The flip side of buckling

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Abstract

Buckling of slender structures under compressive loading is a failure of infinitesimal stability due to a confluence of two factors: the energy density non-convexity and the smallness of Korn's constant. The problem has been well understood only for bodies with simple geometries when the slenderness parameter is well defined. In this paper we present the first rigorous analysis of buckling for bodies with complex geometry. By limiting our analysis to the "near-flip" instability we address the universal features of the buckling phenomenon that depend on neither the shape of the domain nor the degree of constitutive nonlinearity of the elastic material.

1 Introduction

Despite its seemingly straightforward treatment in the introductory engineering courses, buckling instability of slender elastic bodies is known to be tricky. Already the first computation of a critical load for a strut under uniaxial compression by Euler was contested by D'Alembert who claimed that due to flip instability (see Figure 1) the buckling load should be equal to zero [42, p. 258]. In fact, the original Euler's argument [12] left several mathematical questions unanswered. For instance, the formula for the critical load contains the slenderness parameter h even though it is derived from the one-dimensional theory corresponding to the limit $h \to 0$. Furthermore, Euler's apparent use of linear elasticity for the derivation of the critical load formally contradicts the uniqueness theorem of Kirchhoff. Consequently, two different approaches have been pursued in an attempt to treat buckling with full mathematical rigor.

The first approach is to perform, by driving h to zero, a controlled asymptotic dimension reduction, and then study the stability problem in a low-dimensional setting [7]. While this method was used by Euler himself, only recently the original insights into the structure of the limiting energy functional for rods and plates have been confirmed by the rigorous analysis implying global minimization of the energy [15, 30, 32, 33]. The formal problem remains,

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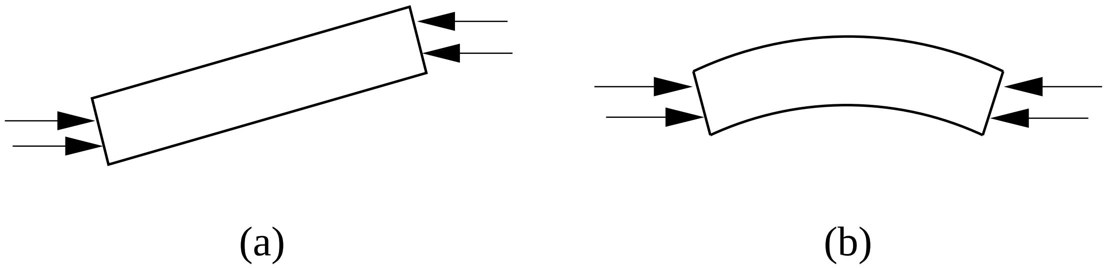

Figure 1: The two instability modes: (a) flip, (b) buckling

however, with the equally rigorous treatment of the local minima of the energy which are often essential in structural stability. Another problem concerns the necessity to deal with compressive prestress which lowers bending rigidity of the structure and ultimately triggers the buckling instability. The prestress enters the linearized elasticity problem in a nontrivial way since, somewhat paradoxically, the linearized strain during buckling is large while the finite strain is small [3, 17, 41]. Despite those remaining analytical problems, the advanced engineering theories of buckling which are based on semi-empirical low-dimensional models for slender bodies, usually account for the above physical effects correctly and give adequate predictions for the critical loads (e.g. [4, 24]). The problem is that those theories remain largely intuitive and are therefore restricted to simple geometries. Most importantly, the exact domain of their application is unclear.

The second approach to buckling is to deal directly with an instability at finite h. In this case, the critical load can be found from the analysis of the positive definiteness of the second variation for a 3D elasticity problem [9, 14, 18, 26, 34, 39]. The ensuing eigenvalue computations are notoriously tedious and specific to the particular geometry and energy density structure. Although such studies, usually possible only for simple shapes and constitutive laws, expose for thicker bodies a sequence of transitions between buckling and barreling, and establish an important link between buckling and surface instabilities, they conceal, in the case of slender bodies, a direct relation between the Euler's buckling and the D'Alembert's flip, and obscure an intuitively appealing relation between the critical load and the Korn constant. The theories which deal with finite slenderness but stop short of solving the full eigenvalue problem are usually focused exclusively on the bounds for the critical load which can be made explicit, again, only for bodies with simple geometry [1, 10, 11, 21].

In this paper we present a treatment of buckling instability which can be placed in between the two above approaches. More precisely, we conduct the analysis of the second variation for the full elastic problem and then take the limit $h \to 0$. The fact that we perform the dimension reduction in the stability conditions, instead of the energy itself, distinguishes our approach from most other mathematical papers on the subject (see the recent review in [25]). We find it necessary to deal directly with the conditions of instability because of our focus on local rather than global minimization of the energy. The latter is the target of the powerful methods based on Gamma-convergence which may or may not be adequate for buckling depending on whether the bifurcation is supercritical or subcritical. An approach similar to ours, but based on formal asymptotic expansions, has been applied previously to prismatic bars under specific constitutive assumptions in [37]. Here, we go further and and obtain rigorous theorems that do not depend on the material model and are largely independent of either the details of loading or the geometry of the domain, encompassing therefore the problem of imperfection sensitivity.

Our approach is based on the new interpretation of buckling bifurcation as a *delayed* flip instability taking place when the compressive dead loads are supplemented by additional constraints that keep the flip from occurring at infinitesimal compression. The appropriate additional constraints are expected to generate a genuinely mixed type loading because in purely hard device buckling is forbidden, while in soft device it degenerates into flip. We show that the delayed flip instability, i.e. near-flip buckling, involves the interplay of two factors. On the one hand, the structure in purely soft device is unstable at arbitrarily small compressive loads due to the intrinsic nonconvexity of the energy density function. On the other hand, the mixed type boundary conditions promote stability. Buckling is then understood as failure of the stabilizing force to overcome the destabilizing force. While Korn's constant is a known characteristic of the stabilizing effect of traction-dominated loading and slenderness of the domain, the parameter which fully characterizes the destabilizing effect of compressive loading has been unknown and is introduced in this paper for the first time.

We recall that the peculiarity of dead compressive loading was first realized by Signorini [38] who has found that in soft device even for small loads there are multiple equilibria. While all those equilibrium configurations have the Cauchy-Green strain tensor $\boldsymbol{C}$ close to $\boldsymbol{I}$ (the identity matrix) only one of them has the deformation gradient $\boldsymbol{F}$ close to $\boldsymbol{I}$. More recently, the corresponding bifurcation has been studied in full detail [5, 6]. In particular, it was found that the branch converging to $\boldsymbol{I}$ is not always stable and that in the case of a compressed strut, the instability manifests itself through the rigid rotation (flip). In the present paper we extend these ideas to the case of mixed loading for slender bodies with complex geometry and explain how the flip bifurcation transforms into buckling. Along the way we re-evaluate buckling from the perspective of the non-commutativity of the two apparently well understood asymptotic procedures: linearization and dimension reduction. From the times of Signorini, linearization has been known to be formally ill defined in the vicinity of a bifurcation point [28], and in the case of near-flip buckling it is the dimension reduction that is responsible for the ultimate shrinking of the domain where the linearization is legitimate [29]. As we make clear, the non-uniformity of the classical linear elastic limit is due to the fact that the work of the pre-stress is neglected during standard linearization while it clearly dominates the energy in the dimension reduction limit.

The inherent nonlinear nature of buckling is concealed by the dependence of the critical load exclusively on the linear elastic moduli. Yet, it has long been realized that buckling results from the energy density non-convexity which, in turn, follows from material frame indifference [2, 19, 42]. The apparent constitutive linearity of the Euler's theory of the critical load arises from the fact that in the case of near-flip buckling the prohibited full linearization can be replaced by a procedure that we call "constitutive linearization" (see [3, 41] for the earlier insights). In fact, our proof of the asymptotic equivalence between the full nonlinear and the constitutively linearized theories of near-flip buckling can be viewed as a rigorous extension of the Föppl-von Kármán theory of buckling for plates and rods [8, 13, 16, 35, 44] to bodies with complex geometries. Furthermore, we show that the engineering theory generates satisfactory predictions for the critical loads only in the cases which we interpret as "smooth" or "Euler" buckling. In those cases our main results are related to the domain of applicability of the conventional approaches. More strikingly, we provide the counterexamples showing that the engineering formulas may fail in the "non Eulerian" cases including buckling of the rather common "multi-element" structures.

The paper is organized as follows. In Section 2 we introduce the main players: the intrinsic energy nonconvexity and the Korn constant. The prototypical flip instability of D'Alembert is introduced in Section 3. Buckling of slender bodies or the "near-flip" buckling is defined and studied in Section 4. In Section 5 we demonstrate that behind the universality of the near-flip buckling lies the possibility to perform partial (or constitutive) linearization of the problem. We use the term "buckling equivalence" or B-equivalence for situations when the asymptotics of the critical load can be computed directly from the partially linearized problem. Section 6 contains the preliminary definition of compressiveness of the loading device which allows us to prove B-equivalence under special smoothness assumptions encompassing Euler's original setting (Section 7). We then show in Section 8 that Euler buckling is far from being generic. A general study of B-equivalence, presented in Section 9, reveals a more subtle nature of the concept of compressive loads. A convenient sufficient condition for B-equivalence, going far beyond Euler's case, is presented in Section 10. In Section 11 we show that if the loading is compressive in the strong sense, the first term in the h-asymptotics of the critical load can be interpreted as a *generalized Korn constant*. Subsequently we show how this interpretation can be used to derive new bounds on both safe and unsafe loads. In the two technical Appendixes we study existence of the homogeneous trivial branch and present a heuristic justification of the Kirchhoff-Love ansatz for arbitrary anisotropic materials.

Throughout the paper we use standard index-free tensor notation and some other useful conventions. For instance, $\langle f\rangle$ denotes the average of the function f over the domain of its definition, $|\boldsymbol{a}|$ denotes the Euclidean norm for a vector $\boldsymbol{a}$, Frobenius norm $\mathrm{Tr}\,(\boldsymbol{a}\boldsymbol{a}^t)^{1/2}$ for the matrix $\boldsymbol{a}$ and an operator norm for the map $\boldsymbol{a}\colon \mathrm{End}(\mathbb{R}^2)\to\mathrm{End}(\mathbb{R}^2)$. The symbol $\|\boldsymbol{f}\|$ always denotes the L^2 norm of $|\boldsymbol{f}(\boldsymbol{x})|$ and $\|\boldsymbol{f}\|_\infty$ denotes the L^∞ norm of $|\boldsymbol{f}(\boldsymbol{x})|$. The equivalence relation $a(h)\sim b(h)$ is understood in the sense that $\lim_{h\to 0} a(h)/b(h)=1$.

2 Preliminaries

To highlight ideas we limit our exposition to the simplest nontrivial case which is 2D elasticity, even though our approach is largely dimension-independent.[1] We assume that the elastic response of the body is governed by the energy density function $W(\boldsymbol{F})$, which is C^3 in the vicinity of the stress-free state $\boldsymbol{F}=\boldsymbol{I}$. We also assume that $\boldsymbol{F}=\boldsymbol{I}$ is the point of local

[1] We will indicate throughout the text, when 3D analogs of our results differ from their 2D counterparts.

minimum for the energy density $W(\boldsymbol{F})$. In particular,

$$W_{\boldsymbol{F}}(\boldsymbol{I}) = \boldsymbol{0}. \tag{2.1}$$

If we load a body Ω by the dead loads $\boldsymbol{t}(\boldsymbol{x})$, $\boldsymbol{x} \in \partial\Omega$, the resulting deformation $\boldsymbol{y}(\boldsymbol{x})$ is expected to be at least a weak local minimizer of the energy

$$\mathcal{E}(\boldsymbol{y}) = \int_{\Omega} W(\boldsymbol{F}(\boldsymbol{x}))d\boldsymbol{x} - \int_{\partial\Omega} (\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{t}(\boldsymbol{x}))ds, \tag{2.2}$$

where $\boldsymbol{F}(\boldsymbol{x}) = \nabla \boldsymbol{y}(\boldsymbol{x})$ is the deformation gradient, $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}(\boldsymbol{x}) - \boldsymbol{x}$ is the displacement vector and ds is the element of the arc length on $\partial\Omega$. The loading $\boldsymbol{t}(\boldsymbol{x})$ should be equilibrated in the sense that:

$$\int_{\partial\Omega} \boldsymbol{t}(\boldsymbol{x})ds = \boldsymbol{0}, \qquad \int_{\partial\Omega} (\boldsymbol{t}(\boldsymbol{x}), \boldsymbol{S}\boldsymbol{y}(\boldsymbol{x}))ds = 0, \tag{2.3}$$

where

$$\boldsymbol{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.4}$$

is a skew-symmetric matrix. While only surface loads are explicitly indicated here, bulk dead loads can be easily included as well.

In parallel to dead loading we shall also consider an equilibrium under mixed loading when in addition to specifying information on applied boundary tractions we also impose constraints on the boundary displacements, say

$$\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega_1 \subset \partial\Omega.$$

In general, we will assume that $\boldsymbol{y}$ is constrained to belong to the affine subspace $\mathcal{F} \subset W^{1,\infty}(\Omega; \mathbb{R}^2)$ of admissible deformations. In other words, we assume that there exists $\boldsymbol{y}_0 \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ and a subspace $V^0 \subset W^{1,\infty}(\Omega; \mathbb{R}^2)$, such that $\mathcal{F} = \boldsymbol{y}_0 + V^0$.

Both boundary conditions and equilibrium equations can be obtained from the following variational equation

$$\int_{\Omega} (W_{\boldsymbol{F}}(\boldsymbol{F}(\boldsymbol{x})), \nabla\boldsymbol{\varphi})d\boldsymbol{x} - \int_{\partial\Omega} (\boldsymbol{\varphi}(\boldsymbol{x}), \boldsymbol{t}(\boldsymbol{x}))ds = 0 \tag{2.5}$$

satisfied for all $\boldsymbol{\varphi} \in V^0$. Let $\mathsf{L}(\boldsymbol{F}) = W_{\boldsymbol{F}\boldsymbol{F}}(\boldsymbol{F})$ denote the set of tangential elastic moduli. We call the admissible deformation $\boldsymbol{y}(\boldsymbol{x})$ of the reference configuration $\Omega \subset \mathbb{R}^2$ *infinitesimally stable* if it solves the equilibrium equations of elastostatics (2.5) and the second variation of the energy

$$\delta^2\mathcal{E}(\boldsymbol{F}, \boldsymbol{\varphi}) = \int_{\Omega} (\mathsf{L}(\boldsymbol{F}(\boldsymbol{x}))\nabla\boldsymbol{\varphi}, \nabla\boldsymbol{\varphi})d\boldsymbol{x} \tag{2.6}$$

is nonnegative for all $\boldsymbol{\varphi} \in V$, where V is a closure of V^0 in $W^{1,2}(\Omega; \mathbb{R}^2)$. The above conditions are known to be necessary and sufficient for $\boldsymbol{y}(\boldsymbol{x})$ to be a weak local minimizer for the energy (2.2) (with uniform positivity of second variation for sufficiency, see [40, 43]).

A crucial role in what follows will be played by an additional assumption that the energy density function $W(\boldsymbol{F})$ is objective (material frame indifferent). It requires that

$$W(\boldsymbol{RF}) = W(\boldsymbol{F}) \tag{2.7}$$

for all rotation matrices $\boldsymbol{R} \in SO(2)$ and all 2×2 matrices $\boldsymbol{F}$ with positive determinant. The condition (2.7) implies that $\boldsymbol{F} = \boldsymbol{I}$ is not a strict local minimum of $W(\boldsymbol{F})$, in particular the Hessian L_0 of $W(\boldsymbol{F})$ at $\boldsymbol{F} = \boldsymbol{I}$,

$$\mathsf{L}_0 = \mathsf{L}(\boldsymbol{I}) = W_{\boldsymbol{FF}}(\boldsymbol{I}) \tag{2.8}$$

is singular:

$$\mathsf{L}_0 \boldsymbol{S} = \boldsymbol{0}. \tag{2.9}$$

We assume that the degeneracy of L_0 does not go beyond (2.9), meaning that

$$(\mathsf{L}_0 \boldsymbol{\xi}, \boldsymbol{\xi}) \geq \gamma_0 |\boldsymbol{\xi}|^2, \text{ for all } \boldsymbol{\xi} \in \mathrm{Sym}(\mathbb{R}^2), \tag{2.10}$$

where $\mathrm{Sym}(\mathbb{R}^2)$ is the space of symmetric 2×2 matrices and γ_0 is the smallest eigenvalue of L_0 understood as a linear operator on $\mathrm{Sym}(\mathbb{R}^2)$.

The objectivity assumption (2.7) implies that there exists a function $\hat{W}(\boldsymbol{C})$, defined on symmetric positive definite matrices $\boldsymbol{C} = \boldsymbol{F}^t \boldsymbol{F}$ (Green's strain tensor), such that $W(\boldsymbol{F}) = \hat{W}(\boldsymbol{C})$. According to our assumptions, the function $\hat{W}(\boldsymbol{C})$ has a strict local minimum at $\boldsymbol{C} = \boldsymbol{I}$. The straightforward computation gives $W_{\boldsymbol{F}}(\boldsymbol{F}) = 2\boldsymbol{F}\hat{W}_{\boldsymbol{C}}(\boldsymbol{C})$ and

$$(\mathsf{L}(\boldsymbol{F})\boldsymbol{\xi}, \boldsymbol{\xi}) = 2(\hat{W}_{\boldsymbol{C}}(\boldsymbol{C}), \boldsymbol{\xi}^t \boldsymbol{\xi}) + 4(\hat{W}_{\boldsymbol{CC}}(\boldsymbol{C})(\boldsymbol{F}^t \boldsymbol{\xi}), \boldsymbol{F}^t \boldsymbol{\xi}) \tag{2.11}$$

for any 2×2 matrices $\boldsymbol{\xi}$ and $\boldsymbol{F}$.

If $\boldsymbol{C} \approx \boldsymbol{I}$, the second term in the right hand side of (2.11), representing the stabilizing effect of the conventional elastic rigidity, is always non-negative, while the first term may be either positive or negative. In the case of compressive loading, the first term is negative and represents the destabilizing effect of stiffness reduction. The study of the competition between these two terms constitutes the main subject of any theory of buckling.

To clarify the meaning of the two terms in the right hand side of (2.11), we recall that there exist matrices $\boldsymbol{F}$, arbitrarily close to the identity matrix, such that the quadratic form $\boldsymbol{\xi} \mapsto (\mathsf{L}(\boldsymbol{F})\boldsymbol{\xi}, \boldsymbol{\xi})$ is non-convex [20]. Indeed, if $\boldsymbol{F}$ is close to $\boldsymbol{I}$ and $\boldsymbol{\xi} = \boldsymbol{S}$—a skew-symmetric matrix, defined in (2.4), then, by symmetry of $\boldsymbol{C}$, the second term in (2.11) is of order $O(|\boldsymbol{F} - \boldsymbol{I}|^2)$:

$$(\hat{W}_{\boldsymbol{CC}}(\boldsymbol{C})(\boldsymbol{F}^t \boldsymbol{S}), \boldsymbol{F}^t \boldsymbol{S}) = (\hat{W}_{\boldsymbol{CC}}(\boldsymbol{C})((\boldsymbol{F} - \boldsymbol{I})^t \boldsymbol{S}), (\boldsymbol{F} - \boldsymbol{I})^t \boldsymbol{S}) = O(|\boldsymbol{F} - \boldsymbol{I}|^2). \tag{2.12}$$

The first term in (2.11), however, is of order $O(|\boldsymbol{F} - \boldsymbol{I}|)$:

$$2(\hat{W}_{\boldsymbol{C}}(\boldsymbol{C}), \boldsymbol{S}^t \boldsymbol{S}) = 2\mathrm{Tr}\,(\hat{W}_{\boldsymbol{CC}}(\boldsymbol{I})(\boldsymbol{C} - \boldsymbol{I})) + O(|\boldsymbol{F} - \boldsymbol{I}|^2) = \mathrm{Tr}\,(\mathsf{L}_0(\boldsymbol{F} - \boldsymbol{I})) + O(|\boldsymbol{F} - \boldsymbol{I}|^2). \tag{2.13}$$

Therefore, when $\boldsymbol{\xi} = \boldsymbol{S}$ and $\boldsymbol{F}$ is close to the identity matrix, the first term in (2.11) may dominate the second term causing an instability. More specifically, if $\mathrm{Tr}\,(\mathsf{L}_0(\boldsymbol{F} - \boldsymbol{I})) < 0$ (compressive stresses), then $(\mathsf{L}(\boldsymbol{F})\boldsymbol{S}, \boldsymbol{S}) < 0$.

The other two main pre-conditions of buckling—traction-dominated loading and slenderness of the domain—are conveniently characterized by the smallness of Korn's constant. The Korn constant is a non-negative number $K(V)$ associated with the subspace $V \subset W^{1,2}(\Omega;\mathbb{R}^2)$. It is defined as the largest number for which the Korn inequality [22]

$$\int_\Omega |e(\boldsymbol{\varphi})|^2 d\boldsymbol{x} \geq K(V) \int_\Omega |\nabla \boldsymbol{\varphi}|^2 d\boldsymbol{x} \tag{2.14}$$

holds for all $\boldsymbol{\varphi} \in V$, where $e(\boldsymbol{\varphi}) = (\nabla\boldsymbol{\varphi} + (\nabla\boldsymbol{\varphi})^t)/2$ is the symmetrized gradient. One can think of $K(V)$ as the "distance" between V and the 1D subspace spanned by $\boldsymbol{Sx}$.

We will need a slightly extended notion of the Korn constant. If L is a fourth order tensor of the positive definite quadratic form $\boldsymbol{\xi} \mapsto (\mathsf{L}\boldsymbol{\xi}, \boldsymbol{\xi})$ on $\mathrm{Sym}(\mathbb{R}^2)$, then we define $K_{\mathsf{L}}(V)$ as the largest number for which the inequality

$$\int_\Omega (\mathsf{L}e(\boldsymbol{\varphi}), e(\boldsymbol{\varphi})) d\boldsymbol{x} \geq K_{\mathsf{L}}(V) \int_\Omega |\nabla \boldsymbol{\varphi}|^2 d\boldsymbol{x} \tag{2.15}$$

holds for all $\boldsymbol{\varphi} \in V$. The classical Korn constant simply corresponds to $\mathsf{L} = \mathsf{I}$ – the fourth order identity tensor. Sometimes it will be convenient to represent the Korn constant in the variational form:

$$K_{\mathsf{L}}(V) = \inf_{\substack{\boldsymbol{\varphi} \in V \\ \|\nabla\boldsymbol{\varphi}\|=1}} \int_\Omega (\mathsf{L}e(\boldsymbol{\varphi}), e(\boldsymbol{\varphi})) d\boldsymbol{x}. \tag{2.16}$$

Formula (2.16) shows that all Korn-type constants are equivalent in the following sense. Take any pair of positive definite quadratic forms L_1 and L_2. Then there are constants $c(\mathsf{L}_1, \mathsf{L}_2)$ and $C(\mathsf{L}_1, \mathsf{L}_2)$, independent of V, such that

$$c(\mathsf{L}_1, \mathsf{L}_2) K_{\mathsf{L}_1}(V) \leq K_{\mathsf{L}_2}(V) \leq C(\mathsf{L}_1, \mathsf{L}_2) K_{\mathsf{L}_1}(V). \tag{2.17}$$

For this reason the geometric meaning of $K_{\mathsf{L}}(V)$ as a function of V is not different from $K(V)$, except that $K_{\mathsf{L}}(V)$ also depends on the degree of convexity of the quadratic form L. More precisely, if $\mathsf{L}_1 < \mathsf{L}_2$ in the sense of quadratic forms, then $K_{\mathsf{L}_1}(V) < K_{\mathsf{L}_2}(V)$.

3 Flip instability

Consider a body loaded by dead tractions and occupying an equilibrium configuration $\boldsymbol{y}(\boldsymbol{x})$. If we replace this configuration by its rotation through the angle ϵ

$$\boldsymbol{y}_\epsilon(\boldsymbol{x}) = \boldsymbol{R}_\epsilon \boldsymbol{y}(\boldsymbol{x}) \tag{3.1}$$

the associated energy increment can be written, in view of (2.7), as

$$\Delta E = -\int_{\partial\Omega} (\boldsymbol{t}(\boldsymbol{x}), \boldsymbol{R}_\epsilon \boldsymbol{y}(\boldsymbol{x}) - \boldsymbol{y}(\boldsymbol{x})) ds.$$

By using $\boldsymbol{R}_\epsilon = \boldsymbol{I}\cos\epsilon + \boldsymbol{S}\sin\epsilon$ we obtain, via $(2.3)_2$

$$\Delta E = (1-\cos\epsilon)\int_{\partial\Omega}(\boldsymbol{t}(\boldsymbol{x}),\boldsymbol{y}(\boldsymbol{x}))ds. \tag{3.2}$$

If the actual configuration of the body is $\Omega^* = \boldsymbol{y}(\Omega)$ then

$$\int_{\partial\Omega}(\boldsymbol{t}(\boldsymbol{x}),\boldsymbol{y}(\boldsymbol{x}))ds = \int_{\Omega^*}\mathrm{Tr}\,\boldsymbol{\tau}(\boldsymbol{y})d\boldsymbol{y},$$

where $\boldsymbol{\tau}(\boldsymbol{y})$ is the Cauchy stress tensor. Therefore for compressive loading with

$$\mathrm{Tr}\,\langle\boldsymbol{\tau}\rangle < 0, \tag{3.3}$$

the rotations through the infinitesimal angle ϵ will lower the total energy producing flip instability. Despite its infinitesimal character, the flip instability can not be captured by linearized theory because expansion in ϵ in (3.2) starts with quadratic terms.

To study the bifurcation leading to flip instability we need to analyze the second variation of the energy. Assume that the body is subjected to tractions $\boldsymbol{t}(\boldsymbol{x};\lambda)$ parameterized by a small parameter λ. Consider a family of deformations $\boldsymbol{y}_\lambda(\boldsymbol{x})$ that satisfy the equations of equilibrium (2.5) with $V^0 = W^{1,\infty}(\Omega;\mathbb{R}^2)$ and assume that it is sufficiently regular in λ.[2] More precisely, we assume that $\boldsymbol{y}_0(\boldsymbol{x}) = \boldsymbol{x}$ and the function $T\colon \lambda \mapsto \boldsymbol{y}_\lambda(\boldsymbol{x})$ that maps a small neighborhood of zero in $\mathbb{R}$ into $W^{1,\infty}(\Omega;\mathbb{R}^2)$ is of class C^1 (in the norm topology of $W^{1,\infty}$). In other words we assume that there exists a Lipschitz function $\boldsymbol{u}'(\boldsymbol{x})$ such that

$$\boldsymbol{F}_\lambda(\boldsymbol{x}) = \nabla\boldsymbol{y}_\lambda(\boldsymbol{x}) = \boldsymbol{I} + \lambda\boldsymbol{H}'(\boldsymbol{x}) + o(\lambda), \quad \text{as } \lambda\to 0, \tag{3.4}$$

where $\boldsymbol{H}' = \nabla\boldsymbol{u}'$ and $o(\cdot)$ is understood in the sense of uniform convergence. In what follows we will refer to $\boldsymbol{H}'(\boldsymbol{x})$ as the incremental strain and to

$$\boldsymbol{\sigma}'(\boldsymbol{x}) = \mathsf{L}_0 e(\boldsymbol{u}') \tag{3.5}$$

as the incremental stress.

An example is provided by the family $\boldsymbol{y}_\lambda(\boldsymbol{x}) = \boldsymbol{F}_\lambda\boldsymbol{x}$ of deformations corresponding to the loading program

$$\boldsymbol{t}(\boldsymbol{x};\lambda) = \lambda\boldsymbol{P}_0\boldsymbol{n}(\boldsymbol{x}), \tag{3.6}$$

where $\boldsymbol{P}_0$ is a given constant symmetric matrix. For the homogeneous deformation gradient $\boldsymbol{F}_\lambda$ the Euler-Lagrange equation (2.5) reduces to

$$W_{\boldsymbol{F}}(\boldsymbol{F}_\lambda)\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{t}(\boldsymbol{x};\lambda), \quad \boldsymbol{x}\in\partial\Omega,$$

which is equivalent to

$$W_{\boldsymbol{F}}(\boldsymbol{F}_\lambda) = \lambda\boldsymbol{P}_0. \tag{3.7}$$

[2]Note, that we do not assume neither uniqueness nor stability of these deformations.

If $\mathrm{Tr}\,\boldsymbol{P}_0 \neq 0$ the existence of $\boldsymbol{F}_\lambda$ satisfying (3.7) is guaranteed by Lemma A.1 proved in Appendix A.[3]

We are now in a position to formulate the sufficient conditions for flip instability. To emphasize compressive character of the loading we set $\lambda < 0$ throughout the paper.

Theorem **3.1** *Consider dead loading $\boldsymbol{t}(\boldsymbol{x};\lambda)$ and assume that $\boldsymbol{y}_\lambda(\boldsymbol{x})$ is regular in the sense of (3.4). Assume that $\mathrm{Tr}\,\langle\boldsymbol{\sigma}'\rangle > 0$. Then there exists $\varsigma > 0$ such that $\delta^2\mathcal{E}(\boldsymbol{F}_\lambda, \boldsymbol{S}\boldsymbol{x}) < 0$ for all $\lambda \in (-\varsigma, 0)$.*

Proof: Let $\mathsf{L}_\lambda(\boldsymbol{x}) = \mathsf{L}(\nabla \boldsymbol{y}_\lambda(\boldsymbol{x}))$. Substitute $\boldsymbol{\varphi}(\boldsymbol{x}) = \boldsymbol{S}\boldsymbol{x}$ in (2.6) and observe that in view of (2.12) and (2.13)

$$(\mathsf{L}_\lambda(\boldsymbol{x})\boldsymbol{S}, \boldsymbol{S}) = \mathrm{Tr}\,(\mathsf{L}_0(\boldsymbol{F}_\lambda(\boldsymbol{x}) - \boldsymbol{I})) + O(|\boldsymbol{F}_\lambda - \boldsymbol{I}|^2).$$

Then, by (3.4) and (3.5)

$$(\mathsf{L}_\lambda(\boldsymbol{x})\boldsymbol{S}, \boldsymbol{S}) = \lambda \mathrm{Tr}\,(\mathsf{L}_0\boldsymbol{H}'(\boldsymbol{x})) + o(\lambda) = \lambda \mathrm{Tr}\,\boldsymbol{\sigma}'(\boldsymbol{x}) + o(\lambda). \tag{3.8}$$

∎

Equation (3.8) can be regarded as a Taylor expansion of $(\mathsf{L}_\lambda(\boldsymbol{x})\boldsymbol{S}, \boldsymbol{S})$ and therefore the coefficient in front of λ can be expected to represent the *third order elastic moduli*. However, the relevant non-linearity is of purely geometrical nature and therefore this coefficient can be reduced to the expression depending only the second order elastic moduli. One can show that behind this reduction are material-independent relations between second and third order elastic moduli, that can be derived from (2.7) by differentiation.

Observe that the trivial branch may be flip-unstable even without our sufficient conditions being satisfied. Indeed, if we replace λ by $-\lambda^2$ in the simplest case of homogeneous loading (3.6), the incremental stress and strain become equal to zero, while flip instability persists. Therefore our condition $\mathrm{Tr}\,\langle\boldsymbol{\sigma}'\rangle > 0$ performs two tasks: it signifies the compressive nature of the loading and simultaneously identifies its scale. The condition analogous to $\mathrm{Tr}\,\langle\boldsymbol{\sigma}'\rangle > 0$ in Theorem 3.1 in 3D is that at least one of the following three inequalities hold (e.g. [20])

$$\sigma_1 + \sigma_2 > 0, \qquad \sigma_2 + \sigma_3 > 0, \qquad \sigma_3 + \sigma_1 > 0, \tag{3.9}$$

where σ_1, σ_2 and σ_3 are the eigenvalues of $\langle\boldsymbol{\sigma}'\rangle$.

We now turn to the necessary conditions for flip instability. Define

$$m(\lambda) = \inf_{\|\nabla\boldsymbol{\varphi}\|=1} \int_\Omega (\mathsf{L}_\lambda(\boldsymbol{x})\nabla\boldsymbol{\varphi}(\boldsymbol{x}), \nabla\boldsymbol{\varphi}(\boldsymbol{x}))d\boldsymbol{x}. \tag{3.10}$$

If $m(\lambda) > 0$ then the second variation is positive and the trivial branch is stable. If $m(\lambda) < 0$ we can identify the energy-decreasing variation $\boldsymbol{\varphi}_\lambda$. Let $\boldsymbol{\varphi}_\lambda$ be such that $\|\nabla\boldsymbol{\varphi}_\lambda\| = 1$ and

$$\delta^2\mathcal{E}(\boldsymbol{F}_\lambda, \boldsymbol{\varphi}_\lambda) - m(\lambda) = o(\lambda), \tag{3.11}$$

where $\boldsymbol{F}_\lambda = \boldsymbol{F}_\lambda(\boldsymbol{x}) = \nabla\boldsymbol{y}_\lambda(\boldsymbol{x})$. We will call $\boldsymbol{\varphi}_\lambda$ an almost-minimizer[4] for $m(\lambda)$.

[3] In 3D the equivalent condition would be that $\boldsymbol{P}_0$ has no pairs of opposite eigenvalues.

[4] The variational problem (3.10) may have no solutions.

THEOREM **3.2** *Consider dead loading $\boldsymbol{t}(\boldsymbol{x};\lambda)$ generating the trivial branch $\boldsymbol{y}_\lambda(\boldsymbol{x})$ which is regular in the sense of (3.4). Then, either there exists $\varsigma > 0$ such that $m(\lambda) > 0$ for all $\lambda \in (-\varsigma, 0)$ or* $\mathrm{Tr}\,\langle \boldsymbol{\sigma}' \rangle \geq 0$ *and the family of almost-minimizers $\boldsymbol{\varphi}_\lambda$ has a subsequence $\lambda_n \to 0$ such that*

$$\lim_{n\to\infty} \nabla \boldsymbol{\varphi}_{\lambda_n} = \pm \frac{\boldsymbol{S}}{\sqrt{2|\Omega|}}$$

in $L^2(\Omega)$.

PROOF: Assume that that there exists a sequence $\lambda_n \to 0^-$ such that $m(\lambda_n) \leq 0$ for all n. Let $\boldsymbol{\varphi}_n = \boldsymbol{\varphi}_{\lambda_n}$, then

$$\int_\Omega \{(\mathsf{L}_0 e(\boldsymbol{\varphi}_n), e(\boldsymbol{\varphi}_n)) + ((\mathsf{L}_{\lambda_n}(\boldsymbol{x}) - \mathsf{L}_0)\nabla\boldsymbol{\varphi}_n, \nabla\boldsymbol{\varphi}_n)\} d\boldsymbol{x} - m(\lambda_n) = o(\lambda_n). \tag{3.12}$$

Assumption of regularity for the trivial branch implies that

$$|\mathsf{L}_{\lambda_n}(\boldsymbol{x}) - \mathsf{L}_0| \leq C|\lambda_n|$$

for all $\boldsymbol{x} \in \Omega$. We therefore can write

$$\int_\Omega (\mathsf{L}_0 e(\boldsymbol{\varphi}_n), e(\boldsymbol{\varphi}_n)) d\boldsymbol{x} \leq C|\lambda_n|.$$

Since L_0 is a positive definite tensor, there exists $C > 0$ such that for all $n \geq 1$

$$\|e(\boldsymbol{\varphi}_n)\|^2 \leq C|\lambda_n| \to 0, \quad n \to \infty. \tag{3.13}$$

LEMMA **3.3** *Suppose $\|\nabla\boldsymbol{\varphi}_n\| = 1$ and $\|e(\boldsymbol{\varphi}_n)\| \to 0$, as $n \to \infty$. Then there exists a subsequence n_k such that $\nabla\boldsymbol{\varphi}_{n_k} \to \alpha_0 \boldsymbol{S}$ as $k \to \infty$ in $L^2(\Omega; \mathbb{R}^{2\times 2})$, where $|\alpha_0| = 1/\sqrt{2|\Omega|}$.*

PROOF: Let us choose a subsequence (not relabeled) such that $\langle \nabla\boldsymbol{\varphi}_n \rangle \to \boldsymbol{\xi}_0$ as $n \to \infty$. By Korn's inequality there exists a constant K_0 depending only on Ω such that for all $\boldsymbol{\varphi} \in W^{1,2}(\Omega; \mathbb{R}^2)$

$$\|e(\boldsymbol{\varphi}_n) - \langle e(\boldsymbol{\varphi}_n) \rangle\| \geq K_0 \|\nabla\boldsymbol{\varphi}_n - \langle \nabla\boldsymbol{\varphi}_n \rangle\|.$$

Then

$$\|\nabla\boldsymbol{\varphi}_n - \boldsymbol{\xi}_0\| \leq \|\nabla\boldsymbol{\varphi}_n - \langle \nabla\boldsymbol{\varphi}_n \rangle\| + \|\langle \nabla\boldsymbol{\varphi}_n \rangle - \boldsymbol{\xi}_0\| \leq \frac{1}{K_0}\|e(\boldsymbol{\varphi}_n) - \langle e(\boldsymbol{\varphi}_n) \rangle\| + \|\langle \nabla\boldsymbol{\varphi}_n \rangle - \boldsymbol{\xi}_0\|.$$

It follows from (3.13) that $\nabla\boldsymbol{\varphi}_n \to \boldsymbol{\xi}_0$ in L^2 and therefore $e(\boldsymbol{\varphi}_n) \to (\boldsymbol{\xi}_0)_{\mathrm{sym}}$, so that $\boldsymbol{\xi}_0 = \alpha_0 \boldsymbol{S}$. Here $|\alpha_0|$ can be found from the condition that $1 = \|\nabla\boldsymbol{\varphi}_n\| \to \|\boldsymbol{\xi}_0\| = \sqrt{2|\Omega|}|\alpha_0|$. ∎

Now consider a subsequence n_k from Lemma 3.3 and relabel in back into n. The inequality $m(\lambda_n)/\lambda_n \geq 0$ and relation (3.12) imply that

$$\varliminf_{n\to\infty} \frac{1}{\lambda_n} \int_\Omega ((\mathsf{L}_{\lambda_n}(\boldsymbol{x}) - \mathsf{L}_0)\nabla\boldsymbol{\varphi}_n, \nabla\boldsymbol{\varphi}_n)\} d\boldsymbol{x} \geq 0.$$

By way of (3.8) and Lemma 3.3 we can now conclude that $\alpha_0^2 \mathrm{Tr}\,\langle \boldsymbol{\sigma}' \rangle \geq 0$. ∎

To avoid immediate flip instability in a compressive loading one must impose additional constraints that exclude the infinitesimal rotations $\boldsymbol{S}\boldsymbol{x}$. In hard device the instability is eliminated completely, while in mixed device the bifurcation point may shift to small but non-zero loads. Below we will be using the term *near-flip buckling* to characterize this "postponed" instability which shares with flip the essential link to the failure of convexity of the elastic energy, due to objectivity.

4 Near-flip buckling

To distinguish the near-flip buckling from the flip we introduce a new small parameter h. Suppose that both the applied tractions and the domain geometry depend on h. Assume that the admissible set of deformations has the form

$$\mathcal{F}_{h,\lambda} = \overline{\boldsymbol{y}}_h(\boldsymbol{x};\lambda) + V_h^0,$$

where $\overline{\boldsymbol{y}}_h(\boldsymbol{x};\lambda)$ is a given function and V_h^0 is a subspace in $W^{1,\infty}(\Omega_h;\mathbb{R}^2)$ that is independent of λ. Let V_h be the closure of V_h^0 in $W^{1,2}(\Omega_h;\mathbb{R}^2)$. The role of h is to parameterize the sequence of Korn constants $K(V_h)$ representing increasing distance between the current loading configuration and the soft device configuration.

Assume again that there exists a trivial branch $\boldsymbol{y}_{h,\lambda}(\boldsymbol{x})$ and a Lipschitz function $\boldsymbol{u}_h'(\boldsymbol{x})$ such that

$$\boldsymbol{F}_{h,\lambda}(\boldsymbol{x}) = \nabla\boldsymbol{y}_{h,\lambda}(\boldsymbol{x}) = \boldsymbol{I} + \lambda\boldsymbol{H}_h'(\boldsymbol{x}) + o(\lambda), \quad \text{as } \lambda \to 0^-, \tag{4.1}$$

where $\boldsymbol{H}_h' = \nabla\boldsymbol{u}_h'$ and $o(\cdot)$ is understood in the sense of uniform convergence in both $\boldsymbol{x}$ and h. In addition, $\boldsymbol{H}_h'(\boldsymbol{x})$ is assumed to be uniformly bounded as $h \to 0$.[5] We note that in order to compute $\boldsymbol{u}_h'$ it is not necessary to know the function $\boldsymbol{F}_{h,\lambda}(\boldsymbol{x})$; it is sufficient to show that (4.1) holds. Indeed, $\boldsymbol{F}_{h,\lambda}(\boldsymbol{x})$ solves (2.5) with $\boldsymbol{t}(\boldsymbol{x})$ replaced by $\boldsymbol{t}_h(\boldsymbol{x};\lambda)$ and V^0 replaced by V_h^0. Differentiating this equation in λ at $\lambda = 0$ we obtain

$$\int_{\Omega_h} (\mathsf{L}_0 e(\boldsymbol{u}_h'), e(\boldsymbol{\varphi}))d\boldsymbol{x} - \int_{\partial\Omega} (\boldsymbol{t}_h^{\mathrm{lin}}, \boldsymbol{\varphi})ds = 0, \tag{4.2}$$

for all $\boldsymbol{\varphi} \in V_h$. Here $\boldsymbol{u}_h' - \overline{\boldsymbol{u}}_h \in V_h^0$, where

$$\overline{\boldsymbol{u}}_h = \frac{\partial\overline{\boldsymbol{y}}_h}{\partial\lambda}(\boldsymbol{x};0), \qquad \boldsymbol{t}_h^{\mathrm{lin}}(\boldsymbol{x}) = \frac{\partial\boldsymbol{t}_h}{\partial\lambda}(\boldsymbol{x};0).$$

Next we introduce the function

$$m(h,\lambda) = \inf_{\substack{\boldsymbol{\varphi}\in V_h \\ \|\nabla\boldsymbol{\varphi}\|=1}} \int_{\Omega_h} (\mathsf{L}(\boldsymbol{F}_{h,\lambda}(\boldsymbol{x}))\nabla\boldsymbol{\varphi}(\boldsymbol{x}), \nabla\boldsymbol{\varphi}(\boldsymbol{x}))d\boldsymbol{x} \tag{4.3}$$

[5] While we are not aware of a general theorem that would guarantee the existence of such equilibrium deformations $\boldsymbol{y}_{h,\lambda}$ satisfying (4.1), the existence of a trivial branch for the special case of a thin periodic plate was proved in [29].

which defines the stability locus

$$\mathfrak{S} = \{(h,\lambda) \in (0,+\infty) \times (-\infty,0) \colon m(h,\lambda) \geq 0\}.$$

The critical load can now be defined as the smallest in absolute value λ making the trivial branch unstable. More precisely,

$$\lambda(h) = \sup\{\lambda < 0 \colon m(h,\lambda) < 0\}. \tag{4.4}$$

Definition 4.1 *An instability of the trivial branch is called a* near-flip buckling *if*

$$\lambda(h) < 0 \tag{4.5}$$

for sufficiently small h and

$$\lim_{h\to 0} \lambda(h) = 0. \tag{4.6}$$

Below we show that the near-flip buckling is indeed "close" to flip in the sense that the variation $\boldsymbol{\varphi}$ that decreases the energy is close to the space of infinitesimal rotations.

First observe that since the admissibility of a variation

$$\boldsymbol{\varphi} = \boldsymbol{S}\boldsymbol{x} + \boldsymbol{a} \tag{4.7}$$

already leads to flip in a stress-free configuration, the infinitesimal stability of a configuration that is close to the stress-free state should be linked to the order of magnitude of the Korn constant $K_{\mathsf{L}_0}(V_h)$, which vanishes if V_h contains a map (4.7). Thus,

$$|m(h,\lambda) - K_{\mathsf{L}_0}(V_h)| \leq \sup_{\substack{\boldsymbol{\varphi}\in V_h \\ \|\nabla\boldsymbol{\varphi}\|=1}} \left| \int_{\Omega_h} ((\mathsf{L}(\boldsymbol{F}_{h,\lambda}(\boldsymbol{x})) - \mathsf{L}_0)\nabla\boldsymbol{\varphi}, \nabla\boldsymbol{\varphi}) d\boldsymbol{x} \right| \leq C|\lambda| \tag{4.8}$$

and, in particular,

$$\lim_{\lambda\to 0} m(h,\lambda) = \inf_{\substack{\boldsymbol{\varphi}\in V_h \\ \|\nabla\boldsymbol{\varphi}\|=1}} \int_{\Omega_h} (\mathsf{L}_0 e(\boldsymbol{\varphi}), e(\boldsymbol{\varphi})) d\boldsymbol{x} = K_{\mathsf{L}_0}(V_h). \tag{4.9}$$

One can see that if $K(V_h)$ does not vanish as $h \to 0$ (recall that $K(V_h) = 1/2$ in hard device), buckling will be prohibited. We conclude that condition

$$K_{\mathsf{L}_0}(V_h) \to 0, \text{ as } h \to 0 \tag{4.10}$$

is necessary for near-flip buckling. Moreover, we can prove the following bound on the critical load (see also [21]).

LEMMA **4.2** *There exists a constant $c > 0$ such that*

$$\lambda(h) \leq -cK_{\mathsf{L}_0}(V_h). \tag{4.11}$$

PROOF: The inequality (4.8) can be written as

$$K_{\mathsf{L}_0}(V_h) - C|\lambda| \leq m(h,\lambda) \leq K_{\mathsf{L}_0}(V_h) + C|\lambda|, \tag{4.12}$$

where $C > 0$. Taking $\lambda = \lambda(h)$ and using $m(h,\lambda(h)) = 0$, we obtain (4.11) with $c = 1/C$. ∎

The lemma says that the Korn constant provides a lower bound on the magnitude of the critical load. While condition (4.10) is necessary for (4.6), it is surely not sufficient. Indeed, a square body under dead tension has Korn constant zero, but is perfectly stable. This shows that the smallness of the Korn constant expresses only the *potential* susceptibility of the structure to buckling that may occur under an appropriate load (or to flip, if Korn constant is zero). For a given load that potential may or may not be realized.

As a corollary of Lemma 4.2, we conclude that the condition

$$K_{\mathsf{L}_0}(V_h) > 0 \tag{4.13}$$

is sufficient for (4.5). Yet, it is not necessary for the near-flip buckling. Indeed, if the structure is a square under a dead tension with an attached slender arm under a dead compression, then the Korn constant is again zero. At the same time flip instability is ruled out by the large and positive value of the average Cauchy stress (condition (3.3) is violated). The slender arm, however, will buckle at a finite compressive loading. This example confirms that in general we cannot expect the Korn constant alone to give even the order of magnitude for the critical load.

Our next aim is to find, in the case of near-flip buckling, the analog of the condition $\mathrm{Tr}\,\langle \boldsymbol{\sigma}' \rangle > 0$, which has proved to be sufficient for the flip. To this end, we observe that the inequality

$$m_0(\lambda) = \overline{\lim_{h\to 0}}\, m(h,\lambda) < 0,$$

that holds for all sufficiently small $\lambda < 0$ (compressive loads), is a sufficient condition for (4.6). Taking a limit in the inequality (4.8), as $h \to 0$ and using (4.10) we obtain

$$|m_0(\lambda)| \leq C|\lambda|.$$

Thus, to understand the sign of $m_0(\lambda)$ at small $\lambda < 0$, we need to study the sign of

$$m_0' = \underline{\lim}_{\lambda\to 0^-} \frac{m_0(\lambda)}{\lambda}. \tag{4.14}$$

In particular, we may already conclude that the inequality $m_0' > 0$ is sufficient for (4.6).

The sufficient condition we have just derived is not entirely satisfactory because m_0' is not represented in terms of the linear elastic parameters as one would expect, given the smallness of the load. Our next step is therefore to replace the function $m(h,\lambda)$ by a simpler "linearized" expression, which preserves its asymptotics near $(0,0)$.

5 Constitutive linearization

When λ is small, while h is finite, one would normally fully linearize the problem by discarding terms that are infinitesimal in λ and replace $\delta^2\mathcal{E}(\boldsymbol{F}_{h,\lambda},\boldsymbol{\varphi})$ with $\int_{\Omega_h}(\mathsf{L}_0 e(\boldsymbol{\varphi}),e(\boldsymbol{\varphi}))d\boldsymbol{x}$. Indeed, if $\|\nabla\boldsymbol{\varphi}\|=1$ then

$$\left|\delta^2\mathcal{E}(\boldsymbol{F}_{h,\lambda},\boldsymbol{\varphi})-\int_{\Omega_h}(\mathsf{L}_0 e(\boldsymbol{\varphi}),e(\boldsymbol{\varphi}))d\boldsymbol{x}\right|\le C|\lambda|, \tag{5.1}$$

where the constant $C>0$ is independent of h and $\boldsymbol{\varphi}$. The second term on the left hand side of (5.1) is bounded from below by the Korn constant $K_{\mathsf{L}_0}(V_h)$, because $\boldsymbol{\varphi}\in V_h$. If the Korn constant degenerates in the limit of small h (e.g. if condition (4.10) is satisfied), the zeroth order term, $\int_{\Omega_h}(\mathsf{L}_0 e(\boldsymbol{\varphi}),e(\boldsymbol{\varphi}))d\boldsymbol{x}$, may not necessarily dominate the terms that we have discarded in our development of $\delta^2\mathcal{E}(\boldsymbol{F}_{h,\lambda},\boldsymbol{\varphi})$. In other words, the classical linearization procedure may fail when *both* λ and h are small.

In order to recover the full two-parameter structure of the asymptotics of $m(h,\lambda)$ we utilize the view of buckling as a near-flip instability. More precisely, we show that, in parallel with Theorem 3.2 the almost minimizers $\boldsymbol{\varphi}_{h,\lambda}$ for $m(h,\lambda)$ are close to infinitesimal rotations. The mean-square distance from the former to the latter, $\|e(\boldsymbol{\varphi}_{h,\lambda})\|$, is to be regarded as a measure of the deviation of the near-flip buckling from the flip.

Let $\boldsymbol{\varphi}_{h,\lambda}\in V_h$ be an almost-minimizer in (4.3), i.e. $\|\nabla\boldsymbol{\varphi}_{h,\lambda}\|=1$ and

$$\delta^2\mathcal{E}(\boldsymbol{F}_{h,\lambda},\boldsymbol{\varphi}_{h,\lambda})-m(h,\lambda)=o(\lambda), \tag{5.2}$$

where $o(\lambda)$ is understood in the sense of *uniform in h convergence*, as $\lambda\to 0$. Then, according to (5.1) and (5.2),

$$\|e(\boldsymbol{\varphi}_{h,\lambda})\|^2\le\frac{1}{\gamma_0}m(h,\lambda)+C|\lambda|,$$

where $\gamma_0>0$ is the smallest eigenvalue of L_0 regarded as an operator on $\mathrm{Sym}(\mathbb{R}^2)$. Applying inequality (4.8), we obtain the desired estimate

$$\|e(\boldsymbol{\varphi}_{h,\lambda})\|^2\le\frac{1}{\gamma_0}K_{\mathsf{L}_0}(V_h)+C|\lambda| \tag{5.3}$$

meaning, in particular, that

$$\lim_{(h,\lambda)\to(0,0)}\|e(\boldsymbol{\varphi}_{h,\lambda})\|=0. \tag{5.4}$$

Relation (5.4) will allow us to improve the naive linearization attempt (5.1). We recall that according to (2.11),

$$\delta^2\mathcal{E}(\boldsymbol{F}_{h,\lambda},\boldsymbol{\varphi}_{h,\lambda})=T_1+T_2,$$

where

$$T_1=\int_{\Omega_h}4(\hat{W}_{\boldsymbol{CC}}(\boldsymbol{C}_{h,\lambda})(\boldsymbol{F}_{h,\lambda}^t\nabla\boldsymbol{\varphi}_{h,\lambda}),\boldsymbol{F}_{h,\lambda}^t\nabla\boldsymbol{\varphi}_{h,\lambda})d\boldsymbol{x}$$

and

$$T_2=\int_{\Omega_h}2(\hat{W}_{\boldsymbol{C}}(\boldsymbol{C}_{h,\lambda}),(\nabla\boldsymbol{\varphi}_{h,\lambda})^t\nabla\boldsymbol{\varphi}_{h,\lambda})d\boldsymbol{x}.$$

Expanding $\boldsymbol{F}_{h,\lambda}$ by means of (4.1), we obtain the asymptotic expansions of T_1 and T_2, as $(h,\lambda) \to (0,0)$:

$$T_1 = \int_{\Omega_h} (\mathsf{L}_0 e(\boldsymbol{\varphi}_{h,\lambda}), e(\boldsymbol{\varphi}_{h,\lambda}))d\boldsymbol{x} + O(\lambda \|e(\boldsymbol{\varphi}_{h,\lambda})\|) + o(\lambda).$$

and

$$T_2 = \lambda \int_{\Omega_h} (\boldsymbol{\sigma}'_h(\boldsymbol{x}), (\nabla \boldsymbol{\varphi}_{h,\lambda})^t \nabla \boldsymbol{\varphi}_{h,\lambda})d\boldsymbol{x} + o(\lambda).$$

Here

$$\boldsymbol{\sigma}'_h(\boldsymbol{x}) = \mathsf{L}_0 e(\boldsymbol{u}'_h) \tag{5.5}$$

is the incremental stress (cf. (3.5)).

In classical linearization one drops all terms of order λ and smaller. However, if $\nabla \boldsymbol{\varphi}_{h,\lambda}$ were skew-symmetric, relations (2.12) and (2.13) would tell us that it is the "main term" that needs to be discarded. In our case, when $\nabla \boldsymbol{\varphi}_{h,\lambda}$ are close to infinitesimal rotations, *both* the main term and order-λ term need to be retained. Our error estimates show that, unless order-λ term vanishes, the error terms are negligible in the limit $(h,\lambda) \to (0,0)$. Thus,

$$m(h,\lambda) \sim \int_{\Omega_h} (\mathsf{L}_0 e(\boldsymbol{\varphi}_{h,\lambda}), e(\boldsymbol{\varphi}_{h,\lambda}))d\boldsymbol{x} + \lambda \int_{\Omega_h} (\boldsymbol{\sigma}'_h(\boldsymbol{x}), (\nabla \boldsymbol{\varphi}_{h,\lambda})^t \nabla \boldsymbol{\varphi}_{h,\lambda})d\boldsymbol{x}.$$

This analysis suggests that $m(h,\lambda)$ may be replaced by a much simpler functional

$$\widehat{m}_{3\mathrm{D}}(h,\lambda) = \inf_{\substack{\boldsymbol{\varphi} \in V_h \\ \|\nabla \boldsymbol{\varphi}\|=1}} \int_{\Omega_h} \{(\mathsf{L}_0 e(\boldsymbol{\varphi}), e(\boldsymbol{\varphi})) + \lambda(\boldsymbol{\sigma}'_h(\boldsymbol{x}), (\nabla \boldsymbol{\varphi})^t \nabla \boldsymbol{\varphi})\}d\boldsymbol{x}. \tag{5.6}$$

In what follows we refer to this reduction of the problem as *constitutive linearization* because only the material behavior has been linearized, not the geometry. While such approximation is similar in spirit to the nonlinear Föppl-von Kármán theory of plates [8, 13, 16, 35, 44], our functional (5.6) deals with arbitrary geometries and should be rather considered as the extension and formalization of a heuristic linearization approach of Biot [3]).

In 2D (our main case of interest) one can simplify the functional $\widehat{m}_{3\mathrm{D}}(h,\lambda)$ a bit further, by replacing $\nabla \boldsymbol{\varphi}$ with its skew-symmetric part $\nabla \boldsymbol{\varphi} - e(\boldsymbol{\varphi})$ and observing that in 2D

$$(\nabla \boldsymbol{\varphi} - e(\boldsymbol{\varphi}))^t (\nabla \boldsymbol{\varphi} - e(\boldsymbol{\varphi})) = \frac{1}{4} |\nabla \times \boldsymbol{\varphi}|^2 \boldsymbol{I}.$$

Thus, using the identity

$$|\nabla \boldsymbol{\varphi}|^2 = |e(\boldsymbol{\varphi})|^2 + \frac{1}{2} |\nabla \times \boldsymbol{\varphi}|^2,$$

we obtain:

$$(\boldsymbol{\sigma}'_h(\boldsymbol{x}), (\nabla \boldsymbol{\varphi}_{h,\lambda})^t \nabla \boldsymbol{\varphi}_{h,\lambda}) = \frac{1}{2} \mathrm{Tr}\, \boldsymbol{\sigma}'_h(\boldsymbol{x}) |\nabla \boldsymbol{\varphi}_{h,\lambda}|^2 + O(|e(\boldsymbol{\varphi}_{h,\lambda})|). \tag{5.7}$$

Notice that $\mathrm{Tr}\, \boldsymbol{\sigma}'_h$ in (5.7) and $\mathrm{Tr}\, \boldsymbol{\sigma}'$ in (3.8) appear for the same reason: an application of (2.11) with $\boldsymbol{\xi} = \boldsymbol{S}$.

We can now summarize our results. After the partial linearization of the problem, we obtained that

$$m(h,\lambda)=\int_{\Omega_h}\left\{(\mathsf{L}_0 e(\boldsymbol{\varphi}_{h,\lambda}),e(\boldsymbol{\varphi}_{h,\lambda}))+\lambda t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}_{h,\lambda}|^2\right\}d\boldsymbol{x}+O(\lambda\|e(\boldsymbol{\varphi}_{h,\lambda})\|)+o(\lambda), \tag{5.8}$$

as $(h,\lambda)\to(0,0)$, where

$$t_h(\boldsymbol{x})=\frac{1}{2}\mathrm{Tr}\,\boldsymbol{\sigma}'_h(\boldsymbol{x}). \tag{5.9}$$

This suggests that we may replace the original functional $m(h,\lambda)$ with

$$\widehat{m}(h,\lambda)=\inf_{\substack{\boldsymbol{\varphi}\in V_h\\ \|\nabla\boldsymbol{\varphi}\|=1}}\int_{\Omega_h}\left\{(\mathsf{L}_0 e(\boldsymbol{\varphi}),e(\boldsymbol{\varphi}))+\lambda t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}|^2\right\}d\boldsymbol{x}. \tag{5.10}$$

Next we define the analogs of m'_0 and $\lambda(h)$ (see (4.14) and (4.4)) as

$$\widehat{m}'_0=\varliminf_{\lambda\to 0^-}\varliminf_{h\to 0}\frac{\widehat{m}(h,\lambda)}{\lambda} \tag{5.11}$$

and

$$\widehat{\lambda}(h)=\sup\{\lambda<0\colon \widehat{m}(h,\lambda)<0\}. \tag{5.12}$$

Observe that the quantities $\widehat{m}'_0$ and $\widehat{\lambda}(h)$ are expressed in terms of linear elastic parameters. As such they are much simpler to compute than the original quantities m'_0 and $\lambda(h)$. For example, in the special case of homogeneous trivial branch $t_h(\boldsymbol{x})=t_h$, the functional $\widehat{m}(h,\lambda)$ can be computed explicitly: $\widehat{m}(h,\lambda)=K_{\mathsf{L}_0}(V_h)+\lambda t_h$. Therefore $\widehat{m}'_0=\varliminf_{h\to 0}t_h$ and

$$\widehat{\lambda}(h)=-\frac{K_{\mathsf{L}_0}(V_h)}{t_h}.$$

By contrast, $m(h,\lambda)$ cannot always be computed explicitly even in the homogeneous case.

Definition 5.1 *We say that $m(h,\lambda)$ and $\widehat{m}(h,\lambda)$ are* buckling-equivalent *or B-equivalent, if $\widehat{m}'_0=m'_0$ and the corresponding critical loads $\lambda(h)$, $\widehat{\lambda}(h)$ have the same asymptotics as $h\to 0$, i.e.*

$$\lim_{h\to 0}\frac{\widehat{\lambda}(h)}{\lambda(h)}=1. \tag{5.13}$$

One of the goals of the paper is to formulate a criterion for B-equivalence of $m(h,\lambda)$ and $\widehat{m}(h,\lambda)$. To this end we first notice that by (5.8) and (5.3)

$$m(h,\lambda)\ge\widehat{m}(h,\lambda)+o(\lambda), \tag{5.14}$$

as $(h,\lambda)\to(0,0)$. The reverse inequality is also true because the process of passing from $m(h,\lambda)$ to $\widehat{m}(h,\lambda)$ is reversible. Indeed, the argument leading to (5.14) is based on the estimate (5.3) for the almost minimizer $\boldsymbol{\varphi}_{h,\lambda}$ for $m(h,\lambda)$. The estimate (5.3) in turn, follows from regularity of the trivial branch (4.1) and the inequality (5.1). Clearly, the estimate (5.3) also holds for the almost minimizer $\widehat{\boldsymbol{\varphi}}_{h,\lambda}$ of $\widehat{m}(h,\lambda)$ because the inequality (5.1) holds if $\delta^2\mathcal{E}(\boldsymbol{F}_{h,\lambda},\boldsymbol{\varphi})$ is replaced by

$$\int_{\Omega_h}\left\{(\mathsf{L}_0 e(\boldsymbol{\varphi}), e(\boldsymbol{\varphi})) + \lambda t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}|^2\right\} d\boldsymbol{x}.$$

Thus,

$$m(h,\lambda) = \widehat{m}(h,\lambda) + o(\lambda), \tag{5.15}$$

as $(h,\lambda)\to(0,0)$. The relation (5.15) already incorporates the preceding asymptotic analysis and will serve as a main tool for proving B-equivalence.

6 Compressive loads

In Section 4 we have identified $m_0' > 0$ as a sufficient condition for near-flip buckling. In this section we show that this condition may be interpreted as the criterion for existence of a sufficiently compressed slender element. We begin with an introduction of a new implicit measure of compressiveness for the applied loads.

Definition 6.1 *We say that the loading is* compressive *if* $\mathfrak{c} > 0$, *where*

$$\mathfrak{c} = \sup_{\substack{\boldsymbol{\varphi}_h\in V_h\\ \|\nabla\boldsymbol{\varphi}_h\|=1\\ \|e(\boldsymbol{\varphi}_h)\|\to 0}} \overline{\lim_{h\to 0}}\int_{\Omega_h} t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}_h|^2 d\boldsymbol{x}. \tag{6.1}$$

THEOREM **6.2** *If (4.10) and (4.13) hold, and the loading is compressive, then the trivial branch undergoes a near-flip buckling instability in the sense of Definition 4.1. Moreover,*

$$m_0' = \widehat{m}_0' = \mathfrak{c}. \tag{6.2}$$

PROOF: The first equality in (6.2) is an immediate consequence of (5.15). To prove the second equality in (6.2), we recall that an almost-minimizer $\boldsymbol{\varphi}_{h,\lambda}$ for $m(h,\lambda)$ satisfies (5.8). Then, by positive definiteness of L_0, we have

$$m_0' \le \varliminf_{\lambda\to 0^-}\varliminf_{h\to 0}\int_{\Omega_h} t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}_{h,\lambda}|^2 d\boldsymbol{x}.$$

We claim that

$$\varliminf_{\lambda\to 0^-}\varliminf_{h\to 0}\int_{\Omega_h} t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}_{h,\lambda}|^2 d\boldsymbol{x} \le \mathfrak{c}. \tag{6.3}$$

Indeed, there exist sequences $\lambda_n\to 0^-$ and $h_n\to 0^+$ such that

$$\varliminf_{\lambda\to 0^-}\varliminf_{h\to 0}\int_{\Omega_h} t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}_{h,\lambda}|^2 d\boldsymbol{x} = \lim_{n\to\infty}\int_{\Omega_{h_n}} t_{h_n}(\boldsymbol{x})|\nabla\boldsymbol{\varphi}_{h_n,\lambda_n}|^2 d\boldsymbol{x}.$$

Thus, in view of (5.4), the inequality (6.3) holds.

Conversely, suppose that $\boldsymbol{\varphi}_h \in V_h$ is *any* sequence satisfying both $\|\nabla\boldsymbol{\varphi}_h\| = 1$, and $\|e(\boldsymbol{\varphi}_h)\| \to 0$ as $h \to 0$, (such sequences exist, due to condition (4.10)). Using this sequence as test functions in the variational principle (5.10) we obtain

$$\widehat{m}_0' \geq \varliminf_{h\to 0} \int_{\Omega_h} t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}_h|^2 d\boldsymbol{x}.$$

Taking supremum over all such sequences we confirm that $m_0' \geq \mathfrak{c}$. ∎

7 Euler buckling

We can prove B-equivalence of the original and the linearized problems by making additional smoothness assumptions on $\widehat{m}(h,\lambda)$.

THEOREM **7.1** *Assume that for all h sufficiently close to zero the function $\lambda \mapsto \widehat{m}(h,\lambda)$ is differentiable in a neighborhood of $\lambda = 0$ and that the partial derivative $\partial\widehat{m}(h,\lambda)/\partial\lambda$ is continuous at $(0,0)$. Assume further that $\mathfrak{c} > 0$. Then m and $\widehat{m}$ are B-equivalent and*

$$\lambda(h) \sim -\frac{K_{\mathsf{L}_0}(V_h)}{\mathfrak{c}}. \tag{7.1}$$

PROOF: The behavior of the function $\widehat{m}(h,\lambda)$ in the vicinity of $(0,0)$ is described by the Lagrange's mean value theorem

$$\widehat{m}(h,\lambda) = K_{\mathsf{L}_0}(V_h) + \lambda\mathfrak{c}_{h,\lambda}, \tag{7.2}$$

where

$$\mathfrak{c}_{h,\lambda} = \frac{\partial\widehat{m}}{\partial\lambda}(h,\theta(h,\lambda)),$$

and $\theta(h,\lambda) \in (\lambda,0)$. Formula (5.11) gives $\widehat{m}_0' = \partial\widehat{m}(0,0)/\partial\lambda$. Therefore, by Theorem 6.2 and continuity of $\partial\widehat{m}(h,\lambda)/\partial\lambda$ at $(0,0)$, we get

$$\lim_{(h,\lambda)\to(0,0)} \mathfrak{c}_{h,\lambda} = \mathfrak{c}.$$

If $\mathfrak{c} > 0$, then, according to Theorem 6.2, $\widehat{\lambda}(h) < 0$, $\lambda(h) < 0$ and

$$\lim_{h\to 0}\widehat{\lambda}(h) = \lim_{h\to 0}\lambda(h) = 0. \tag{7.3}$$

Substituting $\lambda = \widehat{\lambda}(h)$ into (7.2), dividing by $\widehat{\lambda}(h)$ and passing to the limit as $h \to 0$, we obtain

$$0 = \lim_{h\to 0}\frac{K_{\mathsf{L}_0}(V_h)}{\widehat{\lambda}(h)} + \mathfrak{c}. \tag{7.4}$$

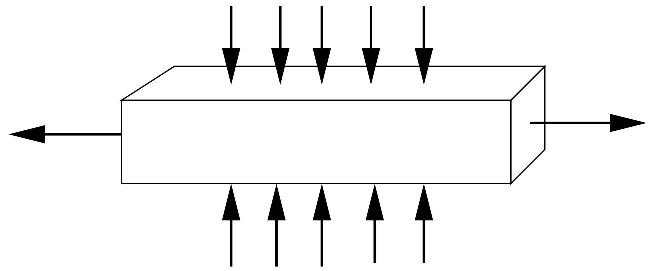

Figure 2: Flip that is not close to buckling

Combining (7.2) and (5.15), we can write that

$$m(h,\lambda) = K_{\mathsf{L}_0}(V_h) + \lambda\mathfrak{c}_{h,\lambda} + o(\lambda), \tag{7.5}$$

as $(h,\lambda) \to (0,0)$. Substituting $\lambda = \lambda(h)$ into (7.5), dividing by $\lambda(h)$ and passing to the limit as $h \to 0$, we obtain

$$0 = \lim_{h\to 0} \frac{K_{\mathsf{L}_0}(V_h)}{\lambda(h)} + \mathfrak{c}.$$

∎

Observe that the formula (7.1) makes explicit the competition between the stabilizing and the destabilizing forces. In particular, the parameter $\mathfrak{c} > 0$ indicates the destabilizing presence of flip instability, while $K_{\mathsf{L}_0}(V_h)$ is a measure of the stabilizing effect of the domain geometry and mixed type loading. In the special case of a homogeneous trivial branch, Theorem 7.1 is applicable and condition $\mathfrak{c} > 0$ is equivalent to $\mathrm{Tr}\,\boldsymbol{\sigma}^* > 0$, where

$$\boldsymbol{\sigma}^* = \lim_{h\to 0} \boldsymbol{\sigma}'_h.$$

Comparing this with Theorem 3.1 we see that in the homogeneous case the sufficient conditions for flip instability and buckling instability coincide. This phenomenon is peculiar to two dimensions. In 3D, the sufficient conditions for buckling is that all three inequalities (3.9) hold, where σ_1, σ_2 and σ_3 are now the eigenvalues of $\boldsymbol{\sigma}^*$. This is different from the sufficient condition for flip, requiring that only one of the inequalities (3.9) hold. Figure 2 shows the loading of a slender structure that is susceptible to flip, but not to buckling. In other words every near-flip buckling instability is close to a flip, but in 3D, unlike in 2D, not every flip instability is close to buckling.

To illustrate the case when the smoothness assumptions in Theorem 4 are justified consider the problem of the Euler buckling (e.g. [9, 25, 26, 39, 45]). Suppose that the domain Ω_h is a rectangle $R_h = [0,1] \times [-h/2, h/2]$ depicted in Figure 3; here h can be viewed as the non-dimensional aspect ratio. The long sides $x_2 = \pm h/2$ are assumed to be free

$$W_{\boldsymbol{F}}(\boldsymbol{F}(\boldsymbol{x}))\boldsymbol{e}_2 = \boldsymbol{0}, \text{ at } x_2 = \pm h/2, \tag{7.6}$$

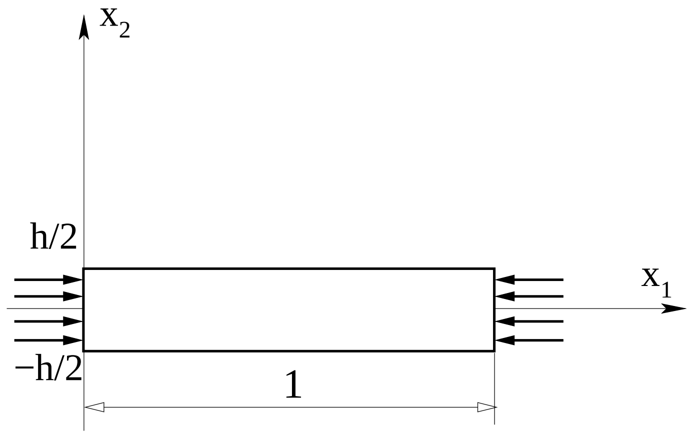

Figure 3: Two-dimensional Euler strut R_h

while on the short sides $x_1 = 0, 1$ we apply compressive dead loading

$$W_{\boldsymbol{F}}(\boldsymbol{F}(\boldsymbol{x}))\boldsymbol{e}_1 = \lambda \boldsymbol{e}_1, \text{ at } x_1 = 0, 1, \quad \lambda < 0. \tag{7.7}$$

If a homogeneous deformation $\boldsymbol{y}_\lambda(\boldsymbol{x})$ satisfies (7.6)–(7.7) then its deformation gradient $\boldsymbol{F}_\lambda$ satisfies (3.7) with $\boldsymbol{P}_0 = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1$:

$$W_{\boldsymbol{F}}(\boldsymbol{F}_\lambda) = \lambda \boldsymbol{e}_1 \otimes \boldsymbol{e}_1. \tag{7.8}$$

Notice that Lemma A.1 in Appendix A guarantees the existence of the smooth in λ function $\boldsymbol{F}_\lambda$ satisfying (7.8). Differentiating (7.8) in λ at $\lambda = 0$ we obtain

$$\boldsymbol{\sigma}'_h = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \qquad t_h = \frac{1}{2}.$$

In order to avoid flip instability, we augment the boundary conditions (7.6)–(7.7) with an additional requirement that the average displacements of the sides $x_1 = 0, 1$ in the x_2 direction are zero[6]:

$$\int_{-h/2}^{h/2} y_2(0, x_2)dx_2 = \int_{-h/2}^{h/2} y_2(1, x_2)dx_2 = 0. \tag{7.9}$$

It is easy to see that the deformation $\boldsymbol{y}_\lambda(\boldsymbol{x}) = \boldsymbol{F}_\lambda \boldsymbol{x}$ satisfies additional boundary conditions (7.9). Indeed, due to frame indifference the matrix $W_{\boldsymbol{F}}(\boldsymbol{F}_\lambda)\boldsymbol{F}_\lambda^t = \lambda \boldsymbol{e}_1 \otimes \boldsymbol{F}_\lambda \boldsymbol{e}_1$ is necessarily symmetric and therefore $\boldsymbol{e}_1$ must be an eigenvector for $\boldsymbol{F}_\lambda$. Then

$$(\boldsymbol{y}_\lambda(1, x_2), \boldsymbol{e}_2) = (\boldsymbol{F}_\lambda \boldsymbol{e}_1, \boldsymbol{e}_2) + x_2(\boldsymbol{F}_\lambda \boldsymbol{e}_2, \boldsymbol{e}_2) = x_2(\boldsymbol{F}_\lambda \boldsymbol{e}_2, \boldsymbol{e}_2) = (\boldsymbol{y}_\lambda(0, x_2), \boldsymbol{e}_2),$$

and (7.9) is satisfied. Next we verify conditions (4.13) and (4.10) that are necessary for near-flip buckling. The space V_h of admissible variations corresponding to the Euler strut with

[6] In his classical paper [12, pp. 102-103] Euler speaks about the column "placed vertically upon the base" and adds that it has "to be so constituted that it can not slip".

loading (7.6)–(7.9) is

$$V_h = \left\{ \boldsymbol{\varphi} \in W^{1,2}(R_h;\mathbb{R}^2) \colon \int_{-h/2}^{h/2} (\boldsymbol{\varphi}(0,x_2), \boldsymbol{e}_2) dx_2 = \int_{-h/2}^{h/2} (\boldsymbol{\varphi}(1,x_2), \boldsymbol{e}_2) dx_2 = 0 \right\}. \quad (7.10)$$

Condition (4.13) is satisfied because V_h does not contain maps of the form $\boldsymbol{Sx}+\boldsymbol{a}$. Condition (4.10) follows from [36], where it was shown that

$$K(V_h^R) = \frac{1}{4}\left(1 - \frac{\pi h}{\sinh(\pi h)}\right) \sim \frac{\pi^2 h^2}{24}.$$

Here

$$V_h^R = \left\{ \boldsymbol{\varphi} \in W^{1,2}(R_h;\mathbb{R}^2) \colon (\boldsymbol{\varphi}(0,x_2), \boldsymbol{e}_2) = (\boldsymbol{\varphi}(1,x_2), \boldsymbol{e}_2) = 0,\ x_2 \in \left[-\frac{h}{2}, \frac{h}{2}\right] \right\},$$

and $V_h^R \subset V_h$.

The asymptotics of the critical load is determined by Theorem 7.1, which gives

$$\lambda(h) \sim -2K_{\mathsf{L}_0}(V_h), \quad (7.11)$$

where we have used that $\mathfrak{c} = \lim_{h\to 0} t_h = 1/2$. Also, according to [36], $\lambda(h) = O(h^2)$. In fact, we can compute the asymptotics of the (anisotropic) Korn constant exactly.

THEOREM **7.2**

$$\lim_{h\to 0} \frac{K_{\mathsf{L}_0}(V_h)}{h^2} = \frac{E\pi^2}{24},$$

where

$$E = (\mathsf{L}_0^{-1}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1), \boldsymbol{e}_1 \otimes \boldsymbol{e}_1)^{-1} \quad (7.12)$$

is the (anisotropic) Young's modulus.[7]

PROOF: The idea of the proof is to bound the asymptotics of the Korn constant from above and from below and show that the bounds agree. The upper bound is obtained by means of the special test function in Korn's inequality (2.15) (the reasoning behind this choice is presented in Appendix B)

$$\boldsymbol{\varphi}_0(\boldsymbol{x}) = \alpha(x_1)\boldsymbol{e}_2 - \alpha'(x_1)x_2\boldsymbol{e}_1 + \frac{1}{2}\alpha''(x_1)\boldsymbol{\nu}x_2^2 - \frac{1}{24}\alpha''(x_1)\boldsymbol{\nu}h^2. \quad (7.13)$$

Here $\boldsymbol{\nu}$ is the anisotropic Poisson's ratio (B.2) and $\alpha(x_1) \in C^2([0,1])$ with $\alpha(0) = \alpha(1) = 0$. Clearly, the map $\boldsymbol{\varphi}_0$, restricted to R_h belongs to the space V_h for all $h > 0$. The ansatz essentially equivalent to (7.13) was first proposed by Kirchhoff in his analysis of the bending

[7] If L_0 is isotropic, i.e $\mathsf{L}_0\boldsymbol{\xi} = \kappa(\mathrm{Tr}\,\boldsymbol{\xi})\boldsymbol{I} + \mu(\boldsymbol{\xi} + \boldsymbol{\xi}^t - (\mathrm{Tr}\,\boldsymbol{\xi})\boldsymbol{I})$, then $E = 4\kappa\mu/(\kappa+\mu)$ is the 2D Young's modulus.

of thin plates [23] and was later generalized for shells by Love [27]. The link between buckling (and therefore bending) and the Korn constant is suggested by Theorem 7.1 that holds for the Euler strut (plate in 3D). The formula (7.1) in Theorem 7.1 implies that minimizers representing buckling modes and the optimal functions in the Korn inequality (2.15) are asymptotically equivalent. Hence, Kirchhoff-Love ansatz for bending should also be appropriate for estimating the Korn constant.

Observe that the gradient of the ansatz $\boldsymbol{\varphi}_0$ can be interpreted as a "parameterized flip",

$$\nabla\boldsymbol{\varphi}_0(\boldsymbol{x}) = \alpha'(x_1)\boldsymbol{S} + O(h). \tag{7.14}$$

In other words, if the slender Euler strut is viewed as a union of loosely connected rigid square blocks of size h, then formula (7.14) interprets buckling as a coherent combination of flips of the individual blocks.

Using $\boldsymbol{\varphi}_0$ as a test function in the generalized Korn's inequality (2.15) we obtain

$$\overline{\lim_{h\to 0}}\frac{K_{\mathsf{L}_0}(V_h)}{h^2} \le \lim_{h\to 0}\frac{\int_{R_h}(\mathsf{L}_0 e(\boldsymbol{\varphi}_0), e(\boldsymbol{\varphi}_0))d\boldsymbol{x}}{h^2\|\nabla\boldsymbol{\varphi}_0\|^2} = \frac{\int_0^1(\mathsf{L}_0\boldsymbol{\Omega}(x_1),\boldsymbol{\Omega}(x_1))dx_1}{24\int_0^1(\alpha'(x_1))^2dx_1}, \tag{7.15}$$

where

$$\boldsymbol{\Omega}(x_1) = \alpha''(x_1)(\boldsymbol{\nu}\odot\boldsymbol{e}_2 - \boldsymbol{e}_1\otimes\boldsymbol{e}_1) \tag{7.16}$$

and $\odot$ denotes the symmetrized tensor product

$$\boldsymbol{a}\odot\boldsymbol{b} = \frac{1}{2}(\boldsymbol{a}\otimes\boldsymbol{b} + \boldsymbol{b}\otimes\boldsymbol{a}). \tag{7.17}$$

Observing that

$$(\mathsf{L}_0(\boldsymbol{\nu}\odot\boldsymbol{e}_2 - \boldsymbol{e}_1\otimes\boldsymbol{e}_1), \boldsymbol{\nu}\odot\boldsymbol{e}_2 - \boldsymbol{e}_1\otimes\boldsymbol{e}_1) = (\mathsf{L}_0^{-1}(\boldsymbol{e}_1\otimes\boldsymbol{e}_1), \boldsymbol{e}_1\otimes\boldsymbol{e}_1)^{-1} = E,$$

we get

$$\overline{\lim_{h\to 0}}\frac{K_{\mathsf{L}_0}(V_h)}{h^2} \le \frac{E\int_0^1(\alpha''(x_1))^2dx_1}{24\int_0^1(\alpha'(x_1))^2dx_1}.$$

However,

$$\min_{\alpha(0)=\alpha(1)}\frac{\int_0^1(\alpha''(x_1))^2dx_1}{\int_0^1(\alpha'(x_1))^2dx_1} = \pi^2, \tag{7.18}$$

and so we obtain

$$\overline{\lim_{h\to 0}}\frac{K_{\mathsf{L}_0}(V_h)}{h^2} \le \frac{E\pi^2}{24}.$$

The reverse inequality can be proved by the application of the dimension reduction lemma, which is a linearized version of the results obtained in [15, 30]. Indeed, if we rescale the domain R_h to $R_1 = [0,1]\times[-1/2,1/2]$ through the change of variables $z_2 = x_2/h$ and introduce the rescaled operators

$$\nabla^h\boldsymbol{u} = \left(\frac{\partial\boldsymbol{u}}{\partial x_1}, \frac{1}{h}\frac{\partial\boldsymbol{u}}{\partial z_2}\right), \qquad e^h(\boldsymbol{u}) = \frac{1}{2}(\nabla^h\boldsymbol{u} + (\nabla^h\boldsymbol{u})^t),$$

with $\boldsymbol{u} = \boldsymbol{u}(x_1, z_2)$, we can prove the following

LEMMA **7.3** *Suppose a sequence $\boldsymbol{u}_h \in W^{1,2}(R_1;\mathbb{R}^2)$ satisfies $\|\nabla^h \boldsymbol{u}_h\| \le C$ and $\|e^h(\boldsymbol{u}_h)\| \le Ch$ for small h and for some constant C independent of h. Then there exists a subsequence (not relabeled) and a function $\alpha(x_1) \in W^{2,2}([0,1])$ such that*

$$\nabla^h \boldsymbol{u}_h \to \alpha'(x_1)\boldsymbol{S} \tag{7.19}$$

in $L^2(R_1;\mathrm{End}(\mathbb{R}^2))$, as $h \to 0$, and

$$\varliminf_{h\to 0} \frac{1}{h^2}\int_{R_1} (\mathsf{L}_0 e^h(\boldsymbol{u}_h), e^h(\boldsymbol{u}_h))dx_1dz_2 \ge \frac{E}{12}\int_0^1 (\alpha''(x_1))^2 dx_1.$$

Here the Young modulus E is given by (7.12).

The proof of Lemma 7.3 is analogous to the proof in [30, Th. 2.1] with $SO(3)$ replaced by $\mathbb{R}\boldsymbol{S}$—the Lie algebra of $SO(2)$, and we only indicate here the main idea: apply Korn inequality to each square of size h (after scaling back from R_1 to R_h) and derive the difference quotient estimate for $\nabla^h \boldsymbol{u}$, [31]. Essentially we need to estimate how closely the deformation in each of the "loosely connected" rigid square block of size h alluded to above, matches the infinitesimal rotation (flip).

Returning to the proof of the Theorem 7.2, we let $\boldsymbol{\varphi}_h \in V_h$ be such that $\|\nabla \boldsymbol{\varphi}_h\| = 1$ and

$$\lim_{h\to 0} \frac{\int_{R_h} (\mathsf{L}_0 e(\boldsymbol{\varphi}_h), e(\boldsymbol{\varphi}_h))d\boldsymbol{x}}{K_{\mathsf{L}_0}(V_h)} = 1.$$

Then the sequence $\boldsymbol{u}_h(x_1, z_2) = \sqrt{h}\boldsymbol{\varphi}_h(x_1, z_2 h)$ satisfies conditions of Lemma 7.3. It follows that there exists $\alpha(x_1) \in W^{2,2}([0,1])$ such that

$$1 = \lim_{h\to 0} \|\nabla \boldsymbol{\varphi}_h\|^2 = \lim_{h\to 0}\int_{R_1} |\nabla^h \boldsymbol{u}_h|^2 dx_1dz_1 = 2\int_0^1 (\alpha'(x_1))^2 dx_1 \tag{7.20}$$

and

$$\varliminf_{h\to 0} \frac{1}{h^2}\int_{R_h} (\mathsf{L}_0 e(\boldsymbol{\varphi}_h), e(\boldsymbol{\varphi}_h))d\boldsymbol{x} = \varliminf_{h\to 0} \frac{1}{h^2}\int_{R_1} (\mathsf{L}_0 e^h(\boldsymbol{u}_h), e^h(\boldsymbol{u}_h))dx_1dz_2 \ge \frac{E}{12}\int_0^1 (\alpha''(x_1))^2 dx_1.$$

Thus,

$$\varliminf_{h\to 0} \frac{K_{\mathsf{L}_0}(V_h)}{h^2} \ge \frac{E\int_0^1 (\alpha''(x_1))^2 dx_1}{24\int_0^1 (\alpha'(x_1))^2 dx_1}. \tag{7.21}$$

Let us show that $\boldsymbol{\varphi}_h \in V_h$ implies that $\alpha(0) = \alpha(1)$. Indeed, (7.19) implies that $\nabla \boldsymbol{v}_h \to \boldsymbol{0}$ in $L^2(R_1)$, where

$$\boldsymbol{v}_h = \boldsymbol{u}_h - \alpha(x_1)\boldsymbol{e}_2.$$

By the Poincaré inequality, there exists a sequence of constant vectors $\boldsymbol{c}_h$ such that $\boldsymbol{v}_h - \boldsymbol{c}_h \to \boldsymbol{0}$ in $W^{1,2}(R_1)$. Therefore,

$$\lim_{h\to 0}\int_{-1/2}^{1/2} (\boldsymbol{v}_h(0,z_2) - \boldsymbol{c}_h, \boldsymbol{e}_2)dz_2 = \lim_{h\to 0}\int_{-1/2}^{1/2} (\boldsymbol{v}_h(1,z_2) - \boldsymbol{c}_h, \boldsymbol{e}_2)dz_2 = 0.$$

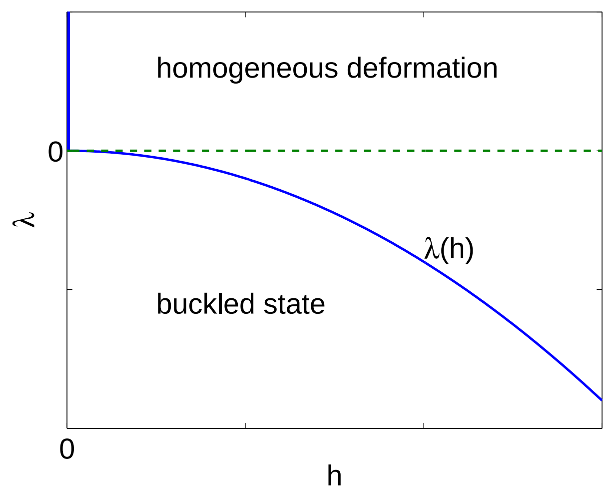

Figure 4: Asymptotic regions of stability in the (h, λ) plane near $(0, 0)$

We also have

$$\int_{-1/2}^{1/2} (\boldsymbol{u}_h(0, z_2), \boldsymbol{e}_2) dz_2 = \int_{-1/2}^{1/2} (\boldsymbol{u}_h(1, z_2), \boldsymbol{e}_2) dz_2 = 0,$$

since $\boldsymbol{u}_h(x_1, x_2/h) \in V_h$. Thus,

$$\alpha(0) = -\lim_{h \to 0} (\boldsymbol{c}_h, \boldsymbol{e}_2) = \alpha(1),$$

and the inequality (7.21) implies together with (7.18) that

$$\lim_{h \to 0} \frac{K_{\mathsf{L}_0}(V_h)}{h^2} \geq \min_{\alpha(0)=\alpha(1)} \frac{E \int_0^1 (\alpha''(x_1))^2 dx_1}{24 \int_0^1 (\alpha'(x_1))^2 dx_1} = \frac{E\pi^2}{24}.$$

∎

Our Theorem 7.2 together with (7.11) gives the explicit asymptotics of the critical load for the Euler strut:

$$\lim_{h \to 0} \frac{\lambda(h)}{h^2} = -\frac{\pi^2 E}{12}. \tag{7.22}$$

Furthermore, the preceding analysis results in the asymptotic stability digram shown in Figure 4. It is instructive to examine the domain of small compressive loadings on this diagram. In the region between the lines $\lambda = 0$ (dotted line) and $\lambda = \lambda(h)$ (thick line) the classical linearization procedure is valid and the trivial solution is unique (the Kirchhoff theorem applies). However, the range of λ corresponding to this classical linearization domain shrinks to zero as $h \to 0$, making the linearization limit non-uniform in h (see also [29]). Notice that any path corresponding to small fixed $\lambda < 0$ and $h \to 0$ eventually enters into the region of the stability diagram where the linearization around a trivial branch does not make sense. Therefore the two limiting procedures, linearization and dimension reduction, do not commute and the critical curve $\lambda = \lambda(h)$ marks the crossover between the two qualitatively different asymptotic regimes.

8 Non-Euler buckling

Theorem 7.1 established B-equivalence between the finite elasticity problem and the constitutively linearized problem when the function $\widehat{m}(h,\lambda)$ is smooth (Euler buckling). In this section we show that for structures containing multiple slender elements the function $\partial\widehat{m}(h,\lambda)/\partial\lambda$ is *never* continuous at $(0,0)$ leading to what we call non-Euler buckling.

To emphasize ideas, we consider a somewhat schematic setting, where all irrelevant difficulties are eliminated. More specifically, we assume that the structure Ω_h is represented by a union of three *disjoint*[8] Euler struts of aspect ratios h, $\sqrt{h}$ and h^2. We will be referring to them as C_1, $C_{1/2}$ and C_2 respectively. The load within each strut is homogeneous but different struts have different loads, so that

$$t_h(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \boldsymbol{x} \in C_1, \\ 10, & \text{if } \boldsymbol{x} \in C_{1/2}, \\ -1, & \text{if } \boldsymbol{x} \in C_2. \end{cases}$$

The most slender strut C_2 is under tension and is therefore stable. The thickest strut $C_{1/2}$ is much more compressed than the strut C_1 of intermediate thickness. Nevertheless it is the latter strut that will buckle first, and so,

$$\lambda(h) \sim -\frac{E\pi^2 h^2}{24}. \tag{8.1}$$

To show that Theorem 7.1 fails to deliver the correct asymptotics for the critical load (8.1) we first observe that the Korn constant for the domain Ω_h is

$$K_{\mathsf{L}_0}(V_{h^2}) \sim \frac{E\pi^2 h^4}{24},$$

with the most slender strut C_2 responsible for the value. At the same time the value $\mathfrak{c} = 10$ is determined by the load on the thickest strut $C_{1/2}$. Then the expression

$$-\frac{K_{\mathsf{L}_0}(V_{h^2})}{\mathfrak{c}} \sim -\frac{E\pi^2 h^4}{240}$$

neither delivers the correct asymptotics to the critical load nor makes any physical sense, as it combines quantities that are produced by the unrelated elements of the structure. To understand the problem we need to examine the function

$$\widehat{m}(h,\lambda) = \min\{K_{\mathsf{L}_0}(V_h) + \lambda, K_{\mathsf{L}_0}(V_{\sqrt{h}}) + 10\lambda, K_{\mathsf{L}_0}(V_{h^2}) - \lambda\}.$$

Here in a convenient abuse of notation, V_{h^α} denotes the space of variations (7.10) for the single Euler strut with aspect ratio h^α; the space of variations corresponding to our problem

[8] Disjointness in this example should not be understood literally. Our conclusions remain valid for an Euler strut which is cracked, meaning that the elements remain attached to the main structure by their slender bases.

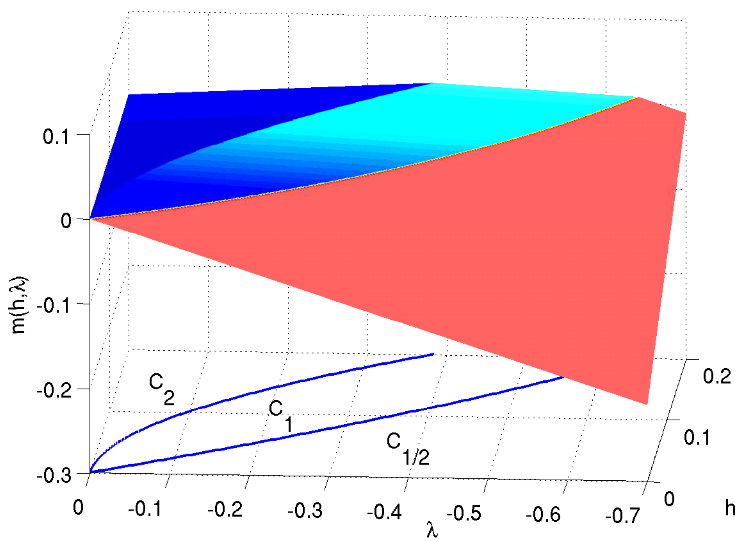

Figure 5: The graph of $\widehat{m}(h,\lambda)$ for the split strut example.

is then $V_h \oplus V_{\sqrt{h}} \oplus V_{h^2}$. We may also write

$$\widehat{m}(h,\lambda) = \begin{cases} K_{\mathsf{L}_0}(V_{h^2}) - \lambda, & \text{if } \lambda \in \left[\frac{1}{2}(K_{\mathsf{L}_0}(V_{h^2}) - K_{\mathsf{L}_0}(V_h)), 0\right] \\ K_{\mathsf{L}_0}(V_h) + \lambda, & \text{if } \lambda \in \left[\frac{1}{9}(K_{\mathsf{L}_0}(V_h) - K_{\mathsf{L}_0}(V_{\sqrt{h}})), \frac{1}{2}(K_{\mathsf{L}_0}(V_{h^2}) - K_{\mathsf{L}_0}(V_h))\right] \\ K_{\mathsf{L}_0}(V_{\sqrt{h}}) + 10\lambda, & \text{if } \lambda \le \frac{1}{9}(K_{\mathsf{L}_0}(V_h) - K_{\mathsf{L}_0}(V_{\sqrt{h}})). \end{cases}$$

It is easy to see that $\widehat{m}(0,\lambda) = 10\lambda$ and $\mathfrak{c} = \partial\widehat{m}(0,0)/\partial\lambda = 10$. The function $\partial\widehat{m}(h,\lambda)/\partial\lambda$ has jump discontinuities along the curves $\lambda = \frac{1}{2}(K_{\mathsf{L}_0}(V_{h^2}) - K_{\mathsf{L}_0}(V_h))$ and $\lambda = \frac{1}{9}(K_{\mathsf{L}_0}(V_h) - K_{\mathsf{L}_0}(V_{\sqrt{h}}))$, shown in Figure 5, violating both assumptions of Theorem 7.1.[9]

We conclude that Theorem 7.1 is not applicable to complex structures with multiple slender elements. Notice, however, that buckling in the above example is determined by the instability of a single Euler strut C_1 and therefore, even though the conditions of the Theorem 7.1 are violated, the problems for m and $\widehat{m}$ *are* B-equivalent. Therefore, we could have solved the problem by applying Theorem 7.1 to the Euler buckling of the strut C_1—the substructure of Ω_h that determines the critical load $\widehat{\lambda}(h)$. The nontrivial question, however, is how to extract such "most vulnerable" substructure in the general case.

[9]If we had assumed that the individual struts were in fact attached to the "main structure" by their slender bases, only the continuity of $\partial\widehat{m}/\partial\lambda$ at $(0,0)$ would have failed.

9 B-equivalence

In this section we relax the smoothness assumptions on $\widehat{m}(h,\lambda)$, and prove a more general B-equivalence theorem that is able to handle the non-Euler buckling discussed in Section 8.

First, to every sequence $\widetilde{\lambda}(h) < 0$ such that $\widetilde{\lambda}(h) \to 0$ as $h \to 0$, we associate a non-negative number[10] (including infinity)

$$p = \lim_{h\to 0} \frac{\widetilde{\lambda}(h)}{\widehat{\lambda}(h)}.$$

We regard two different sequences $\widetilde{\lambda}(h)$ as equivalent if they correspond to the same p. Our goal then is to find sufficient conditions for the sequence $\lambda(h)$ to belong to the class $p = 1$.

The idea is to show that when $h \to 0$, $m(h, \widetilde{\lambda}(h))$ has different asymptotics for sequences $\widetilde{\lambda}(h)$ that belong to different p-classes. The desired information on the asymptotics of $m(h, \widetilde{\lambda}(h))$ can be obtained via the relation (5.15). In what follows we show that $\lambda(h)$ belongs to the class $p = 1$, if the ratio of the main term to the residual in (5.15) is sufficiently large.

For each $p > 0$ we use the sequence $\widetilde{\lambda}(h) = p\widehat{\lambda}(h)$ as a representative of the class p. Then, from (5.15) we obtain

$$m(h, p\widehat{\lambda}(h)) = \widehat{m}(h, p\widehat{\lambda}(h)) + o(\widehat{\lambda}(h)).$$

The first term on the right-hand side is the main term, while the second term represents the residual. We may normalize the residual to obtain

$$\frac{m(h, p\widehat{\lambda}(h))}{\widehat{\lambda}(h)} = f_h(p) + o(1), \tag{9.1}$$

where

$$f_h(p) = \frac{\widehat{m}(h, p\widehat{\lambda}(h))}{\widehat{\lambda}(h)}. \tag{9.2}$$

The plan is to show that when $h \to 0$ the normalized main term $f_h(p)$ depends on p in an "essential way" in the vicinity of $p = 1$.

It is easy to see that functions $f_h(p)$ are uniformly bounded and uniformly Lipschitz continuous on any compact subset of $(0, +\infty)$. Thus, by the Ascoli-Arzela theorem, there exists a subsequence h_k and a continuous function $M(p)$ such that

$$\lim_{k\to\infty} f_{h_k}(p) = M(p) \tag{9.3}$$

uniformly in p on compact subsets of $(0, +\infty)$. From now on we restrict our attention to this subsequence h_k and relabel it back to h to simplify notation.

[10]If the limit does not exists, we restrict h to a convergent subsequence.

To distinguish sequences $\widetilde{\lambda}(h)$ belonging to different classes p, we need the function $M(p)$ to be strictly monotone in the vicinity of $p = 1$. First we observe that

$$N(s) = \lim_{h \to 0} s f_h \left(\frac{1}{s}\right) \tag{9.4}$$

is a convex and non-increasing function on $(0, +\infty)$. Indeed, we can write

$$N(s) = \lim_{h \to 0} n \left(h, \frac{s}{\widehat{\lambda}(h)}\right),$$

where

$$n(h, \tau) = \sup_{\substack{\boldsymbol{\varphi} \in V_h \\ \|\nabla \boldsymbol{\varphi}\| = 1}} \int_{\Omega_h} \left\{\tau(\mathsf{L}_0 e(\boldsymbol{\varphi}), e(\boldsymbol{\varphi})) + t_h(\boldsymbol{x}) |\nabla \boldsymbol{\varphi}|^2\right\} d\boldsymbol{x} \tag{9.5}$$

is a strictly monotone increasing and convex function of τ. Thus, $N(s)$ is also convex and non-increasing function of s. By passing to the limit as $h \to 0$ in (9.4), we obtain

$$M(p) = pN\left(\frac{1}{p}\right). \tag{9.6}$$

This formula implies that $M(p)$ is strictly monotone increasing in the vicinity of $p = 1$ if and only if $N(s)$ is strictly decreasing around $s = 1$. For real-valued, convex, non-increasing functions there is a simple characterization of strict monotonicity near a point.

Lemma **9.1** *Let $N(s)$ be a non-increasing, convex function on $(0, +\infty)$. $N(s)$ is strictly decreasing around $s_0 > 0$ if and only if $N'(s_0+) < 0$.*

Proof: For real-valued convex, non-increasing functions the one-sided derivatives

$$N'(s\pm) = \lim_{\epsilon \to 0^\pm} \frac{N(s + \epsilon) - N(s)}{\epsilon}$$

always exist and are non-positive. If $N'(s_0+) < 0$ then $N'(s_0-) \leq N'(s_0+)$ because $N(s)$ is a convex function and its (one-sided) derivative is therefore a non-decreasing function of s (if we assume that $s- < s+$). But then $N(s)$ is strictly decreasing around s_0.

Assume now that $N'(s_0+) = 0$. Then for any $s > s_0$ we have

$$0 \geq N'(s+) \geq N'(s-) \geq N'(s_0+) = 0.$$

Therefore, $N'(s\pm) = 0$ for every $s > s_0$. We conclude that $N(s) = N(s_0)$ for all $s > s_0$. ∎

Lemma 9.1 implies that $M(p)$ is strictly monotone increasing around $p = 1$ if and only if $\mathfrak{c}^* > 0$, where

$$\mathfrak{c}^* = -N'(1+) = M'(1-). \tag{9.7}$$

One can see that $\mathfrak{c}^*$ is exactly the measure of compressiveness $\mathfrak{c}$ in the example from Section 8 corresponding to the strut C_1. Indeed, a straightforward calculation shows that in this case

$$M(p) = \max\{-p, p-1\} = \begin{cases} -p, & \text{if } p \in [0, 1/2] \\ p-1, & \text{if } p > 1/2. \end{cases}$$

Thus, $\mathfrak{c}^* = M'(1) = 1$.

Clearly, computing $\mathfrak{c}^*$ may be demanding in cases, where $\widehat{m}(h, \lambda)$ is not known explictly. It is important to realize, however, that we only need to verify that $\mathfrak{c}^*$ is positive without having to compute its value. The following alternative, that is a simple corollary of Lemma 9.1 and formula (9.6), is important in both theory and examples.

Corollary 9.2 *Either* $\mathfrak{c}^* > 0$ *and* $M(p) < 0$*, for all* $p \in (0, 1)$*, and* $M(p) > 0$*, for all* $p > 1$ *or* $\mathfrak{c}^* = 0$ *and* $M(p) = 0$ *for all* $p \in (0, 1]$*.*

In order to complete the theory we need to show that B-equivalence indeed takes place when the asymptotics of $\widehat{m}(h, p\widehat{\lambda}(h))$ depends on p in an "essential way".

THEOREM **9.3** *Assume that* $\mathfrak{c}^* > 0$*. Then* $m(h, \lambda)$ *and* $\widehat{m}(h, \lambda)$ *are B-equivalent.*

PROOF: According to Corollary 9.2, $\mathfrak{c}^* > 0$ implies that $M(p) < 0$ for all $p \in (0, 1)$ and $M(p) > 0$ for all $p > 1$. Let us choose any $p > 1$. Then

$$\lim_{h \to 0} \frac{m(h, p\widehat{\lambda}(h))}{\widehat{\lambda}(h)} = \lim_{h \to 0} \frac{\widehat{m}(h, p\widehat{\lambda}(h))}{\widehat{\lambda}(h)} = M(p) > 0,$$

due to (9.1). Therefore, there exists $h_0 > 0$ such that $m(h, p\widehat{\lambda}(h)) < 0$ for all $h < h_0$. It follows from the definition of $\lambda(h)$ that $\lambda(h) \geq p\widehat{\lambda}(h)$. Thus,

$$\overline{\lim_{h \to 0}} \frac{\lambda(h)}{\widehat{\lambda}(h)} \leq p.$$

We conclude that

$$\overline{\lim_{h \to 0}} \frac{\lambda(h)}{\widehat{\lambda}(h)} \leq 1,$$

since, $p > 1$ was arbitrary.

Now, let $p \in (0, 1)$. Then $M(p) < 0$ and, arguing as before, we conclude that there exists $h_0 > 0$ such that $m(h, p\widehat{\lambda}(h)) > 0$ for all $h < h_0$. At this point, however, we cannot conclude that $\lambda(h) \leq p\widehat{\lambda}(h)$ because the definition of $\lambda(h)$ does not imply that $m(h, \lambda)$ is negative for all $\lambda < \lambda(h)$ and $m(h, \lambda)$ does not have to enjoy the same monotonicity properties as $\widehat{m}(h, \lambda)$. Let us show, however, that the desired limit inequality nevertheless holds. Arguing *ad absurdum*, we assume that there is a sequence $h_n \to 0$ such that $\lambda(h_n) > p\widehat{\lambda}(h_n)$. According to the definition (4.4) of $\lambda(h)$ there exists $\lambda_n \in (p\widehat{\lambda}(h_n), \lambda(h_n))$ such that $m(h_n, \lambda_n) < 0$. Hence, by (5.15)

$$\overline{\lim_{n\to\infty}} \frac{\widehat{m}(h_n, \lambda_n)}{\lambda_n} = \overline{\lim_{n\to\infty}} \frac{m(h_n, \lambda_n)}{\lambda_n} \geq 0.$$

But

$$\frac{\widehat{m}(h, \lambda)}{\lambda} = n\left(h, \frac{1}{\lambda}\right),$$

where $n(h, \tau)$, defined by (9.5), is a strictly monotone increasing function. Therefore,

$$\frac{\widehat{m}(h_n, \lambda_n)}{\lambda_n} < \frac{\widehat{m}(h_n, p\widehat{\lambda}(h_n))}{p\widehat{\lambda}(h_n)},$$

since $\lambda_n > p\widehat{\lambda}(h_n)$. Passing to the limit in the last inequality, we obtain

$$0 \leq \overline{\lim_{n\to\infty}} \frac{\widehat{m}(h_n, \lambda_n)}{\lambda_n} \leq \frac{M(p)}{p} < 0.$$

The contradiction above shows that there exists $h_0 > 0$, such that $\lambda(h) \leq p\widehat{\lambda}(h)$ for all $h < h_0$. Thus,

$$\underline{\lim_{h\to 0}} \frac{\lambda(h)}{\widehat{\lambda}(h)} \geq p$$

for all $p < 1$. ∎

The measure of compressiveness $\mathfrak{c}^*$ has the same meaning as $\mathfrak{c}$, except that it is generated by the "substructure" of Ω_h that is responsible for the critical load, while $\mathfrak{c}$ corresponds to the most compressed slender element. For the Euler strut those two measures of compressiveness coincide, while for the split Euler strut from Section 8 they are different. Intuitively, it is obvious that $\mathfrak{c} \geq \mathfrak{c}^*$, however to prove this inequality rigorously one needs to formalize the "extraction of substructure" procedure.

LEMMA **9.4** *Assume that $\widehat{\lambda}(h) < 0$ for all small enough h and*

$$\lim_{h\to 0} \widehat{\lambda}(h) = 0. \tag{9.8}$$

Then

$$\mathfrak{c} \geq \mathfrak{c}^* \geq 0. \tag{9.9}$$

PROOF: Let $\boldsymbol{\varphi}_h \in V_h$ be an almost-minimizer for $\widehat{m}(h, \widehat{\lambda}(h))$ such that $\|\nabla \boldsymbol{\varphi}_h\| = 1$ and

$$\lim_{h\to 0} \int_{\Omega_h} \left\{ \frac{(\mathsf{L}_0 e(\boldsymbol{\varphi}_h), e(\boldsymbol{\varphi}_h))}{\widehat{\lambda}(h)} + t_h(\boldsymbol{x})|\nabla \boldsymbol{\varphi}_h|^2 \right\} d\boldsymbol{x} = M(1) = 0. \tag{9.10}$$

According to (5.3), $\|e(\boldsymbol{\varphi}_h)\| \to 0$, as $h \to 0$. Using $\boldsymbol{\varphi}_h$ as a test function for $\widehat{m}(h, p\widehat{\lambda}(h))$ we obtain

$$f_h(p) \geq \int_{\Omega_h} \left\{ \frac{(\mathsf{L}_0 e(\boldsymbol{\varphi}_h), e(\boldsymbol{\varphi}_h))}{\widehat{\lambda}(h)} + p t_h(\boldsymbol{x}) |\nabla \boldsymbol{\varphi}_h|^2 \right\} d\boldsymbol{x}.$$

Passing to the limit as $h \to 0$ and using (9.10) we obtain, for $p < 1$, that

$$M(p) \geq (p-1) \overline{\lim_{h \to 0}} \int_{\Omega_h} t_h(\boldsymbol{x}) |\nabla \boldsymbol{\varphi}_h|^2 d\boldsymbol{x}.$$

Thus,

$$\mathfrak{c}^* = \lim_{p \to 1^-} \frac{M(p)}{p-1} \leq \overline{\lim_{h \to 0}} \int_{\Omega_h} t_h(\boldsymbol{x}) |\nabla \boldsymbol{\varphi}_h|^2 d\boldsymbol{x} \leq \mathfrak{c}.$$

To prove the last inequality in (9.9) we recall that $N(s)$ is non-increasing. This implies that $M(p)$ is non-negative, when $p > 1$ and non-positive, when $p < 1$. Thus, $\mathfrak{c}^* \geq 0$. ∎

10 Failure of B-equivalence

In this section we give examples showing that the variational problems for m and $\widehat{m}$ can be non B-equivalent even in the case when $\widehat{\lambda}(h) \to 0$. Obviously, in this situation we must have $\mathfrak{c}^* = 0$, i.e. the integral

$$\int_{\Omega_h} t_h(\boldsymbol{x}) |\nabla \boldsymbol{\varphi}_{h,\lambda}|^2 d\boldsymbol{x}$$

must be small. This can be achieved if $t_h(\boldsymbol{x})$ is either small or very oscillatory. The application of oscillatory forces will not result in the oscillatory stress field because of the St. Venant's principle (that expresses the smoothing properties of Green's functions of elliptic operators). Therefore, we concentrate on the case when $t_h(\boldsymbol{x})$ is uniformly small.

Consider the rectangular domain R_h as in the Euler's example in Section 7 and the loading given by (7.8), with λ replaced by $\rho\lambda h + \lambda^2$. The parameter $\rho > 0$ is fixed, but arbitrary. Let $\lambda(h)$ and $\widehat{\lambda}(h) = 2K_{\mathsf{L}_0}(V_h)$ denote the critical load of the Euler strut and its constitutively linearized counterpart, respectively. Let $\boldsymbol{F}_\lambda$ be the homogeneous deformation gradient of the trivial branch for the Euler strut from Section 7. If $\widetilde{\boldsymbol{F}}_{h,\lambda}$ and $\widetilde{\lambda}(h)$ are the trivial branch and the critical load for our example with re-parameterized loading, then

$$\widetilde{\boldsymbol{F}}_{h,\lambda} = \boldsymbol{F}_{\lambda(\lambda+\rho h)}$$

and

$$\widetilde{\lambda}(h) = -\frac{1}{2}\left(\rho h - \sqrt{\rho^2 h^2 + 4\lambda(h)}\right).$$

In other words $\widetilde{\lambda}(h)$ is the larger (smaller in absolute value) of the two negative roots of

$$x^2 + x\rho h = \lambda(h), \tag{10.1}$$

provided

$$\rho > \frac{2\sqrt{-\lambda(h)}}{h} = \rho_c.$$

If $\rho < \rho_c$ the strut never gets sufficiently compressed to buckle.

In order to determine t_h for our example we differentiate

$$W_{\boldsymbol{F}}(\widetilde{\boldsymbol{F}}_{h,\lambda}) = (\rho\lambda h + \lambda^2)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \tag{10.2}$$

in λ at $\lambda = 0$ to obtain

$$\widetilde{\boldsymbol{\sigma}}_h = \rho h \boldsymbol{e}_1 \otimes \boldsymbol{e}_1.$$

Therefore,

$$t_h = \frac{\rho h}{2},$$

and $\mathfrak{c} = 0$. The constitutively linearized critical load $\widehat{\widehat{\lambda}}(h)$ for our example is related to the constitutively linearized critical load $\widehat{\lambda}(h)$ for Euler strut via

$$\widehat{\widehat{\lambda}}(h) = \frac{\widehat{\lambda}(h)}{\rho h}.$$

If $0 < \rho < \rho_c$ then constitutively linearized problem predicts a buckling load $\widehat{\widehat{\lambda}}(h) \to 0$ while, in fact, the strut does not buckle at all. We also observe that if $\rho > \rho_c$ then

$$\lim_{h\to 0} \frac{\widetilde{\lambda}(h)}{\widehat{\widehat{\lambda}}(h)} = \frac{6\rho\left(\rho - \sqrt{\rho^2 - \frac{E\pi^2}{3}}\right)}{E\pi^2} > 1$$

and therefore m and $\widehat{m}$ are not B-equivalent. Taking into account (7.22) and (5.13) we obtain $\widehat{\widehat{\lambda}}(h) \to 0$, as $h \to 0$ and thus, by Lemma 9.4, $0 = \mathfrak{c} \geq \mathfrak{c}^* \geq 0$. Therefore, $\mathfrak{c}^* = 0$.

The above example can be viewed as a generalization of our remark at the end of Section 3 showing the possibility of a similar degeneracy in the case of flip. Here again the theory fails because it implicitly assumes that λ gives the scale of magnitude of the applied loads. As in the case of the flip, the condition $\mathfrak{c}^* > 0$ plays a dual role: it ensures B-equivalence of m and $\widehat{m}$, and endows the parameter λ with the expected physical meaning.

Notice that in our example the surface $\widehat{m}(h, \lambda)$ was smooth and there was no difference between $\mathfrak{c}^*$ and $\mathfrak{c}$. To show how the same scaling degeneracy presents itself in the non smooth case, where $\mathfrak{c} \neq \mathfrak{c}^*$, we consider two Euler struts, one with aspect ratio h^α, the other with aspect ratio h. The load on the first strut will be given by (7.8), with λ replaced by $-\lambda^2$, while the load on the second strut will be exactly as in (7.8). The first strut will buckle when the load is

$$-\sqrt{-\lambda(h^\alpha)} \sim -\frac{\pi h^\alpha \sqrt{E}}{2\sqrt{3}},$$

while the second column will buckle at a load

$$\lambda(h) \sim -\frac{E\pi^2 h^2}{12}.$$

If $\alpha < 2$ and h is sufficiently small, the second strut will buckle first.

Now let us compute $\mathfrak{c}^*$. We easily see that

$$t_h(\boldsymbol{x}) = \begin{cases} 0, & \boldsymbol{x} \in \text{first strut} \\ \dfrac{1}{2}, & \boldsymbol{x} \in \text{second strut} \end{cases}$$

and

$$\widehat{m}(h,\lambda) = \min\{K_{\mathsf{L}_0}(V_{h^\alpha}), K_{\mathsf{L}_0}(V_h) + \frac{1}{2}\lambda\}.$$

A simple calculation yields, $M(p) = (p-1)/2$, if $0 < \alpha \le 1$, and

$$M(p) = \max\{0, \frac{1}{2}(p-1)\} = \begin{cases} 0, & \text{if } p \in [0,1] \\ \dfrac{1}{2}(p-1), & \text{if } p > 1, \end{cases}$$

if $\alpha \in (1,2)$. Thus, when $\alpha \in (0,1]$, we have $\mathfrak{c}^* = 1/2 > 0$, while $\mathfrak{c}^* = M'(1-) = 0$, when $\alpha \in (1,2)$. At the same time, we have $\mathfrak{c} = 1/2 > 0$ for all $\alpha > 0$. The problems m and $\widehat{m}$ are B-equivalent for all $\alpha \in (0,2)$, but $\mathfrak{c}^* = 0$ for $\alpha \in (1,2)$. Hence, B-equivalence does not imply $\mathfrak{c}^* = 0$.

To summarize, even when the loading on a slender element that determines the critical buckling load is compressive and scales "nicely" with λ, it does not guarantee that $\mathfrak{c}^*$ is positive. In fact, the condition $\mathfrak{c}^* > 0$ requires that λ determines the scale of the load, not only on the element that buckles first, but also on the elements that are more slender.

11 Simple sufficient conditions for B-equivalence

It is desirable to have conditions that guarantee positivity of $\mathfrak{c}^*$ without the full knowledge of the function $M(p)$. Intuitively it is clear that if the most slender element in the structure is under compression, then $\mathfrak{c}^* > 0$. To express this condition we introduce a new measure of compressiveness

$$\mathfrak{c}_K = \sup_{\substack{\|e(\boldsymbol{\varphi}_h)\|^2 = O(K(V_h)) \\ \|\nabla\boldsymbol{\varphi}_h\|=1}} \overline{\lim_{h\to 0}} \int_{\Omega_h} t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}_h|^2 d\boldsymbol{x}. \tag{11.1}$$

The idea behind introducing $\mathfrak{c}_K$ is that condition $\|e(\boldsymbol{\varphi}_h)\|^2 = O(K(V_h))$ forces us to include only those variations $\boldsymbol{\varphi}_h$ that are supported on the most slender element(s). Then, the condition $\mathfrak{c}_K > 0$ says that one of the most slender elements in the structure is under compressive load.[11]

[11]If Ω_h contains multiple elements of maximal slenderness then $\mathfrak{c}_K$ corresponds to the most compressed one of those. The element that buckles first will necessarily have maximal slenderness, but in general can still be different from the one that generates $\mathfrak{c}_K$.

The parameter $\mathfrak{c}_K$ has several useful properties. First, its definition (11.1) implies that

$$\mathfrak{c} \geq \mathfrak{c}_K. \tag{11.2}$$

Then, in view of the second example in Section 10, condition $\mathfrak{c}_K > 0$ ensures that λ is the correct loading scale on the most slender element. Moreover we can prove

LEMMA **11.1** *The following three statements are equivalent*

(a) $\widehat{\lambda}(h) = O(K(V_h))$.

(b) $\mathfrak{c}_K \geq \mathfrak{c}^* > 0$.

(c) $\mathfrak{c}_K > 0$.

PROOF: Let us show that $(a) \Rightarrow (b)$. By our assumption

$$M(0) = \lim_{h\to 0} \frac{K_{\mathsf{L}_0}(V_h)}{\widehat{\lambda}(h)} < 0. \tag{11.3}$$

Therefore, by Corollary 9.2, $\mathfrak{c}^* > 0$. The proof of $\mathfrak{c}_K \geq \mathfrak{c}^*$ simply repeats the proof of the inequality (9.9) in Lemma 9.4. We only need to point out that the almost-minimizer $\boldsymbol{\varphi}_h$ for $\widehat{m}(h, \widehat{\lambda}(h))$ always satisfies $\|e(\boldsymbol{\varphi}_h)\|^2 = O(|\widehat{\lambda}(h)|)$. The test function $\boldsymbol{\varphi}_h$ is admissible in the definition (11.1) of $\mathfrak{c}_K$, since by assumption $\widehat{\lambda}(h) = O(K(V_h))$.

The implication $(b) \Rightarrow (c)$ is straightforward, so it remains to prove that $(c) \Rightarrow (a)$. Arguing *ad absurdum* we assume that there is a subsequence (not relabeled) such that $K(V_h)/\widehat{\lambda}(h) \to 0$, as $h \to 0$. Let $\boldsymbol{\varphi}_h \in V_h$ be a sequence of test functions such that $\|\nabla\boldsymbol{\varphi}_h\| = 1$, $\|e(\boldsymbol{\varphi}_h)\|^2 = O(K(V_h))$ and

$$\lim_{h\to 0} \int_{\Omega_h} t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}_h|^2 d\boldsymbol{x} = \mathfrak{c}_K.$$

Using the above sequence $\boldsymbol{\varphi}_h$ as a test function in the definition (5.10) of $\widehat{m}$ and applying the identity $\widehat{m}(h, \widehat{\lambda}(h)) = 0$, we get

$$0 = \frac{\widehat{m}(h, \widehat{\lambda}(h))}{\widehat{\lambda}(h)} \geq \varliminf_{h\to 0} \int_{\Omega_h} t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}_h|^2 d\boldsymbol{x} = \mathfrak{c}_K > 0.$$

This contradiction shows that $\widehat{\lambda}(h) = O(K(V_h))$. ∎

Even though positivity of $\mathfrak{c}_K$ ensures positivity of $\mathfrak{c}^*$, it is still possible for the element that buckles first to be different from the element responsible for the value of the parameter $\mathfrak{c}_K$. Therefore inequality (b) in Lemma 11.1 may be strict. Consider for example a structure Ω_h consisting of two disjoint Euler struts under compression with aspect ratios h and $2h$. If $t_h(\boldsymbol{x})$ on the second strut is twice as large as on the first one: $t_2 = 2t_1$, then, according to results of Section 7, the first strut will buckle first, resulting in $\mathfrak{c}^* = t_1$. At the same time $\mathfrak{c}_K = t_2 = 2\mathfrak{c}^*$.

If we add a third Euler strut of aspect ratio $\sqrt{h}$ with the value of $t_h(\boldsymbol{x})$ equal to $t_3 > t_2$, then we get $\mathfrak{c} = t_3 > \mathfrak{c}_K$. Thus, even within the context of Lemma 11.1, all the numbers $\mathfrak{c}$, $\mathfrak{c}_K$ and $\mathfrak{c}^*$ may be distinct. Observe also, that under the smoothness assumptions of Theorem 7.1, we have $\mathfrak{c}^* = \mathfrak{c} = \mathfrak{c}_K$. The first equality follows from the proof of Theorem 7.1, where in this case $M(p) = \mathfrak{c}(p-1)$. The last equality is the consequence of (11.2) and Lemma 11.1(b).

A new sufficient condition of B-equivalence is provided by the following

THEOREM **11.2** *If $\mathfrak{c}_K > 0$ then $m(h,\lambda)$ and $\widehat{m}(h,\lambda)$ are B-equivalent and $\lambda(h) = O(K(V_h))$.*

PROOF: If $\mathfrak{c}_K > 0$ then, according to Lemma 11.1, $\mathfrak{c}^* > 0$ and $\widehat{\lambda}(h) = O(K(V_h))$. In that case Theorem 9.3 ensures that the functionals $m(h,\lambda)$ and $\widehat{m}(h,\lambda)$ are buckling-equivalent, and therefore, $\lambda(h) = O(K(V_h))$. ▮

Corollary 11.3 *If $t_h(\boldsymbol{x}) > t_0 > 0$ then $m(h,\lambda)$ and $\widehat{m}(h,\lambda)$ are B-equivalent.*

We remark that the different measures of compressiveness and slenderness that have appeared so far in the paper are in one way or another associated with the function $M(p)$. For instance, the quantities $\mathfrak{c}_K$ and $K_{\mathsf{L}_0}(V_h)$ are associated with the behavior of $M(p)$ at $p = 0$. Indeed, by Lemma 11.1, $\mathfrak{c}_K > 0$ if and only if $M(0) < 0$, because

$$M(0) = \lim_{h\to 0} \frac{K_{\mathsf{L}_0}(V_h)}{\widehat{\lambda}(h)}.$$

The constant $\mathfrak{c} = \widehat{m}_0'$ is associated with the behavior of $M(p)$ at $p = \infty$ because it corresponds to the limit $h \to 0$ at fixed λ. In fact,

$$N(0) \ge \widehat{m}_0' \ge \lim_{p\to\infty} \frac{M(p)}{p} = N(0+). \tag{11.4}$$

The equality in the second position is secured if the element that buckles first is the most compressed one. If the second inequality (11.4) is strict, we may interpret it as a statement that $\widehat{m}_0'$ is determined by the behavior of $M(p)$ "beyond $p = \infty$". Finally, the quantities $\mathfrak{c}^*$ and $\widehat{\lambda}(h)$ are associated with the behavior of $M(p)$ at $p = 1$ by virtue of their definitions,.

12 Critical load as a generalized Korn constant

The definition of Korn's constant can be generalized as follows. Let Ω be an open and bounded subset of $\mathbb{R}^2$ and let V be a closed subspace of $W^{1,2}(\Omega;\mathbb{R}^2)$. Suppose that $t(\boldsymbol{x})$ is an arbitrary L^∞ function on Ω. We say that $t(\boldsymbol{x})$ is V-positive if there exists $\boldsymbol{\varphi} \in V$ such that

$$\int_\Omega t(\boldsymbol{x})|\nabla\boldsymbol{\varphi}|^2 d\boldsymbol{x} > 0.$$

Let L be a fourth order elasticity tensor. Define

$$K_{\mathsf{L}}[t(\boldsymbol{x});V] = \inf_{\substack{\boldsymbol{\varphi}\in V \\ \int_\Omega t(\boldsymbol{x})|\nabla\boldsymbol{\varphi}|^2 d\boldsymbol{x}=1}} \int_\Omega (\mathsf{L}e(\boldsymbol{\varphi}), e(\boldsymbol{\varphi}))d\boldsymbol{x}. \tag{12.1}$$

Clearly, if $t(\boldsymbol{x}) = 1$, we obtain the definition (2.15) of the Korn constant $K_{\mathsf{L}}(V)$. If $t(\boldsymbol{x})$ is not V-positive then the infimum in (12.1) is taken over an empty set, and $K_{\mathsf{L}}[t(\boldsymbol{x});V] = +\infty$.

THEOREM **12.1** *Let $\widehat{\lambda}(h) < 0$ be the critical load corresponding to $\widehat{m}(h,\lambda)$. Then t_h is V_h-positive and*

$$\widehat{\lambda}(h) = -K_{\mathsf{L}_0}[t_h(\boldsymbol{x});V_h].$$

PROOF: Suppose $\lambda < 0$, is such that $\widehat{m}(h,\lambda) < 0$. Then there exists $\boldsymbol{\varphi} \in V_h$, such that $\|\nabla\boldsymbol{\varphi}\| = 1$ and

$$\int_{\Omega_h} \{(\mathsf{L}_0 e(\boldsymbol{\varphi}), e(\boldsymbol{\varphi})) + \lambda t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}|^2\}d\boldsymbol{x} < 0. \tag{12.2}$$

In particular,

$$\int_{\Omega_h} t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}|^2 d\boldsymbol{x} > 0, \tag{12.3}$$

and t_h is V_h-positive. By using the definition (12.1) we obtain

$$\int_{\Omega_h} (\mathsf{L}_0 e(\boldsymbol{\varphi}), e(\boldsymbol{\varphi}))d\boldsymbol{x} \geq K_{\mathsf{L}_0}[t_h(\boldsymbol{x});V_h] \int_{\Omega_h} t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}|^2 d\boldsymbol{x}.$$

Applying this inequality to (12.2) we get

$$0 > (K_{\mathsf{L}_0}[t_h(\boldsymbol{x});V_h] + \lambda) \int_{\Omega_h} t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}|^2 d\boldsymbol{x}.$$

In view of (12.3), the inequality

$$\lambda < -K_{\mathsf{L}_0}[t_h(\boldsymbol{x});V_h]$$

is satisfied whenever $\widehat{m}(h,\lambda) < 0$. By definition of $\widehat{\lambda}(h)$ there exists a sequence $\lambda_n < \widehat{\lambda}(h)$ such that $\widehat{m}(h,\lambda_n) < 0$ and such that $\lambda_n \to \widehat{\lambda}(h)$ as $n \to \infty$. Thus, we conclude that

$$\widehat{\lambda}(h) \leq -K_{\mathsf{L}_0}[t_h(\boldsymbol{x});V_h]. \tag{12.4}$$

To prove equality in (12.4), let $\boldsymbol{\varphi}_h^{(n)} \in V_h$ be a minimizing sequence in the definition of $K_{\mathsf{L}_0}[t_h(\boldsymbol{x});V_h]$, i.e.

$$\int_{\Omega_h} t_h(\boldsymbol{x})|\nabla\boldsymbol{\varphi}_h^{(n)}|^2 d\boldsymbol{x} = 1$$

and

$$\lim_{n\to\infty} \int_{\Omega_h} (\mathsf{L}_0 e(\boldsymbol{\varphi}_h^{(n)}), e(\boldsymbol{\varphi}_h^{(n)}))d\boldsymbol{x} = K_{\mathsf{L}_0}[t_h(\boldsymbol{x});V_h].$$

Then, substituting

$$\widehat{\boldsymbol{\varphi}}_h^{(n)} = \frac{\boldsymbol{\varphi}_h^{(n)}}{\|\nabla \boldsymbol{\varphi}_h^{(n)}\|}$$

into (5.10) we get

$$\widehat{m}(h, \lambda) \leq \frac{1}{\|\nabla \boldsymbol{\varphi}_h^{(n)}\|^2} \left(\int_{\Omega_h} (\mathsf{L}_0 e(\boldsymbol{\varphi}_h^{(n)}), e(\boldsymbol{\varphi}_h^{(n)})) d\boldsymbol{x} + \lambda \right).$$

If $\lambda < 0$ is such that $\widehat{m}(h, \lambda) > 0$, then for every $n \geq 1$

$$\lambda > -\int_{\Omega_h} (\mathsf{L}_0 e(\boldsymbol{\varphi}_h^{(n)}), e(\boldsymbol{\varphi}_h^{(n)})) d\boldsymbol{x}.$$

Passing to the limit, as $n \to \infty$, we obtain

$$\lambda \geq -K_{\mathsf{L}_0}[t_h(\boldsymbol{x}); V_h]. \tag{12.5}$$

If the inequality (12.4) is strict then there exists λ_0 such that $\widehat{\lambda}(h) < \lambda_0 < -K_{\mathsf{L}_0}[t_h(\boldsymbol{x}); V_h]$. Therefore $\widehat{m}(h, \lambda_0) > 0$, by definition of $\widehat{\lambda}(h)$. Thus, λ_0 satisfies (12.5), which is a contradiction. ▮

An important property of $K_{\mathsf{L}}[t(\boldsymbol{x}); V]$ is its monotone dependence on t.

THEOREM **12.2** *Suppose $t_1(\boldsymbol{x}) \leq t_2(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$ and suppose that $t_1(\boldsymbol{x})$ is V-positive. Then*

$$K_{\mathsf{L}}[t_1(\boldsymbol{x}); V] \geq K_{\mathsf{L}}[t_2(\boldsymbol{x}); V].$$

PROOF: Clearly, $t_2(\boldsymbol{x})$ is V-positive, if $t_1(\boldsymbol{x})$ is. Moreover, if $\boldsymbol{\varphi} \in V$ is such that

$$\int_\Omega t_1(\boldsymbol{x}) |\nabla \boldsymbol{\varphi}|^2 d\boldsymbol{x} = 1, \tag{12.6}$$

then

$$\int_\Omega t_2(\boldsymbol{x}) \left| \nabla \left(\frac{\boldsymbol{\varphi}}{\alpha} \right) \right|^2 d\boldsymbol{x} = 1,$$

where

$$\alpha^2 = \int_\Omega t_2(\boldsymbol{x}) |\nabla \boldsymbol{\varphi}|^2 d\boldsymbol{x} \geq 1.$$

Then

$$K_{\mathsf{L}}[t_2(\boldsymbol{x}); V] \leq \frac{1}{\alpha^2} \int_\Omega (\mathsf{L} e(\boldsymbol{\varphi}), e(\boldsymbol{\varphi})) d\boldsymbol{x} \leq \int_\Omega (\mathsf{L} e(\boldsymbol{\varphi}), e(\boldsymbol{\varphi})) d\boldsymbol{x}.$$

To prove the theorem we must now take infimum over all $\boldsymbol{\varphi} \in V$ satisfying (12.6). ▮

The monotonicity of the generalized Korn constant allows one to derive bounds on the critical buckling load. For example, the inequality $t_h(\boldsymbol{x}) \le \|t_h\|_\infty$ implies that

$$\widehat{\lambda}(h) \le -\frac{K_{\mathsf{L}_0}(V_h)}{\|t_h\|_\infty}. \tag{12.7}$$

If in addition we know that $t_h(\boldsymbol{x}) \ge t_0 > 0$ then

$$-\frac{K_{\mathsf{L}_0}(V_h)}{t_0} \le \widehat{\lambda}(h) \le -\frac{K_{\mathsf{L}_0}(V_h)}{\|t_h\|_\infty}. \tag{12.8}$$

The inequalities (12.8) supplement results of Lemma 11.1 and Corollary 11.3. The second inequality in (12.8) is equivalent to the $(h, \lambda) \to (0, 0)$ asymptotics of the best finite h bound obtained in [10, 11]. In the homogeneous case the bounds (12.8) collapse, providing an explicit formula for the asymptotics of the critical load as we have illustrated in the case of Euler's strut.

A Trivial branch

Here we study the question whether for small $\lambda \neq 0$ the equation (3.7) has a unique smooth solution $\boldsymbol{F}_\lambda$ in the vicinity of $\boldsymbol{F} = \boldsymbol{I}$. The question is nontrivial because the implicit function theorem cannot be applied directly due to the fact that $\mathsf{L}_0 = W_{\boldsymbol{F}\boldsymbol{F}}(\boldsymbol{I})$ has rank $3 < 4$.

LEMMA **A.1** *Let $\boldsymbol{P}_0 \in \mathrm{Sym}(\mathbb{R}^2)$ be such that $\mathrm{Tr}\,\boldsymbol{P}_0 \neq 0$. Then there exists a neighborhood $\mathcal{N}$ of $\boldsymbol{I}$ in $\mathrm{End}(\mathbb{R}^2)$ where*

$$W_{\boldsymbol{F}}(\boldsymbol{F}) = \lambda \boldsymbol{P}_0 \tag{A.1}$$

has a unique solution $\boldsymbol{F}_\lambda$ for all sufficiently small $\lambda \neq 0$. In addition, $\boldsymbol{F}_\lambda$ is as smooth as $W_{\boldsymbol{F}}(\boldsymbol{F})$, and

$$\lim_{\lambda \to 0} \boldsymbol{F}_\lambda = \boldsymbol{I}. \tag{A.2}$$

PROOF: The objectivity of the energy $W_{\boldsymbol{F}}$ implies that the matrix $W_{\boldsymbol{F}}(\boldsymbol{F})\boldsymbol{F}^t$ is symmetric. Then according to (A.1) the matrix $\boldsymbol{P}_0\boldsymbol{F}^t$ must also be symmetric. Define

$$\mathcal{L} = \{\boldsymbol{F} \in \mathrm{End}(\mathbb{R}^2) \colon \boldsymbol{P}_0\boldsymbol{F}^t = \boldsymbol{F}\boldsymbol{P}_0\}.$$

The space $\mathcal{L}$ is three-dimensional for any $\boldsymbol{P}_0 \neq \boldsymbol{0}$[12] and the function

$$\boldsymbol{G}(\boldsymbol{F}, \lambda) = W_{\boldsymbol{F}}(\boldsymbol{F})\boldsymbol{F}^t - \lambda \boldsymbol{P}_0\boldsymbol{F}^t$$

maps the neighborhood of $\boldsymbol{I} \times \{0\}$ in $\mathcal{L} \times \mathbb{R}$ into the neighborhood of $\boldsymbol{0}$ in $\mathrm{Sym}(\mathbb{R}^2)$. In particular, $\boldsymbol{G}(\boldsymbol{I}, 0) = \boldsymbol{0}$ and

$$\boldsymbol{G}_{\boldsymbol{F}}(\boldsymbol{I}, 0) = \mathsf{L}_0,$$

[12] In 3D the subspace $\mathcal{L}$ is 6-dimensional, unless $\mathrm{rank}(\boldsymbol{P}_0) \le 1$.

where L_0 is understood as a map between $\mathcal{L}$ and $\mathrm{Sym}(\mathbb{R}^2)$. One can check that $\mathrm{Skew}(\mathbb{R}^2)\cap\mathcal{L} = \{\mathbf{0}\}$ if and only if $\mathrm{Tr}\,\boldsymbol{P}_0 \neq 0$. Thus, the map $\mathsf{L}_0\colon \mathcal{L} \to \mathrm{Sym}(\mathbb{R}^2)$ is a bijection and the implicit function theorem is applicable. So, there exists a smooth function $\boldsymbol{F}_\lambda$ such that for small λ it satisfies (A.2) and $\boldsymbol{G}(\boldsymbol{F}_\lambda, \lambda) = \mathbf{0}$. Thus, for small λ the matrices $\boldsymbol{F}_\lambda$ are invertible. Therefore, $\boldsymbol{F}_\lambda$ satisfies (A.1). ∎

Both conditions on $\boldsymbol{P}_0$ in Lemma A.1 are necessary. Indeed, if $\boldsymbol{F}_\lambda$ is a solution of (A.1), then, differentiating (A.1), with $\boldsymbol{F} = \boldsymbol{F}_\lambda$ at $\lambda = 0$, we get $\mathsf{L}_0\boldsymbol{F}_0' = \boldsymbol{P}_0$, from which it follows that $\boldsymbol{P}_0$ must be a symmetric. If $\mathrm{Tr}\,\boldsymbol{P}_0 = 0$, the question of solvability of (A.1) depends on the particular form of W, and below we present an example when solution does not exist.

Fix any symmetric $\boldsymbol{P}_0 \neq \mathbf{0}$ such that $\mathrm{Tr}\,\boldsymbol{P}_0 = 0$. Then there exists a symmetric matrix $\boldsymbol{M}_0$ such that $\boldsymbol{M}_0\boldsymbol{P}_0$ is not symmetric and $(\boldsymbol{M}_0, \boldsymbol{P}_0) > 0$. Now let L_0 represent a matrix of positive definite quadratic form on $\mathrm{Sym}(\mathbb{R}^2)$ such that $\mathsf{L}_0\boldsymbol{M}_0 = \boldsymbol{P}_0$. It is easy to verify that such a matrix always exists.

Lemma **A.2** *Let* $W(\boldsymbol{F}) = \dfrac{1}{4}(\mathsf{L}_0(\boldsymbol{F}^t\boldsymbol{F} - \boldsymbol{I}), \boldsymbol{F}^t\boldsymbol{F} - \boldsymbol{I})$, *where* L_0 *has been defined above. Then there is no* C^2 *function* $\boldsymbol{F}_\lambda$ *satisfying (A.2) and solving (A.1).*

Proof: Suppose, on the contrary, that the desired function $\boldsymbol{F}_\lambda$ does exist. Then, for our particular choice of W

$$\boldsymbol{F}_\lambda(\mathsf{L}_0(\boldsymbol{F}_\lambda^t\boldsymbol{F}_\lambda - \boldsymbol{I})) = \lambda\boldsymbol{P}_0. \tag{A.3}$$

If we substitute the asymptotics

$$\boldsymbol{F}_\lambda = \boldsymbol{I} + \lambda\boldsymbol{F}_1 + \lambda^2\boldsymbol{F}_2 + o(\lambda^2)$$

into (A.3) we obtain, equating terms of order λ and λ^2

$$2\mathsf{L}_0\boldsymbol{F}_1 = \boldsymbol{P}_0, \qquad \mathsf{L}_0(\boldsymbol{F}_1^t\boldsymbol{F}_1 + 2\boldsymbol{F}_2) + 2\boldsymbol{F}_1(\mathsf{L}_0\boldsymbol{F}_1) = \mathbf{0}.$$

It follows from the first equation above and non-degeneracy of L_0 that $\boldsymbol{F}_1 = \dfrac{1}{2}\boldsymbol{M}_0 + \boldsymbol{N}$, where $\boldsymbol{N}$ is a skew-symmetric matrix. The second equation implies that the matrix

$$\boldsymbol{F}_1(\mathsf{L}_0\boldsymbol{F}_1) = \frac{1}{4}\boldsymbol{M}_0\boldsymbol{P}_0 + \frac{1}{2}\boldsymbol{N}\boldsymbol{P}_0$$

is symmetric. In 2D one can show that if $\boldsymbol{P}_0$ is symmetric and trace-free and $\boldsymbol{N}$ is skew-symmetric then $\boldsymbol{N}\boldsymbol{P}_0$ is again symmetric and trace-free. Thus, the symmetry of $\boldsymbol{F}_1(\mathsf{L}_0\boldsymbol{F}_1)$ is equivalent to the symmetry of $\boldsymbol{M}_0\boldsymbol{P}_0$, which is false by construction. ∎

B Justification of the Kirchhoff-Love ansatz

First recall that the almost minimizers $\boldsymbol{\varphi}_h$ in the definition of the Korn constant (2.16) satisfy Lemma 7.3. Moreover, strong convergence in L^2 of the rescaled gradients $\sqrt{h}\nabla^h\boldsymbol{\varphi}_h$ prevents oscillatory behavior. Thus, without loss of generalitywe may assume that the almost minimizer $\boldsymbol{\varphi}_h$ depends on h smoothly and we can expand $\boldsymbol{\varphi}_h(\boldsymbol{x})$ in the powers of (x_2, h):

$$\boldsymbol{\varphi}_h(\boldsymbol{x}) = \boldsymbol{\phi}(x_1) + \boldsymbol{\psi}(x_1)x_2 + \boldsymbol{\eta}_1(x_1)h + \frac{1}{2}\boldsymbol{\xi}(x_1)x_2^2 + \frac{1}{2}\boldsymbol{\eta}_2(x_1)h^2 + \boldsymbol{\eta}_{12}(x_1)hx_2 + O(h^3).$$

We have

$$\nabla\boldsymbol{\varphi}_h(\boldsymbol{x}) = \boldsymbol{\phi}'(x_1) \otimes \boldsymbol{e}_1 + \boldsymbol{\psi}(x_1) \otimes \boldsymbol{e}_2 + O(h)$$

and

$$e(\boldsymbol{\varphi}_h) = (\boldsymbol{\phi}'(x_1) + x_2\boldsymbol{\psi}'(x_1) + h\boldsymbol{\eta}_1'(x_1)) \odot \boldsymbol{e}_1 + (\boldsymbol{\psi}(x_1) + x_2\boldsymbol{\xi}(x_1) + h\boldsymbol{\eta}_{12}(x_1)) \odot \boldsymbol{e}_2 + O(h^2),$$

where $\odot$ is the symmetrized tensor product, defined in (7.17). If $\boldsymbol{\varphi}_h$ is the optimal test function in the Korn inequality, then the leading term in $e(\boldsymbol{\varphi}_h)$ must vanish, while the one in $\nabla\boldsymbol{\varphi}_h$ must remain of order 1. It then follows that $\boldsymbol{\phi}(x_1) = \alpha(x_1)\boldsymbol{e}_2$, while $\boldsymbol{\psi}(x_1) = -\alpha'(x_1)\boldsymbol{e}_1$. In that case $\nabla\boldsymbol{\varphi}_h(\boldsymbol{x}) = \alpha'(x_1)\boldsymbol{S} + O(h)$, while

$$e(\boldsymbol{\varphi}_h) = x_2\boldsymbol{\Omega}(x_1) + h\boldsymbol{\Xi}(x_1) + O(h^2),$$

where

$$\boldsymbol{\Omega}(x_1) = \boldsymbol{\xi}(x_1) \odot \boldsymbol{e}_2 - \alpha''(x_1)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$$

and

$$\boldsymbol{\Xi}(x_1) = \boldsymbol{\eta}_1'(x_1) \odot \boldsymbol{e}_1 + \boldsymbol{\eta}_{12}(x_1) \odot \boldsymbol{e}_2.$$

Then we obtain

$$\int_{R_h} (\mathsf{L}_0 e(\boldsymbol{\varphi}_h), e(\boldsymbol{\varphi}_h))d\boldsymbol{x} = \frac{h^3}{12}\int_0^1 \{(\mathsf{L}_0\boldsymbol{\Omega}(x_1), \boldsymbol{\Omega}(x_1)) + 12(\mathsf{L}_0\boldsymbol{\Xi}(x_1), \boldsymbol{\Xi}(x_1))\}dx_1 + O(h^4). \quad \text{(B.1)}$$

Observe that the leading term in the asymptotics of $\|\nabla\boldsymbol{\varphi}_h\|^2 = 2h\|\alpha'\|^2 + O(h^2)$ depends only on $\alpha(x_1)$. Therefore, in order to minimize $\int_{R_h}(\mathsf{L}_0 e(\boldsymbol{\varphi}_h), e(\boldsymbol{\varphi}_h))d\boldsymbol{x}$, while keeping $\|\nabla\boldsymbol{\varphi}_h\|$ fixed, we need to minimize the h^3 term in (B.1) with respect to $\boldsymbol{\xi}(x_1)$, $\boldsymbol{\eta}_1(x_1)$ and $\boldsymbol{\eta}_{12}(x_1)$. The minimum is achieved at $\boldsymbol{\eta}_1(x_1) = \mathbf{0}$ and $\boldsymbol{\eta}_{12}(x_1) = \mathbf{0}$. Thus we arrive at the ansatz

$$\boldsymbol{\varphi}_0(\boldsymbol{x}) = \alpha(x_1)\boldsymbol{e}_2 - \alpha'(x_1)x_2\boldsymbol{e}_1 + \frac{1}{2}\boldsymbol{\xi}(x_1)x_2^2 + \frac{1}{2}\boldsymbol{\eta}_2(x_1)h^2,$$

subject to the constraint that $\boldsymbol{\varphi}_0 \in V_h$. It follows that we need to require that $\alpha(0) = \alpha(1) = 0$ and $(\boldsymbol{\eta}_2(x_1), \boldsymbol{e}_2) = -(\boldsymbol{\xi}(x_1), \boldsymbol{e}_2)/12$ at $x_1 = 0, 1$. Observe, that $\boldsymbol{\eta}_2$ does not enter the estimate for the Korn constant. Therefore, it will be convenient to choose $\boldsymbol{\eta}_2(x_1) = -\boldsymbol{\xi}(x_1)/12$.

Finally, we need to choose $\boldsymbol{\xi}(x_1)$ in such a way as to minimize $(\mathsf{L}_0\boldsymbol{\Omega}(x_1), \boldsymbol{\Omega}(x_1))$. Performing the minimization explicitly we obtain $\boldsymbol{\xi}(x_1) = \alpha''(x_1)\boldsymbol{\nu}$, where

$$\boldsymbol{\nu} = \boldsymbol{A}(\boldsymbol{e}_2)^{-1}\boldsymbol{A}(\boldsymbol{e}_1, \boldsymbol{e}_2)\boldsymbol{e}_1 \tag{B.2}$$

is the anisotropic Poisson's ratio.[13] Here we used the standard notation for the acoustic form

$$(\boldsymbol{A}(\boldsymbol{m}, \boldsymbol{n})\boldsymbol{u}, \boldsymbol{v}) = (\mathsf{L}_0(\boldsymbol{m} \otimes \boldsymbol{u}), \boldsymbol{n} \otimes \boldsymbol{v})$$

and the acoustic tensor $\boldsymbol{A}(\boldsymbol{n}) = \boldsymbol{A}(\boldsymbol{n}, \boldsymbol{n})$.

We finish by mentioning one curious effect of the anisotropy of L_0. Recall that according to the classical "Kirchhoff's hypothesis" the transversal fibers in a bent strut remain straight and orthogonal to the deformed middle surface. In our 2D setting the images of the transversal fibers $x_1 = x_1^0$ (see Figure 3) under the incremental displacements $\boldsymbol{u}_\epsilon = \epsilon\boldsymbol{\varphi}_0(\boldsymbol{x})$ are given by the parametric equations:

$$\begin{cases} X_1(t) &= x_1^0 - \epsilon\alpha'(x_1^0)t + \dfrac{\epsilon}{2}\alpha''(x_1^0)\nu_1(t^2 - h^2/12), \\ X_2(t) &= t + \epsilon\alpha(x_1^0) + \dfrac{\epsilon}{2}\alpha''(x_1^0)\nu_2(t^2 - h^2/12), \end{cases}$$

where $t \in [-h/2, h/2]$. The orthogonality of these curves to the deformed midline in the limit $h \to 0$ is readily verified. A simple calculation shows that the curvature of the lines $\boldsymbol{X}(t)$ in the limit $h \to 0$ is $\epsilon\alpha''(x_1^0)\nu_1$. Thus, the curvature of the deformed cross-sections in the limit $h \to 0$ is zero if and only if $\nu_1 = 0$. The formula in the footnote 13 shows that this condition is indeed satisfied for all isotropic tensors L_0. However, for a generic anisotropic material the curvature will be different from zero because of the axial Poisson effect ($\nu_1 \neq 0$). Therefore the Kirchhoff-Love ansatz $\boldsymbol{\varphi}_0$ given by (7.13) requires the transversal fibers to bend in an anisotropic material.

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