Microstructures minimizing the energy of a two phase elastic composite in two space dimensions. I: the confocal ellipse construction.

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Abstract

For modeling coherent phase transformations and for applications to structural optimization, it is of interest to identify microstructures with minimal energy or maximal stiffness. We present a new and appealingly simple class of extremal microstructures, which we call the *confocal ellipse construction,* for the case of a two-dimensional elastic composite made from two isotropic elastic materials. When the macroscopic stress

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and strain are isotropic, our construction reduces to the well-known "coated sphere" microstructure.

1   Introduction.

Extremal elastic composites arise in a number of different applications. In the phase transition literature, elastic energy minimization is widely believed to explain the shapes of coherent precipitates, see e.g. (Kaganova and Roitburd, 1987; Khachaturyan, 1983; Kohn and Lu, in preparation; Lee, Barnett and Aaronson, 1977). In recent work on structural optimization, the increasingly popular "homogenization method" makes use of extremal composites as structural components, see e.g. (Allaire and Kohn, 1993c; Bendsøe and Kikuchi, 1988; Jog, Haber and Bendsøe, 1992; Lurie and Cherkaev, 1986; Murat and Tartar, 1985). And in the modeling of damage, one recent approach is based on elastic energy minimization (Francfort and Marigo, 1993).

Fundamental to all these applications is the task of identifying composites which minimize or maximize the effective energy $(C^*\xi, \xi)$ at a given (typically anisotropic) strain ξ. For composites made from two "well-ordered" component materials there is now a general algorithm for evaluating the extremal energy and giving examples of extremal microstructures (Allaire and Kohn, 1993b; Allaire and Kohn, 1993a; Allaire and Kohn, 1994; Avellaneda, 1987b; Kohn and Lipton, 1988). This algorithm works even when the component materials are anisotropic, so long as the two Hooke's laws are "well-ordered." The microstructures it produces are "sequential laminates," created by a hierarchical procedure based on repeated lamination at well-separated length scales.

But elastically extremal microstructures are typically not unique. Consider for example the Hashin-Shtrikman lower bound on the bulk modulus of a composite made from two isotropic components. (This is a special case of the problem described above: it amounts to minimizing (C^*I, I), where I is the identity matrix.) The general theory provides an extremal sequentially laminated composite (see also Francfort and Murat, 1986). However, the optimal bulk modulus bound is also achieved by the well-known "concentric sphere construction" of Hashin (1962), which fills space with scaled copies of a coated sphere. The concentric sphere microstructure is of course totally different from a sequential laminate. It is in a sense quite complicated, since it requires the use of spheres of many sizes, ranging to

infinitesimal. But in another sense it is extremely simple, since everything is determined by a single basic unit — the coated sphere.

The goal of this article is to provide an anisotropic generalization of the concentric sphere construction. We shall show that an analogous *confocal ellipse construction* minimizes the effective energy $(C^*\xi, \xi)$ at a given anisotropic strain ξ, among mixtures of two isotropic materials with specified volume fractions, in two space dimensions. The orientation of the confocal ellipses depends on ξ, as does the eccentricity; this dependence will be made explicit. The construction works only for a certain range of ξ, specified by (4.5) below. As ξ approaches the boundary of this range, the ellipses become increasingly eccentric, yielding a microstructure which is locally more or less layered. For ξ outside the admissible set (4.5), the optimal microstructure is in fact obtained by simple layering.

Our work seems to be the first application of the confocal ellipse construction to elasticity. But this construction has previously been used to give examples of optimal microstructures for certain problems of effective conductivity (Bergman, 1980; Bergman, 1982; Milton, 1980; Milton, 1981a; Milton, 1981b; Milton, 2002; Tartar, 1985; Zhikov, 1991). It is no accident that the same microstructure succeeds in achieving optimality in both settings: one can show that our 2D elasticity problem is equivalent to a certain 2D conductivity problem, in the sense that the two have the same optimality conditions (Grabovsky, 1996). However, the arguments in this paper do not depend upon the link to conductivity. Rather, we shall establish the optimality of the confocal ellipse construction by means of a simple and direct calculation.

We wish to highlight some features of the confocal ellipse construction. One is the fact that the strain in each "core" ellipse is constant. Now, it has been known since the work of Eshelby that for an isolated elliptical inclusion in an infinite plane loaded at infinity, the strain in the inclusion is constant. When there are two or more elliptical inclusions, however, they interact, and the strain fields inside are generally not constant. The confocal ellipse construction shows that it is nevertheless possible to have *infinitely many inclusions, all perfectly elliptical in shape, with the property that the strain in each is constant.* Of course, to achieve this behavior it is crucial that the ellipses be positioned properly, and that their eccentricity bear the right relation to the average strain.

The confocal ellipse construction may be relevant to the equilibrium shapes of coherent precipitates in crystalline solids. Indeed, several authors have explained the shapes of cer-

tain precipitates in the low volume fraction regime, by treating them as isolated ellipsoidal inclusions and optimizing with respect to eccentricity and orientation, e.g. (Kaganova and Roitburd, 1987; Kardonski and Roitburd, 1972; Lee et al., 1977; Pineau, 1976). At larger volume fractions it is natural to expect the inclusions to interact, leading to shapes which are no longer elliptical. But in the context of the present paper, we see that even at large volume fraction an optimal configuration *can* consist of ellipses — though it does not have to.

The confocal ellipse construction has an interesting interpretation in the setting of structural optimization (Allaire and Kohn, 1993c; Kohn and Strang, 1986). Briefly, it gives an example of an optimal design problem which has an extremely simple solution. Indeed, let Ω be a region in the plane, and consider the problem of filling Ω with two isotropic elastic materials, using given volume fractions, so as to minimize the elastic energy associated with the affine boundary displacement $u = \xi x$. If Ω is an ellipse, with eccentricity in the appropriate relation to ξ, then the solution to this problem is to place the more rigid material in a single, confocal inclusion at the center of Ω.

Which optimal microstructure is "better," the confocal ellipse construction or the sequential laminate? The answer depends, of course, on the criterion by which they are judged. But we wish to emphasize an important distinction between the two constructions: the confocal ellipse geometry achieves the optimal bound *exactly*, whereas the sequential laminate achieves it only *approximately*. Indeed, the formula used for computing the effective behavior of a sequential laminate assumes a separation of scales, i.e. it applies in the limit as the ratio between successive length scales tends to infinity. An actual mixture can approach this behavior by having a large ratio between successive length scales, but it will always fall slightly short of the mark. The confocal ellipse construction does not suffer from this difficulty. It does require the use of small inclusions; but it is fully determined by dividing space into two measurable sets, one occupied by the "inclusion" phase and the other by the "matrix" phase, without the need for any limiting process.

It is natural to wonder whether the confocal ellipse construction can be generalized to the case of anisotropic component materials. We shall address this question in (Grabovsky, 1996). The answer is generically "no." The determining consideration is the Hooke's law tensor of the matrix material. For "most" anisotropic Hooke's laws, it is impossible to achieve optimality using a construction similar to confocal ellipses. Thus elastic anisotropy

serves to break the degeneracy of the problem considered here, favoring sequential lamination over confocal ellipses.

Another natural question is whether there is a similar confocal ellipsoid construction in three space dimensions. The answer is affirmative. This topic is addressed in (Grabovsky, 1996), using an argument based on optimality conditions and on known results from conductivity (Bergman, 1980; Bergman, 1982; Milton, 1980; Milton, 1981a; Milton, 1981b; Milton, 2002; Tartar, 1985; Zhikov, 1991). The argument used in the present paper is fundamentally two-dimensional, since it makes use of complex variables.

A third question is whether there might be other interesting optimal microstructures, besides sequential laminates and the confocal ellipse construction. The answer is a resounding "yes." A very different and highly ordered alternative is provided by the *Vigdergauz microstructure* (Vigdergauz, 1986; Vigdergauz, 1989; Vigdergauz, 1994), which consists of a periodic array of appropriately shaped inclusions. Our paper (Grabovsky and Kohn, 1995) gives a detailed exposition of this microstructure and its properties.

A fourth question concerns the Hooke's law C^* of the confocal ellipse microstructure. It is natural to ask whether C^* can be made explicit. We have no such formula – all we know explicitly is $\langle C^*\xi, \xi \rangle$ for the special value of ξ that determined the microstructure. Indeed, we doubt that the tensor C^* is fully determined: different ways of packing space with coated ellipses probably yield different Hooke's laws. This difficulty is well–known when $\xi = I$. A common solution is to use a realizable effective medium theory such as the coherent potential approximation (Milton, 1985) or the differential effective medium theory (Avellaneda, 1987a). These give more or less explicit formulas for C^* associated with *particular* ways of filling space with coated ellipses. In this respect the elasticity problem is quite different from conductivity, where the effective behavior of the confocal ellipse microstructure is computable and independent of the packing geometry (Bergman, 1980; Bergman, 1982; Milton, 1980; Milton, 1981a; Milton, 1981b; Milton, 2002; Tartar, 1985; Zhikov, 1991).

2   Formulation of the problem.

Let us consider a bounded open set E in R^2 with Lipschitz boundary. Assume that it is "made" of two isotropic elastic materials with Hooke's laws C_1 and C_2. Then at any point

$x \in E$ the Hooke's law is given by the 4th order tensor

$$C(x) = C_1 \chi_1(x) + C_2 \chi_2(x).$$

where $\chi_1(x)$ and $\chi_2(x)$ are the indicator functions of the sets occupied by materials 1 and 2 respectively, with

$$\chi_1(x) + \chi_2(x) = 1.$$

Each material is characterized by a bulk modulus k_i and a shear modulus μ_i, and the associated Hooke's law C_i is

$$C_i \xi = 2\mu_i \left(\xi - \frac{1}{2}(\mathbf{Tr}\xi)I \right) + k_i(\mathbf{Tr}\xi)I \tag{2.1}$$

for any symmetric 2×2 matrix ξ, where I is the identity matrix. We may assume that $\mu_1 \neq \mu_2$, since if $\mu_1 = \mu_2$ the effective behavior is completely determined and independent of the microstructure (Francfort and Tartar, 1991; Hill, 1963; Hill, 1964; Lurie, Cherkaev and Fedorov, 1982). We shall suppose that the two materials are numbered so that $\mu_1 > \mu_2$.

The vector of displacements v satisfies the equilibrium equation in E:

$$\nabla \cdot \big(C(x)e(v) \big) = 0, \tag{2.2}$$

where

$$e(v)_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

In this article we are mostly concerned with a particular case when the boundary condition is a prescribed affine displacement on ∂E:

$$v = \xi x, \qquad x \in \partial E \tag{2.3}$$

where ξ is 2×2 symmetric matrix. Our goal is to minimize the elastic energy over all choices of $\chi_1(x)$ with fixed average value

$$\theta_1 = < \chi_1 > = \int_E \chi_1(x)dx. \tag{2.4}$$

Let $QW(\xi)$ denote the minimal energy of the structure with the given affine boundary condition:

$$QW(\xi) = \inf_{<\chi_1>=\theta_1} \inf_{v|_{\partial E}=\xi x} \int_E \big(C(x)e(v), e(v) \big) dx. \tag{2.5}$$
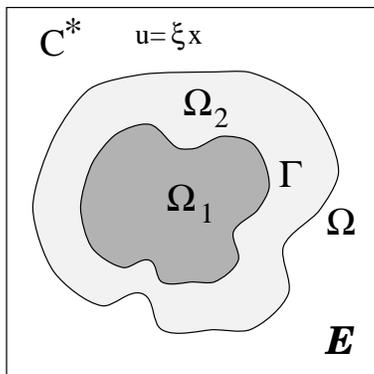
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Figure 1: The concentric "something" construction.

The value of $QW(\xi)$ is well known by now. It does not depend on the domain E, and it is the same for the periodic problem with average strain ξ. A formula for $QW(\xi)$ is given explicitly in (Allaire and Kohn, 1993a; Gibiansky and Cherkaev, 1984).

When $\xi = I$ there is a well-known construction for achieving $QW(\xi)$, the "concentric circle microstructure" introduced by Hashin (1962). The goal of this article is to give a similar construction for anisotropic ξ. In other words, we look for a pair of smooth, simply connected domains one embedded in the other such that when the ensemble is placed in an optimal effective medium C^*, strained uniformly, it does not disturb the elastic field outside (see Figure 1). This property permits one to construct extremal mixtures of C_1 and C_2 by packing space with scaled copies of the basic ensemble. This is exactly how the concentric circle construction works, only now the two domains are no longer circles, since ξ is not isotropic. We will show that for fixed ξ the ensemble $\Omega = \Omega_1 \cup \Omega_2$ has to be a particular ellipse and the inner inclusion Ω_1 has to be a confocal ellipse inside Ω.

This construction is not always possible, as we will see later. The regime of ξ for which it is possible is defined by (4.5) below. Roughly, our construction works if the hydrostatic part of ξ is large compared to its deviatoric part. When (4.5) doesn't hold, it is easy to see why there can be no analogue of the concentric sphere construction. It can be shown (see e.g. Grabovsky, 1996) that for such ξ a mixture achieving the optimal lower bound must lead to a strain field taking only two values. As a consequence, any smooth part of the interface in an optimal microstructure must be a straight line.

The key to our solution of this problem is the observation that the extremal fields must

satisfy certain optimality conditions (Milton, 2002). These conditions were explicitly derived in (Grabovsky, 1996). It turns out that the form of the optimality conditions depends on the value of the average strain ξ. There are three different regimes. In two of them the fields in both phases have to be constant. In the third regime, however, the fields in the matrix phase are only partially determined and they do not have to be constant. It is the optimality conditions for this regime that we are going to apply. We list them here for easy reference.

In phase **1**

$$e(v) = \varepsilon_0 I = \text{constant}, \tag{2.6}$$

where

$$\varepsilon_0 = \frac{(\mu_2 + k_2)\mathbf{Tr}\xi}{2(\mu_2 + \theta_1 k_2 + \theta_2 k_1)};$$

in phase **2**

$$\mathbf{div}v = d, \tag{2.7}$$

where

$$d = \frac{(k_1 + \mu_2)\mathbf{Tr}\xi}{(\mu_2 + \theta_1 k_2 + \theta_2 k_1)};$$

and in the whole of E

$$\mathbf{curl}(v) = 0. \tag{2.8}$$

We will show that these conditions uniquely determine the shape of Ω and the inclusion. However we must emphasize that the uniqueness is in part due to our restriction of the topology of the microstructure.

In the next two sections we will use complex variables in order to solve the inverse problem (2.6), (2.7), (2.8), using the approach of Cherepanov (1974). From now on $v(x)$ will denote the solution of (2.5) restricted to a single copy of Ω.

3   Reduction to complex variables.

First we will quickly review the complex variable theory for the 2-D isotropic elasticity equations (Muskhelishvili, 1953). Then we will formulate the inverse elasticity problem in complex variables.

If the Hooke's law C is isotropic with bulk modulus k and shear modulus μ and a vector field u solves

$$\nabla \cdot \big(Ce(u)\big) = 0, \tag{3.1}$$

in a connected open set Σ in R^2, then there exist two analytic functions ϕ and ψ of $z = x+iy$ in Σ, such that

$$u_1 + iu_2 = (\frac{1}{k} + \frac{1}{2\mu})\phi(z) - \frac{1}{2\mu}(\overline{\psi}(z) + z\overline{\Phi}(z)), \tag{3.2}$$

where $\Phi = \phi'$. The associated stress $\sigma = Ce(u)$ is given by

$$\left. \begin{array}{c} \sigma_{11} + \sigma_{22} = 4\Re e \Phi(z) \\[2mm] \sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2\big(\bar{z}\Phi'(z) + \Psi(z)\big) \end{array} \right\} \tag{3.3}$$

where $\Psi = \psi'$.

It is easy to check that the two potentials ϕ and ψ are unique up to an additive constant. More precisely, if ϕ_0 and ψ_0 is a pair of complex potentials then any other pair ϕ and ψ is given by

$$\phi = \phi_0 + c,$$

$$\psi = \psi_0 + \Lambda\bar{c},$$

where c is an arbitrary complex constant and

$$\Lambda = \frac{2\mu + k}{k}.$$

Now consider two domains Ω_1 and Ω_2 as in Figure 1, separated by a smooth interface Γ. Let

$$C(x) = \left\{ \begin{array}{ll} C_1 & \text{if } x \in \Omega_1 \\ C_2 & \text{if } x \in \Omega_2, \end{array} \right.$$

where C_1 and C_2 are isotropic Hooke's laws given by (2.1). Suppose v solves (2.2). Then (3.2) and (3.3) hold in each of the sets Ω_1 and Ω_2, and we also have the following interface conditions on Γ:

$$[\phi(z) + \overline{\psi}(z) + z\overline{\Phi}(z)] = \text{const} \tag{3.4}$$

$$[(\frac{1}{k} + \frac{1}{2\mu})\phi(z) - \frac{1}{2\mu}(\overline{\psi}(z) + z\overline{\Phi}(z))] = 0, \qquad z \in \Gamma. \tag{3.5}$$

Here square brackets denote the jump across the interface (e.g. $[\phi] = \phi_2 - \phi_1$).

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Now we can use the above formulas together with the optimality conditions (2.6), (2.7), (2.8) in order to determine the complex potentials. We find that

$$\phi_1 = k_1 \varepsilon_0 z, \qquad \psi_1 = 0, \qquad (3.6)$$

$$\phi_2(z) = \frac{1}{2} d k_2 z, \qquad (3.7)$$

choosing the arbitrary additive constants for each pair of complex potentials to be zero. We have not specified the potential ψ_2 because the optimality conditions provide no information about it. We need to use the interface conditions instead to determine ψ_2. Using the continuity of tractions (3.4) at the interface Γ we obtain

$$\psi_2(z) = c\bar{z}, \quad z \in \Gamma, \qquad (3.8)$$

where

$$c = \frac{\mu_2(k_1 - k_2)\mathbf{Tr}\xi}{\mu_2 + \theta_2 k_1 + \theta_1 k_2}. \qquad (3.9)$$

One can easily check that (3.8) is consistent with the continuity of displacements (3.5).

Now we must identify the boundary condition on $\partial\Omega$ in order to find ψ_2 there. Since we desire a generalization of concentric sphere construction, the set $\Omega = \Omega_1 \cup \Omega_2$ should not disturb the uniform strain field $e(u) = \xi$ if placed in the optimal effective medium C^*. It follows, by continuity of displacements, that the boundary condition on $\partial\Omega$ is:

$$v = \xi x, \quad x \in \partial\Omega$$

or after a simple calculation

$$v_1 + iv_2 = \frac{1}{2}(z\mathbf{Tr}\xi - (\xi_{22} - \xi_{11} - 2i\xi_{12})\bar{z}).$$

Therefore from (3.2) we can find ψ_2 on $\partial\Omega$:

$$\psi_2(z) = bz + \theta_1 c\bar{z}, \quad z \in \partial\Omega, \qquad (3.10)$$

where

$$b = \mu_2(\xi_{22} - \xi_{11} + 2i\xi_{12}) \qquad (3.11)$$

and c is defined in (3.9).

In the above calculation we have not used the continuity of tractions across $\partial\Omega$. To check that it holds we can argue that any solution to the problem (3.8), (3.10) provides

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Figure 2: Conformal mapping.

a test function for the standard energy variational principle. On the other hand, our test field satisfies the optimality conditions (2.6), (2.7) and (2.8). Therefore, the value of the energy functional on this test field assumes its absolute minimum (Grabovsky, 1996). Thus we conclude that such a test field must be the true field, satisfying the equations of elastic equilibrium (2.2). Therefore, for such a field the continuity of tractions must hold on all interfaces automatically.

We have reduced our problem to that of finding an analytic function in the annulus-like region bounded by Γ and $\partial\Omega$ satisfying the boundary conditions (3.8), (3.10). It turns out that this problem can be solved fairly easily.

4 Solution of the inverse problem.

In this section we find the remaining unknown complex potential ψ_2 and the shape of Ω and the inclusion using the technique introduced by Cherepanov (1974) for the case of holes. To begin, we represent our annulus-like region as the conformal image of the circular annulus formed by concentric circles K_1 and K_2 with radii r and R $(R > r)$ respectively under mapping $w(\zeta)$ (see Figure 2).

We adopt the notation $\psi(\zeta) = \psi_2(w(\zeta))$. Then the relations (3.8) and (3.10) become

$$\left.\begin{array}{l} \psi(\zeta) = c\overline{w(\zeta)}, \quad \zeta \in K_1 \\[2mm] \psi(\zeta) = bw(\zeta) + \theta_1 c\overline{w(\zeta)}, \quad \zeta \in K_2. \end{array}\right\} \qquad (4.1)$$

Now we represent $w(\zeta)$ by its Laurent series in the annulus,

$$w(\zeta) = \sum_{n=-\infty}^{\infty} w_n \zeta^n,$$

and substitute it into (4.1) using the fact that $\overline{\zeta}$ equals r^2/ζ on K_1 and R^2/ζ on K_2. Then, using the uniqueness of analytic continuation, we obtain:

$$\psi(\zeta) = \theta_1 c \sum_{n=-\infty}^{\infty} \overline{w}_n \frac{R^{2n}}{\zeta^n} + b \sum_{n=-\infty}^{\infty} w_n \zeta^n = c \sum_{n=-\infty}^{\infty} \overline{w}_n \frac{r^{2n}}{\zeta^n}.$$

Equating the coefficients at ζ^{-n} we get:

$$\theta_1 c \overline{w}_n R^{2n} + b w_{-n} = c \overline{w}_n r^{2n}. \tag{4.2}$$

We first notice that for $n = 0$ we have $b w_0 = \theta_2 c \overline{w}_0$. As we will see later the existence condition for the problem we are solving now is $|b| < \theta_2 |c|$, therefore we conclude that $w_0 = 0$.

Consider first the case $b = 0$ (which means that ξ is a multiple of identity). Then

$$\theta_1 c \overline{w}_n R^{2n} = c \overline{w}_n r^{2n}$$

and thus for all n such that $w_n \neq 0$ we must have

$$\theta_1 R^{2n} = r^{2n}.$$

Therefore there exists a unique n_0 such that $w_{n_0} \neq 0$. Since we want a 1 to 1 mapping between the annuli we must have $n_0 = 1$ or $n_0 = -1$ and, $w(\zeta) = w_1 \zeta$ or $w(\zeta) = w_{-1}/\zeta$ while $r^2/R^2 = \theta_1$. In both cases we arrive at concentric circles.

Now consider the remaining case $b \neq 0$, and let

$$q = \frac{b}{c}.$$

Then we can rewrite (4.2) as

$$w_{-n} = \frac{\overline{w}_n (r^{2n} - \theta_1 R^{2n})}{q}$$

for all $n \in Z$. Then

$$w_n = \frac{\overline{w}_{-n}(r^{-2n} - \theta_1 R^{-2n})}{q} = \frac{w_n(\theta_1 R^{2n} - r^{2n})(\theta_1 R^{-2n} - r^{-2n})}{|q|^2}$$

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and therefore for any n for which $w_n \neq 0$ we have

$$|q|^2 = {\theta_1}^2 + 1 - \theta_1 \left(\frac{r^{2n}}{R^{2n}} + \frac{R^{2n}}{r^{2n}} \right).$$

This is a quadratic equation with respect to $X = R^{2n}/r^{2n}$:

$$\theta_1{}^2 + 1 - \theta_1 \left(X + \frac{1}{X} \right) = |q|^2. \tag{4.3}$$

For the construction to exist this equation must have two distinct real roots X and $1/X$. One must be greater than 1 the other must be less than 1. In order for this to be true we must have

$$X + \frac{1}{X} = \frac{\theta_1{}^2 + 1 - |q|^2}{\theta_1} > 2$$

or

$$|q|^2 < \theta_2{}^2. \tag{4.4}$$

In the original parameters this existence condition has the form

$$\left[(\xi_{22} - \xi_{11})^2 + 4\xi_{12}^2 \right]^{1/2} < \theta_2 \frac{|k_1 - k_2| \cdot |\xi_{11} + \xi_{22}|}{\mu_2 + \theta_1 k_2 + \theta_2 k_1}, \tag{4.5}$$

If (4.5) holds then there is a unique n_0 such that $w_{n_0} \neq 0$ and $w_{-n_0} \neq 0$. But n_0 has to be 1 (or -1) because we need a 1 to 1 mapping between the annuli. So we obtain

$$w(\zeta) = w_1 \zeta + \frac{\overline{w}_1 (r^2 - \theta_1 R^2)}{q} \frac{1}{\zeta}, \tag{4.6}$$

which solves the problem.

5 Analysis of the solution.

Now we are ready to answer the question posed at the beginning of this article, what is the shape of the domains Ω_1 and Ω_2 (Figure 1) corresponding to the elastic energy minimum. The conformal image of circles K_1 and K_2 under the map w given by (4.6) provides the answer. Let us study the map w in detail. Let $w_1 = |w_1| e^{i\alpha}$. Then

$$w(\zeta) = |w_1| \left(e^{i\alpha} \zeta + \frac{r^2 (1 - \theta_1 X)}{q} \frac{1}{e^{i\alpha} \zeta} \right).$$

Notice that $1 - \theta_1 X > 0$ because from (4.3)

$$1 - \theta_1 X = \frac{|q|^2 X}{X - \theta_1} > 0, \tag{5.1}$$

as $X > 1$. Now let β be such that

$$q = |q|e^{i\beta}$$

(q is complex in general). Then

$$w(\zeta) = 2|w_1|rpe^{-i\beta/2}J(\frac{e^{i\gamma}}{rp}\zeta),$$

where

$$p = \sqrt{\frac{1 - \theta_1 X}{|q|}}, \quad \gamma = \alpha + \beta/2,$$

and

$$J(z) = \frac{1}{2}(z + \frac{1}{z})$$

is the Joukovsky function. Now it is clear how the function $w(\zeta)$ acts. The map $\zeta' = (e^{i\gamma}/rp)\zeta$ transforms concentric circles K_1 and K_2 with radii r and R into concentric circles with radii $1/p$ and $\sqrt{X}/p$. The Joukovsky function maps these concentric circles into confocal ellipses with axes

$$a_1 = \frac{1}{2}(p + \frac{1}{p}) \quad b_1 = \frac{1}{2}(p - \frac{1}{p}) \quad a_2 = \frac{1}{2}(\frac{\sqrt{X}}{p} + \frac{p}{\sqrt{X}}) \quad b_2 = \frac{1}{2}(\frac{\sqrt{X}}{p} - \frac{p}{\sqrt{X}}).$$

Then these ellipses are scaled arbitrarily and rotated by the angle $-\beta/2$. Scaling does not change strains and stresses in the phases. Thus what matters is only the ratio of the axes. We can simplify and scale the values of a_i and b_i:

$$a_1^2 = \theta_1(\theta_2 + |q|)^2(1 + \theta_1 + |q|)^2;$$

$$b_1^2 = \theta_1({\theta_2}^2 - |q|^2)((1 + \theta_1)^2 - |q|^2);$$

$$a_2^2 = (\theta_2 + |q|)^2((1 + \theta_1)^2 - |q|^2);$$

$$b_2^2 = ({\theta_2}^2 - |q|^2)(1 + \theta_1 + |q|)^2.$$

In order to describe the orientation of the ellipses we need a simple lemma.

Lemma 1 *Let ξ be a symmetric 2×2 matrix. Let $a(\xi) = \xi_{22} - \xi_{11} + 2i\xi_{12} = |a|e^{i\phi}$. Then the direction $\mathrm{Arg}(z) = -\phi/2$ is the eigendirection of ξ corresponding to the smaller eigenvalue.*

Proof. Introduce the notation: $y = \xi x$, $z = x_1 + ix_2$, $w = y_1 + iy_2$. Direct calculation shows that

$$w = \frac{1}{2}(z\,\mathbf{Tr}\xi - \bar{a}\bar{z})$$

Taking $z = e^{-i\phi/2}$ we obtain $w = kz$, where $k = \min\{\xi_1, \xi_2\}$, and ξ_1, ξ_2 are the eigenvalues of ξ. $\square$

Now we apply this lemma to the matrix

$$\eta = \frac{\mu_2}{c}\xi$$

and notice that $q = a(\eta)$. Then it easily follows that rotation by angle $-\beta/2$ causes the principal directions of ellipses to coincide with the eigendirections of ξ. Moreover, the direction of the longer axes of the ellipses corresponds to the eigenvalue with smaller modulus if $k_1 > k_2$ (the materials are well ordered), and to the eigenvalue with larger modulus otherwise.

6 Relevance to phase transitions.

As we mentioned in the introduction, energy minimization is a generally accepted tool for explaining the shapes of coherent precipitates. (See (Socrate and Parks, 1993) for a recent critical review). In the low volume fraction limit the shape and orientation of various precipitates have been explained by such a criterion. The argument in this article shows that energy minimization is consistent with elliptical precipitates even when the volume fraction is not small. However, in this case all length scales must occur and the particle locations must be strongly correlated.

The validity of this conclusion may not be immediately clear, because in the modeling of precipitation processes, the two phases usually have different stress-free strains. In the case when the two phases also have different Hooke's laws this problem is equivalent to ours, but with a new average strain tensor ξ'. This fact is known among experts (see e.g. Benveniste and Dvorak, 1990; Levin, 1967), but perhaps it is not yet widely appreciated. In the present context we can formulate the problem with a nonzero transformation strain as follows. Let $\varepsilon(x) = \varepsilon_1\chi_1(x) + \varepsilon_2\chi_2(x)$ be the stress free strain at the point x. Then the equilibrium equations of elasticity have the following form:

$$\begin{cases} \nabla \cdot \Big(C(x)\big(e(v) - \varepsilon(x)\big)\Big) = 0, \\ v = \xi x, \quad x \in \partial\Omega. \end{cases} \tag{6.1}$$

We claim that there is a new average strain ξ' such that the solution v' of

$$
\begin{cases}
\nabla \cdot \big(C(x)e(v')\big) = 0, \\
v' = \xi'x, \quad x \in \partial\Omega
\end{cases}
\tag{6.2}
$$

is related to v by

$$
v = v' + \xi x - \xi'x.
\tag{6.3}
$$

We choose ξ' such that it satisfies

$$
C(x)(\varepsilon(x) + \xi - \xi') = \text{constant},
$$

or equivalently

$$
\xi' = \xi + (C_1 - C_2)^{-1}(C_1\varepsilon_1 - C_2\varepsilon_2).
$$

One verifies that for this choice of ξ' the solutions of Euler equations (6.1) and (6.2) are related by (6.3).

One might try to use our results to predict the shapes of coherent precipitates at large volume fraction. Unfortunately, as we know, energy minimization does not determine an optimal microstructure uniquely. Therefore energy minimization alone cannot be sufficient to explain the shapes of coherent precipitates. There are various selection mechanisms which might prefer one elastically optimal microstructure over another. One is surface energy — widely believed to be significant for modeling of coherent phase transformations (see e.g. Grinfeld, 1991; Khachaturyan, 1983; Thompson, Su and Voorhees, 1994; Voorhees, 1992), and also considered in recent work on structural optimization (Haber, Jog and Bendsøe, 1994). Another is dynamics — which might favor an optimal microstructure with a larger basin of attraction; our understanding of this issue is very limited, but see (Swart and Holmes, 1992; Thompson et al., 1994) for some relevant work. A third mechanism is the effect of geometrical nonlinearity — see (Ball and James, 1992; Bhattacharya, 1993; Kohn, 1991) for examples involving martensitic transformation, where geometrical nonlinearity breaks some of the degeneracy of predictions based on a linear strain analysis. A fourth mechanism is elastic anisotropy, which seems to favor "rank-two lamination" over the confocal ellipse construction (Grabovsky, 1996).

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