eta\eta\eta}W_{,\eta\eta z} = \frac{1}{4}(\psi'(z)^2)'(\phi''(\eta)^2)' + (\phi'''(\eta)^2)'(\psi(z)^2)' \pm (\phi'''(\eta)\phi''(\eta))'\psi(z)\psi''(z)$$
$$\mp \phi'''(\eta)^2(\psi(z)\psi'(z))' \pm 2\phi'''(\eta)^2\psi'(z)^2.$$

This shows that

$$\int_0^{2\pi}\int_0^L W_{,\eta\eta\eta}W_{,\eta\eta z}d\eta dz = \pm 2\int_0^{2\pi}\int_0^L \phi'''(\eta)^2\psi'(z)^2 d\eta dz \neq 0.$$

Now, Theorem 2.6 applies, and we conclude that $\lambda_{\rm cl}(h)$ is a buckling load.

We now prove the lower bound in (4.17). Let

$$\mathfrak{C}_h^0(\boldsymbol{\phi}) = \int_{\mathcal{C}_h}\langle\boldsymbol{\sigma}^0, \nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi}\rangle d\boldsymbol{x}.$$

Observe that

$$-\mathfrak{C}_h^0(\boldsymbol{\phi}) = -\widetilde{\mathfrak{C}}_h(\boldsymbol{\phi}) + R_h(\boldsymbol{\phi}),$$

where

$$-\widetilde{\mathfrak{C}}_h(\boldsymbol{\phi}) = E(\|(\nabla\boldsymbol{\phi})_{rz}\|^2 + \|(\nabla\boldsymbol{\phi})_{\theta z}\|^2) - \frac{sE}{\nu+1}((\nabla\boldsymbol{\phi})_{rz}, (\nabla\boldsymbol{\phi})_{r\theta}),$$

and

$$R_h(\boldsymbol{\phi}) = E\|(\nabla\boldsymbol{\phi})_{zz}\|^2 - \frac{sE}{\nu+1}\{((\nabla\boldsymbol{\phi})_{\theta z}, (\nabla\boldsymbol{\phi})_{\theta\theta}) + (\nabla\boldsymbol{\phi})_{z\theta}, (\nabla\boldsymbol{\phi})_{zz})\}.$$

By Theorem 3.3 we estimate

$$|R_h| \le C(\|e(\boldsymbol{\phi})\|^2 + \|e(\boldsymbol{\phi})\|\|\nabla\boldsymbol{\phi}\|) \le \frac{C\|e(\boldsymbol{\phi})\|^2}{\sqrt{K(V_h)}},$$

where the constant C is independent of $\boldsymbol{\phi}\in V_h$. Let

$$\widetilde{\mathfrak{R}}(h,\boldsymbol{\phi}) = -\frac{\mathfrak{S}_h(\boldsymbol{\phi})}{\widetilde{\mathfrak{C}}_h(\boldsymbol{\phi})}.$$

Then

$$\left|\frac{1}{\widetilde{\mathfrak{R}}(h,\boldsymbol{\phi})} - \frac{1}{\mathfrak{R}(h,\boldsymbol{\phi})}\right| = \frac{|R_h|}{\mathfrak{S}_h(\boldsymbol{\phi})} \le \frac{C}{\sqrt{K(V_h)}} \le Ch^{-3/4}.$$

Recalling that we have prove that $\lambda_{\rm cl}(h)$ is the buckling mode and $\lambda_{\rm cl}(h) \le Ch^{5/4}$, we conclude, by Theorem 2.11, that the pair $(\widetilde{\mathfrak{R}}(h,\boldsymbol{\phi}), V_h)$ is B-equivalent to the pair $(\mathfrak{R}(h,\boldsymbol{\phi}), V_h)$. By Theorem 3.3 we obtain, applying the Cauchy-Schwarz inequality,

$$|((\nabla\boldsymbol{\phi})_{rz}, (\nabla\boldsymbol{\phi})_{\theta r})| \le \|(\nabla\boldsymbol{\phi})_{rz}\|\|\nabla\boldsymbol{\phi}\| \le \frac{C\|e(\boldsymbol{\phi})\|}{\sqrt{h}}\frac{\|e(\boldsymbol{\phi})}{\sqrt{K(V_h)}}\| \le \frac{C\|e(\boldsymbol{\phi})\|^2}{h^{5/4}}.$$

Applying Theorem 3.3 and (2.1) we obtain

$$\widetilde{\mathfrak{R}}(h,\boldsymbol{\phi}) \ge \frac{\alpha_{\mathsf{L}_0}\|e(\boldsymbol{\phi})\|^2}{C\|e(\boldsymbol{\phi})\|^2(h^{-1} + h^{-1/2} + |s|h^{-5/4})} \ge \frac{Ch^{5/4}}{|s| + h^{1/4}}. \tag{4.19}$$

□

We remark that the scaling law $\sigma_{\rm cr} \sim h^{5/4}$ has been obtained in [12] by methods similar to the ones used to obtain (1.1).

5 Conclusions

We have demonstrated that the theory of near-flip buckling from [16] can be made applicable to buckling of shells. In the case of axially compressed circular cylindrical shells it reveals the mechanism of scaling instability caused by load imperfections. There is an interplay between the slenderness of the domain as measured by the Korn constant and the stress in a "perfect" trivial branch. The scaling of the Korn constant is insensitive to the details of boundary conditions, as evidenced by the compactly supported optimal ansatz (3.12). An immediate consequence of our Rayleigh quotient characterization of buckling load (2.18) is the *safe load* estimate:

$$\lambda_{\rm cl}(h)=\inf_{\boldsymbol{\phi}\in\mathcal{A}_h}\mathfrak{R}(h,\boldsymbol{\phi})\ge\inf_{\boldsymbol{\phi}\in V_h}\frac{\alpha_{\mathsf{L}_0}\|e(\boldsymbol{\phi})\|^2}{\|\boldsymbol{\sigma}_h\|_\infty\|\nabla\boldsymbol{\phi}\|^2}=\frac{\alpha_{\mathsf{L}_0}K(V_h)}{\|\boldsymbol{\sigma}_h\|_\infty}. \tag{5.1}$$

For structures whose buckling loads are proportional to $K(V_h)$, such as rods and plates, the inequalities in (5.1) are asymptotically sharp. In particular, the scaling law of the buckling load of such a structure would be insensitive to small perturbations of the stress tensor $\boldsymbol{\sigma}_h$, and hence, such structures should not exhibit sensitivity to imperfections.

However, theoretical buckling loads may be significantly different from $K(V_h)$ due to special structure of the stress tensor $\boldsymbol{\sigma}_h$ in a trivial branch. For example, the critical stress for axially compressed circular cylindrical shells, given by (1.1), is much higher than $K(V_h)$, due to the special structure (3.7) of $\boldsymbol{\sigma}_h$.

Now, let us identify general conditions that lead to scaling instability in slender bodies. According to our definition of slenderness (Definition 2.5) there exists $\boldsymbol{\phi}_h\in V_h$ such that some of the components of $\nabla\boldsymbol{\phi}_h$ in some, possibly curvilinear, coordinate system, are much larger than $|e(\boldsymbol{\phi}_h)|$, as $h\to 0$, due to cancellations. There is a potential for scaling instability if the stress in a trivial branch "activates" only those components of $\nabla\boldsymbol{\phi}_h$ that are much smaller that $|\nabla\boldsymbol{\phi}_h|$. For example, according to inequalities (3.10)–(3.11), (θz) and (rz) components of $\nabla\boldsymbol{\phi}_h$ should be much smaller than $|\nabla\boldsymbol{\phi}_h|$. The scaling $\lambda_{\rm cr}\sim h$ in the formula (1.1) is a consequence of (3.7), since

$$\langle\boldsymbol{\sigma}^h,\nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi}\rangle\sim|(\nabla\boldsymbol{\phi})_{rz}|^2+|(\nabla\boldsymbol{\phi})_{\theta z}|^2+|(\nabla\boldsymbol{\phi})_{zz}|^2\sim\frac{1}{h}|e(\boldsymbol{\phi})|^2,$$

provided $\boldsymbol{\phi}$ saturates inequalities (3.10)–(3.11). Generalizing this observation we can say that there is a potential for scaling instability if the stress $\boldsymbol{\sigma}_h$ in a linearized trivial branch has the property

$$\lim_{h\to 0}\frac{K(V_h)}{\lambda_{\rm cl}(h)}=0, \tag{5.2}$$

where $\lambda_{\rm cl}(h)$ is given by (2.18). In that case it is easy to perturb $\boldsymbol{\sigma}_h$ (artificially) so as to violate (5.2). If imperfections of load cause such a perturbation in the stress in the trivial branch, then scaling instability will be observed. This was exactly how the scaling instabilities exhibited in Sections 4.1 and 4.2 were produced. Another important observation is that scaling laws come from Korn-type inequalities for gradient components. The scaling exponents of the corresponding Korn constants are functions of certain "essential" features of spaces V_h. That means that (a) scaling of Korn constants can only be changed by altering radically the type of loading and (b) possible set of the scaling exponents is finite and predetermined by the space V_h. (In our example it is $\sim h$, $\sim h^{5/4}$, $\sim h^{3/2}$.) This means, in particular, that the observed scaling laws will not depend on the minutiae of an imperfection, but rather on the buckling mode with the highest exponent that a given imperfection activates.

In Section 4.2 we encountered a situation where imperfections of load could not perturb the stress tensor arbitrarily, (see Theorem 4.3). Its structure, given by (4.8) resulted in the jump from $\lambda_{\rm cl}(h) \sim h$ in (5.2) for the purely axial load to $\lambda_{\rm cl}(h) \sim h^{5/4}$ for a load with a torsion component. In experiments described in [26], where imperfections of shape were controlled, the observed buckling load fell at the upper end of the historically accumulated data. This suggests that the new scaling $h^{5/4}$ could be at play there. In fact, the scaling exponents computed from experimental data reported in [22] range from 1.3 to 1.49, whose lower end comes close to 5/4 and upper end to 3/2.

We conjecture that imperfections of shape (to which the theory could potentially be applied) would add a hoop stress (i.e. $(\theta\theta)$-component) to $\boldsymbol{\sigma}_h$ and achieve the absolute lower bound (5.1) that scales as $h^{3/2}$. This is exactly what happened in Section 4.1.

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A Non-linear trivial branch for a neo-Hookean material.

We note that in order to derive the asymptotics of the critical load in the presence of torsion only the explicit linear trivial branch (4.14) is needed. However, for the result to be legitimate we also need to know the *existence* of the regular trivial branch, and not its explicit representation. At the moment we lack tools for establishing existence of regular trivial branches, nor can we exhibit one entirely explicitly for a material whose energy density satisfies our regularity assumptions. Here we derive an explicit form of the regular trivial branch for an incompressible neo-Hookean material in the way of providing “evidence” for our assumption of existence of the regular trivial branch for compressible materials.

The strain energy density function for a neo-Hookean solid has the form

$$W(\boldsymbol{F}) = \frac{E}{6}(|\boldsymbol{F}|^2 - 3), \qquad \det \boldsymbol{F} = 1. \tag{A.1}$$

We are looking for a trivial branch in a cylindrical shell, given in cylindrical coordinates by

$$y_r = \psi(r)\cos(\alpha z), \qquad y_\theta = \psi(r)\sin(\alpha z), \qquad y_z = (1-\lambda)z, \tag{A.2}$$

where $\psi(r)$ also depends on α, λ and h. When $\alpha = 0$ we expect that $\psi(r)$ will reduce to $(a(\lambda)+1)r$, as in (3.4). We remark that, the ansatz (A.2) should also work for isotropic compressible materials with strain energy density

$$W(\boldsymbol{F}) = \frac{1}{2}|\boldsymbol{F}|^2 + H(\det \boldsymbol{F}), \quad H'(1) = -1, \quad H''(1) > \frac{1}{3},$$

except the resulting non-linear second order boundary value problem for $\psi(r)$

$$\begin{cases} \psi'' + \dfrac{\psi'}{r} - \left(\alpha^2 + \dfrac{1}{r^2}\right)\psi = (1-\lambda)\dfrac{\psi}{r}\dfrac{d}{dr}\left[H'\left((1-\lambda)\dfrac{\psi\psi'}{r}\right)\right], & r \in I_h, \\ \psi' + (1-\lambda)\dfrac{\psi}{r}H'\left((1-\lambda)\dfrac{\psi\psi'}{r}\right) = 0, & r = 1 \pm \dfrac{h}{2} \end{cases} \tag{A.3}$$

cannot be solved explicitly.

Returning to the neo-Hookean solid (A.1) we must have

$$\det(\nabla \boldsymbol{y}) = (1-\lambda)\psi'(r)\frac{\psi(r)}{r} = 1,$$

and hence

$$\psi(r) = \sqrt{\frac{r^2}{1-\lambda} + \beta} \tag{A.4}$$

for some $\beta > -1$.

The Piola-Kirchhoff stress function is

$$\boldsymbol{P}(\boldsymbol{F}) = \frac{E}{3}\left(\boldsymbol{F} - \frac{3\hat{p}}{E}\mathrm{cof}(\boldsymbol{F})\right),$$

where the Lagrange multiplier $\hat{p}$ plays the role of pressure. For $\boldsymbol{y}$, given by (A.2) and $\boldsymbol{F} = \nabla \boldsymbol{y}$ we compute

$$\boldsymbol{F}^T\boldsymbol{F} = \begin{bmatrix} (\psi'(r))^2 & 0 & 0 \\ 0 & \frac{\psi(r)^2}{r^2} & \frac{\alpha\psi(r)^2}{r} \\ 0 & \frac{\alpha\psi(r)^2}{r} & \alpha^2\psi(r)^2 + (1-\lambda)^2 \end{bmatrix}.$$

The traction-free condition $\boldsymbol{P}\boldsymbol{e}_r = \boldsymbol{0}$ on $r = 1 \pm h/2$ can be written as

$$\boldsymbol{F}^T\boldsymbol{F}\boldsymbol{e}_r = p\boldsymbol{e}_r, \quad r = 1 \pm \frac{h}{2}, \qquad p = 3\hat{p}/E.$$

The formula for $\boldsymbol{F}^T\boldsymbol{F}$, together with $\det \boldsymbol{F} = 1$, implies that

$$p(r,\theta,z) = (\psi'(r))^2, \quad r = 1 \pm \frac{h}{2}. \tag{A.5}$$

This suggests that it is reasonable to look for the trivial branch for which the function $p(r,\theta,z)$ depends only on r. Under this assumption we compute

$$\frac{3}{E}\boldsymbol{P} = \begin{bmatrix} s_1(r)\cos(\alpha z) & -s_2(r)\sin(\alpha z) & -s_3(r)\sin(\alpha z) \\ s_1(r)\sin(\alpha z) & s_2(r)\cos(\alpha z) & s_3(r)\cos(\alpha z) \\ 0 & q_1(r) & q_2(r) \end{bmatrix},$$

where

$$s_1 = \psi' - \frac{p}{\psi'}, \quad s_2 = \frac{\psi}{r} - \frac{rp}{\psi}, \quad s_3 = \alpha\psi, \quad q_1 = \frac{\alpha r p}{1-\lambda}, \quad q_2 = 1 - \lambda - \frac{p}{1-\lambda}.$$

It follows that $\nabla \cdot \boldsymbol{P} = \mathbf{0}$ results in a single ODE for $p(r)$:

$$(rs_1)' = s_2 + \alpha r s_3. \tag{A.6}$$

Substituting (A.4) for $\psi(r)$ into (A.6) and solving for $p(r)$ we obtain

$$p(r) = \frac{1}{2(1-\lambda)}\left(\ln\left(\frac{1}{1-\lambda} + \frac{\beta}{r^2}\right) - r^2\alpha^2 - \frac{\beta(1-\lambda)}{r^2 + \beta(1-\lambda)} + \gamma\right)$$

for some constant of integration γ. The traction-free boundary conditions (A.5) become

$$\frac{r^2}{r^2 + \beta(1-\lambda)} = \ln\left(\frac{1}{1-\lambda} + \frac{\beta}{r^2}\right) - r^2\alpha^2 + \gamma - 1, \quad r = 1 \pm \frac{h}{2}. \tag{A.7}$$

To simplify notation we denote

$$\Phi(r;\lambda,\beta) = \ln\left(\frac{1}{1-\lambda} + \frac{\beta}{r^2}\right) - \frac{r^2}{r^2 + \beta(1-\lambda)}.$$

Then (A.7) can be written as

$$\begin{cases} \alpha^2\left(1+\frac{h}{2}\right)^2 = \Phi\left(1+\frac{h}{2};\lambda,\beta\right) + \gamma - 1, \\ \alpha^2\left(1-\frac{h}{2}\right)^2 = \Phi\left(1-\frac{h}{2};\lambda,\beta\right) + \gamma - 1 \end{cases} \tag{A.8}$$

Eliminating γ from (A.8) we obtain

$$\alpha^2 = \frac{1}{2h}\left(\Phi\left(1+\frac{h}{2};\lambda,\beta\right) - \Phi\left(1-\frac{h}{2};\lambda,\beta\right)\right).$$

when h is small

$$\alpha^2 \approx \frac{1}{2}\Phi'(1;\lambda,\beta) = -\frac{\beta(1-\lambda)(2+\beta(1-\lambda))}{(1+\beta(1-\lambda))^2}.$$

Thus, when $(h,\lambda) \to (0,0)$, $\beta \approx -\alpha^2/2$. We conclude that α, and, therefore, β must go to zero, as $\lambda \to 0$, since otherwise, the trivial branch $\boldsymbol{y}(\boldsymbol{x};h,\lambda)$, given by (A.2), (A.4) will not emanate from the undeformed configuration. The regularity of the trivial branch in λ demands that $\alpha(h,\lambda) \sim \alpha_0(h)\lambda$, as $\lambda \to 0$. Thus, for an arbitrary fixed parameter $\beta_0 > 0$ we set[11] $\beta = -\beta_0^2\lambda^2/2$, resulting in the explicit expression for the parameter α:

$$\alpha(\lambda,h) = \sqrt{\frac{\Phi(1+h/2;\lambda,-\beta_0^2\lambda^2/2) - \Phi(1-h/2;\lambda,-\beta_0^2\lambda^2/2)}{2h}}. \tag{A.9}$$

[11]Recall that we are investigating imperfections of load where the boundary conditions at $z = L$ are not fully specified. This gives us just enough freedom to choose β_0 arbitrarily.

Hence, the non-linear trivial branch has the form

$$y_r = \sqrt{\frac{r^2}{1-\lambda} - \frac{\beta_0^2\lambda^2}{2}}\cos(\alpha(\lambda,h)z), \quad y_\theta = \sqrt{\frac{r^2}{1-\lambda} - \frac{\beta_0^2\lambda^2}{2}}\sin(\alpha(\lambda,h)z), \quad y_z = (1-\lambda)z, \tag{A.10}$$

where $\beta_0 > 0$ is a constant and $\alpha(\lambda, h)$ is given by (A.9). We compute

$$\left.\frac{\partial\alpha}{\partial\lambda}\right|_{\lambda=0} = \frac{4\beta_0}{4-h^2}, \qquad \left.\frac{\partial\psi}{\partial\lambda}\right|_{\lambda=0} = \frac{r}{2}.$$

Therefore, the linearized trivial branch displacement $\boldsymbol{u}^h$ is given by

$$u_r^h = \left.\frac{\partial y_r}{\partial\lambda}\right|_{\lambda=0} = \frac{r}{2}, \qquad u_\theta^h = \left.\frac{\partial y_\theta}{\partial\lambda}\right|_{\lambda=0} = \frac{4\beta_0 rz}{4-h^2}, \qquad u_z^h = \left.\frac{\partial y_z}{\partial\lambda}\right|_{\lambda=0} = -z.$$

The corresponding linear stress and its $h \to 0$ limit are

$$\boldsymbol{\sigma}_h = E\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{4\beta_0 r}{3(4-h^2)} \\ 0 & \frac{4\beta_0 r}{3(4-h^2)} & -1 \end{bmatrix}, \qquad \boldsymbol{\sigma}^0 = E\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\beta_0}{3} \\ 0 & \frac{\beta_0}{3} & -1 \end{bmatrix}.$$

These agree with formulas (4.14), (4.16) for $\nu = 1/2$.

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