An application of the general theory of exact relations to fiber-reinforced conducting composites with Hall effect.

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Abstract

In this paper we present the exhaustive and non-redundant list of all microstructure-independent relations for effective conductivity of fiber-reinforced composites with Hall effect. Our results are independent of the number of constituents and the degree of their anisotropy. We use this to apply our results to both polycrystals and two-phase composites. Our tool is the general theory of exact relations for composite materials developed in collaboration with Graeme Milton and Daniel Sage.

1 Introduction

In [14] Graeme Milton, Daniel Sage and the author have developed a general theory (see also [21, Chapter 17] for a compact account) that has the power to identify every single microstructure-independent relation for effective tensors of composites in a wide variety of physical contexts, regardless of the number of constituent materials or the degree of their anisotropy. This theory has been applied to polycrystalline composites in [10] to 2D conductivity and elasticity, in [15] to 3D elasticity and 2D and 3D piezo-electricity, and in [14] to 3D pyro-electricity, thermo-electricity, thermo-elasticity and thermo-piezo-electricity. In [11] the theory has been applied in the non-polycrystalline case of 2D conductivity with Hall effect. In this paper we report on the results of a massive research effort[1] [12] that builds on [11] and applies the general theory to fiber-reinforced conducting composites with Hall effect. In this case the number of (infinite families of) solutions to the equations of general theory is overwhelming. Nevertheless, thanks to the efforts of the three REU teams, led by the author, all of the solutions have been computed, organized and reduced to a small number of

[1] This worked spanned 3 Summers and involved 14 talented undergraduates under the NSF funded program "Research Experience for Undergraduates" (REU).

fundamental facts, presented in this paper. Here we apply our exact results to two examples: a polycrystal and a two-phase composite.

In the framework of the homogenization theory for periodic composites [2, 16, 21], the microstructure of a fiber-reinforced conducting composite is described mathematically by an L^∞ function $\boldsymbol{L}(\boldsymbol{x})$ taking values in the set of 3×3 positive-definite matrices and defined on a 2D period cell $Q = [0, 1]^2$. The anti-symmetric part of the local conductivity matrix $\boldsymbol{L}$ is due to the assumed presence of the magnetic field $\boldsymbol{h}$ and the non-vanishing 3×3 matrix of Hall coefficients $\boldsymbol{R}$, introduced in [7]. If $\boldsymbol{C}$ denotes the symmetric part of $\boldsymbol{L}$, then the current density $\boldsymbol{j}$ produced inside such a material by the electric field $\boldsymbol{e}$ is given by

$$\boldsymbol{j} = \boldsymbol{C}\boldsymbol{e} + (\boldsymbol{R}\boldsymbol{h}) \times \boldsymbol{e}. \tag{1.1}$$

If the magnetic field $\boldsymbol{h}$ is weak, both the conductivity tensor $\boldsymbol{C}$ and the Hall tensor $\boldsymbol{R}$ can be assumed to be independent of the magnetic field. If the magnetic field is strong then the dependence of $\boldsymbol{C}$ on $\boldsymbol{h}$ can not be neglected. Indeed, our formulas indicate that even if $\boldsymbol{C}$ and $\boldsymbol{R}$ do not depend on $\boldsymbol{h}$ originally, the effective tensor $\boldsymbol{C}^*$ of a composite may exhibit strong dependence on $\boldsymbol{h}$, when $|\boldsymbol{h}|$ is large.

Suppose that the periodic composite occupies a domain $\Omega \subset \mathbb{R}^3$. Then the conductivity of the material at the point $(x_1, x_2, x_3) \in \Omega$ is assumed to be equal to $\boldsymbol{L}(\boldsymbol{x}/\epsilon)$, where $\boldsymbol{x} = (x_1, x_2)$, the function $\boldsymbol{L}(\boldsymbol{x})$ is extended doubly-periodically to $\mathbb{R}^2$ with period cell $Q = [0, 1]^2$, and a small parameter ϵ is a typical length scale of the periodic microstructure in the plane transversal to the fibers.

The independence of the microstructure of x_3 makes it convenient to represent all tensors in block form separating directions that are parallel and transversal to the fibers. Thus, we write

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{\Lambda} & \boldsymbol{p} \\ \boldsymbol{q} & \alpha \end{bmatrix},$$

where $\boldsymbol{\Lambda}$ is the 2×2 positive definite matrix of the transversal conductivity; $\alpha > 0$ is the conductivity along the fibers and $\boldsymbol{p}$ and $\boldsymbol{q}$ are $\mathbb{R}^2$ vectors linking fields along and across the fibers.

The general cell problem for periodic conducting composites is

$$\mathrm{Div}(\boldsymbol{L}(\mathrm{Grad}(\phi) + \widehat{\boldsymbol{\xi}})) = 0, \tag{1.2}$$

where Div and Grad operators are divergence and gradient, respectively, in (x_1, x_2, x_3) and $\widehat{\boldsymbol{\xi}} \in \mathbb{R}^3$ is an arbitrary vector. It is not difficult to show that the independence of $\boldsymbol{L}(\boldsymbol{x})$ from the x_3 variable implies that the solution ϕ of (1.2) is also independent of x_3. In that case we may rewrite the cell problem (1.2) in terms of the block-components of $\boldsymbol{L}(\boldsymbol{x})$:

$$\nabla \cdot (\boldsymbol{\Lambda}(\boldsymbol{x})(\nabla\phi + \boldsymbol{\xi}) + \xi_3 \boldsymbol{p}(\boldsymbol{x})) = 0, \tag{1.3}$$

where $\boldsymbol{\xi} \in \mathbb{R}^2$, $(\boldsymbol{\xi}, \xi_3) = \widehat{\boldsymbol{\xi}}$ and $\nabla\cdot$ and ∇ are the divergence and gradient operators, respectively, in $\boldsymbol{x} = (x_1, x_2)$. The solution to (1.3) can be conveniently written in terms of the generalized Helmholtz projection operator $\Gamma_{\mathbf{\Lambda}}$ on $L^2(Q;\mathbb{R}^2)$. The action of $\Gamma_{\mathbf{\Lambda}}$ on an arbitrary function $\boldsymbol{f} \in L^2(Q;\mathbb{R}^2)$ is defined by

$$\Gamma_{\mathbf{\Lambda}}\boldsymbol{f} = \nabla\psi,$$

where $\psi(\boldsymbol{x})$ is the unique (up to an additive constant) Q-periodic solution of

$$-\nabla\cdot\mathbf{\Lambda}(\boldsymbol{x})\nabla\psi = \nabla\cdot\boldsymbol{f}(\boldsymbol{x}).$$

We may also extend the definition of $\Gamma_{\mathbf{\Lambda}}$ from $L^2(Q;\mathbb{R}^2)$ to $L^2(Q;\mathrm{End}(\mathbb{R}^2))$. The action of $\Gamma_{\mathbf{\Lambda}}$ on an arbitrary function $\boldsymbol{F} \in L^2(Q;\mathrm{End}(\mathbb{R}^2))$ is defined by

$$(\Gamma_{\mathbf{\Lambda}}\boldsymbol{F})\boldsymbol{\xi} = \Gamma_{\mathbf{\Lambda}}(\boldsymbol{F}\boldsymbol{\xi})$$

for any $\boldsymbol{\xi} \in \mathbb{R}^2$.

If $\phi(\boldsymbol{x})$ solves (1.3) then it depends linearly on $\widehat{\boldsymbol{\xi}}$, and therefore,

$$\nabla\phi = (\Gamma_{\mathbf{\Lambda}}\mathbf{\Lambda})\boldsymbol{\xi} + \xi_3(\Gamma_{\mathbf{\Lambda}}\boldsymbol{p}). \tag{1.4}$$

In general, the effective tensor $\boldsymbol{L}^*$ is defined by its action on an arbitrary vector $\widehat{\boldsymbol{\xi}}$ by

$$\boldsymbol{L}^*\widehat{\boldsymbol{\xi}} = \int_{\widehat{Q}} \boldsymbol{L}(\mathrm{Grad}(\phi) + \widehat{\boldsymbol{\xi}})dx_1dx_2dx_3. \tag{1.5}$$

Recall that neither $\boldsymbol{L}$ nor ϕ depend on x_3. Therefore, we get

$$\boldsymbol{L}^*\widehat{\boldsymbol{\xi}} = \begin{bmatrix} \mathbf{\Lambda}^*\boldsymbol{\xi} + \xi_3\boldsymbol{p}^* \\ (\boldsymbol{q}^*,\boldsymbol{\xi}) + \xi_3\alpha^* \end{bmatrix} = \begin{bmatrix} \langle\mathbf{\Lambda}(\boldsymbol{x})\nabla\phi\rangle + \langle\mathbf{\Lambda}\rangle\boldsymbol{\xi} + \xi_3\langle\boldsymbol{p}\rangle \\ \langle(\boldsymbol{q}(\boldsymbol{x}),\nabla\phi)\rangle + (\langle\boldsymbol{q}\rangle,\boldsymbol{\xi}) + \xi_3\langle\alpha\rangle \end{bmatrix}, \tag{1.6}$$

where $\langle\cdot\rangle$ denote the average over the period cell Q. Here and elsewhere in the paper we use the inner product notation $(\cdot,\cdot)$ to denote the dot product of two vectors or matrices. Substituting (1.4) into (1.6) and equating coefficients at $\boldsymbol{\xi}$ and ξ_3, we obtain

$$\begin{aligned} \mathbf{\Lambda}^* &= \langle\mathbf{\Lambda}\rangle + \langle\mathbf{\Lambda}\Gamma_{\mathbf{\Lambda}}\mathbf{\Lambda}\rangle, \qquad & \boldsymbol{p}^* &= \langle\boldsymbol{p}\rangle + \langle\mathbf{\Lambda}\Gamma_{\mathbf{\Lambda}}\boldsymbol{p}\rangle, \\ \alpha^* &= \langle\alpha\rangle + \langle(\Gamma_{\mathbf{\Lambda}}\boldsymbol{p},\boldsymbol{q})\rangle, \qquad & \boldsymbol{q}^* &= \langle\boldsymbol{q}\rangle + \langle\mathbf{\Lambda}^T\Gamma_{\mathbf{\Lambda}^T}\boldsymbol{q}\rangle. \end{aligned} \tag{1.7}$$

Suppose that we have made a composite with two materials with conductivity tensors $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$. Suppose also that we know the volume fraction of each material, i.e. we know the average $\langle\boldsymbol{L}\rangle$. If nothing else is known about the microstructure of the composite, what can we tell about the possible values of the effective conductivity? This problem is called the G-closure problem [18, 25]. Generically, the supplied information about the composite constrains the effective tensor to lie in a bounded region of the tensor space described by a set of inequalities. Occasionally, when the conductivity tensors of constituents satisfy a special relation, or possess a special symmetry, the effective tensor has to satisfy an equation that is *independent of the microstructure*. In this case we speak of *exact relations*.

Another situation covered by our theory is the relations between effective tensors of two composites that are made with different materials but share the same microstructure. If there exists a function $\mathbf{\Phi}$ mapping 3×3 positive-definite matrices into 3×3 positive-definite matrices such that $(\mathbf{\Phi}(\boldsymbol{L}(\boldsymbol{x})))^* = \mathbf{\Phi}(\boldsymbol{L}^*)$ then we say that the map $\mathbf{\Phi}$ describes a *link*.[2]

In the next section we list a non-redundant set of all links and exact relations for fiber-reinforced composites with Hall effect. Even though our list is not particularly long, the total number of ways in which various facts on the list can be combined to produce new exact relations and links is staggering. Most (but not all) facts on our list could be easily proved directly from equations (1.7). This is not the point. It is the reverse operation—generating a complete non-redundant set of exact relations and links in the present context that is possible only through the application of the general theory of exact relations. In this connection we must mention the work of Bergman and Strelniker, who extended the ideas of duality of Dykhne [9], Keller [17], Mendelson [19] and Milton [20] and obtained new links and new exact reactions in the context of two-component composites [4, 3, 5, 23, 24] and three-component composites [6] (see also [1]).

2 Exact relations and links

The first three items below are links (first two are infinite families of links) of the form $\boldsymbol{L}' = \mathbf{\Phi}(\boldsymbol{L})$ described above. The maps $\mathbf{\Phi}$ corresponding to these links are given explicitly.

1. Let $\boldsymbol{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{\Psi}(\mathbf{\Lambda}) = \dfrac{(\mathbf{\Lambda} - r_0\boldsymbol{S})^T}{\det(\mathbf{\Lambda} - r_0\boldsymbol{S})}$. Then the link is

$$\begin{aligned} \mathbf{\Lambda}' &= \tau_0\mathbf{\Psi}(\mathbf{\Lambda}) - r_0'\boldsymbol{S}, \\ \boldsymbol{p}' &= \boldsymbol{p}_0' + \mu_0\mathbf{\Psi}(\mathbf{\Lambda})(\boldsymbol{p} - \boldsymbol{p}_0)^\perp, \\ \boldsymbol{q}' &= \boldsymbol{q}_0' + \nu_0\mathbf{\Psi}(\mathbf{\Lambda})^T(\boldsymbol{q} - \boldsymbol{q}_0)^\perp, \\ \alpha' &= \alpha_0 + \tfrac{\mu_0\nu_0}{\tau_0}\left\{(\mathbf{\Psi}(\mathbf{\Lambda})(\boldsymbol{p} - \boldsymbol{p}_0)^\perp, (\boldsymbol{q} - \boldsymbol{q}_0)^\perp) - \alpha\right\}, \end{aligned} \tag{2.1}$$

where $\boldsymbol{a}^\perp = \boldsymbol{S}\boldsymbol{a} = (-a_2, a_1)$. Here r_0, r_0', τ_0, μ_0, ν_0, α_0 are arbitrary constants and $\boldsymbol{p}_0$, $\boldsymbol{q}_0$, $\boldsymbol{p}_0'$, $\boldsymbol{q}_0'$ are arbitrary vectors in $\mathbb{R}^2$.

2.

$$\begin{aligned} \mathbf{\Lambda}' &= \tau_0\mathbf{\Lambda} + r_0'\boldsymbol{S}, \\ \boldsymbol{p}' &= \mathbf{\Lambda}\boldsymbol{p}_0 + \mu_0\boldsymbol{p} - \boldsymbol{p}_0', \\ \boldsymbol{q}' &= \mathbf{\Lambda}^T\boldsymbol{q}_0 + \nu_0\boldsymbol{q} - \boldsymbol{q}_0', \\ \alpha' &= \tau_0^{-1}\{\mu_0\nu_0\alpha + \mu_0(\boldsymbol{p}, \boldsymbol{q}_0) + \nu_0(\boldsymbol{q}, \boldsymbol{p}_0) + (\mathbf{\Lambda}\boldsymbol{p}_0, \boldsymbol{q}_0)\} - \alpha_0, \end{aligned} \tag{2.2}$$

The link (2.2) is a limiting case of the link (2.1), when some of the parameters in (2.1) go to infinity.

[2] More generally, a link is a relation between *some* of the components of $(\mathbf{\Phi}(\boldsymbol{L}(\boldsymbol{x})))^*$ and $\mathbf{\Phi}(\boldsymbol{L}^*)$.

3. The effective tensor $\boldsymbol{L}^*$ enjoys the "transpose symmetry", [22, Proposition 2],

$$(\boldsymbol{L}^T)^* = (\boldsymbol{L}^*)^T. \tag{2.3}$$

4. $\boldsymbol{\Lambda}^*$ depends only on $\boldsymbol{\Lambda}(\boldsymbol{x})$ and not on $\boldsymbol{p}$, $\boldsymbol{q}$ or α. It is computed as an effective conductivity of the 2D periodic composite with local conductivity $\boldsymbol{\Lambda}(\boldsymbol{x})$. Also, $\boldsymbol{p}^*$ depends only on $\boldsymbol{\Lambda}(\boldsymbol{x})$ and $\boldsymbol{p}(\boldsymbol{x})$.

5. If the constituents of the fiber-reinforced composite do not exhibit transversal Hall effect, i.e. $\boldsymbol{\Lambda}(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x})$ is symmetric, then

$$\begin{bmatrix} \boldsymbol{\sigma} & \boldsymbol{p}+\boldsymbol{q} \\ \boldsymbol{q} & \alpha \end{bmatrix}^* = \begin{bmatrix} \boldsymbol{\sigma}^* & \boldsymbol{p}^*+\boldsymbol{q}^* \\ \boldsymbol{q}^* & \alpha^* \end{bmatrix}, \tag{2.4}$$

   where

$$\begin{bmatrix} \boldsymbol{\sigma} & \boldsymbol{p} \\ \boldsymbol{q} & \alpha \end{bmatrix}^* = \begin{bmatrix} \boldsymbol{\sigma}^* & \boldsymbol{p}^* \\ \boldsymbol{q}^* & \alpha^* \end{bmatrix}.$$

6. If $\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{p}_0$ then

$$\boldsymbol{p}^* = \boldsymbol{p}_0, \qquad \alpha^* = \langle\alpha\rangle. \tag{2.5}$$

7. If $\boldsymbol{\Lambda}(\boldsymbol{x})\boldsymbol{e}_0 = \boldsymbol{j}_0$ then

$$\boldsymbol{\Lambda}^*\boldsymbol{e}_0 = \boldsymbol{j}_0, \qquad (\boldsymbol{q}^*, \boldsymbol{e}_0) = (\langle\boldsymbol{q}\rangle, \boldsymbol{e}_0). \tag{2.6}$$

8. If $\boldsymbol{\Lambda}(\boldsymbol{x}) = \boldsymbol{\Lambda}_0$ then

$$\boldsymbol{\Lambda}^* = \boldsymbol{\Lambda}_0, \qquad \boldsymbol{p}^* = \langle\boldsymbol{p}\rangle, \qquad \boldsymbol{q}^* = \langle\boldsymbol{q}\rangle. \tag{2.7}$$

The properties (2.1) and (2.2) are not immediately readable off the formulas (1.7), even though (2.2) can be proved by substituting the expressions for $\boldsymbol{\Lambda}'$, $\boldsymbol{p}'$, $\boldsymbol{q}'$ and α' into (1.7). The remaining microstructure-independent relations are easy to prove using formulas (1.7). However, it is not clear how starting with (1.7) one can *generate* all items on our list.

The transpose symmetry (2.3) is a consequence of the easily proved property of the operator $\Gamma_{\boldsymbol{\Lambda}}$:

$$\langle(\Gamma_{\boldsymbol{\Lambda}}\boldsymbol{p}, \boldsymbol{q})\rangle = \langle(\Gamma_{\boldsymbol{\Lambda}^T}\boldsymbol{q}, \boldsymbol{p})\rangle.$$

The properties in item 4 above and (2.4) are clearly readable off (1.7). However, (1.7) contains a bit more microstructure-independent information than (2.4). Namely, it is the linear dependence of $\boldsymbol{p}^*$ on $\boldsymbol{p}$, in the case when $\boldsymbol{\Lambda}(\boldsymbol{x})$ is not symmetric and the dependence of α^* on $\langle\alpha\rangle$ that do not follow from item 4 and (2.4). In order for our theory of exact relations to pick up these properties, we should have considered links between *three* uncoupled problems.

Finally, the properties (2.5) and (2.6) follow from (1.7) and the fact that $\Gamma_{\boldsymbol{\Lambda}}\boldsymbol{p}_0 = \boldsymbol{0}$, while The property (2.7) follows from (1.7) and the relation $\langle\Gamma_{\boldsymbol{\Lambda}}\boldsymbol{p}\rangle = 0$.

3 Example: a polycrystal

Suppose that the ohmic conductivity of a single crystal is $\boldsymbol{C}_0$ and the Hall tensor, defined in (1.1) is $\boldsymbol{R}_0$. If the magnetic field $\boldsymbol{h}$ is weak, it is reasonable to assume, as we argued in the Introduction, that tensors $\boldsymbol{C}_0$ and $\boldsymbol{R}_0$ are independent of $\boldsymbol{h}$. Then, if we rotate the crystal, but not the magnetic field, by a rotation $\widehat{\boldsymbol{Q}} \in SO(3)$, the ohmic conductivity of the crystal will be $\widehat{\boldsymbol{Q}}\boldsymbol{C}_0\widehat{\boldsymbol{Q}}^T$ and its Hall tensor $\widehat{\boldsymbol{Q}}\boldsymbol{R}_0\widehat{\boldsymbol{Q}}^T$. Assume that the polycrystalline texture $\widehat{\boldsymbol{Q}}(\boldsymbol{x})$ has cylindrical symmetry. In other words

$$\widehat{\boldsymbol{Q}}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{Q}(\boldsymbol{x}) & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix},$$

where $\boldsymbol{Q}(\boldsymbol{x}) \in SO(2)$ for each $\boldsymbol{x} \in Q$. For general $\boldsymbol{h}$ the formula (1.1) gives a complicated expression for the local Hall conductivity $\boldsymbol{L}(\boldsymbol{x})$. However, if the magnetic field $\boldsymbol{h}$ is directed along the fibers, the formula for $\boldsymbol{L}(\boldsymbol{x})$ simplifies and we obtain $\boldsymbol{L}(\boldsymbol{x}) = \widehat{\boldsymbol{Q}}(\boldsymbol{x})\boldsymbol{L}_0\widehat{\boldsymbol{Q}}(\boldsymbol{x})^T$, where $\boldsymbol{L}_0$ is defined by its action on an arbitrary vector $\boldsymbol{e} \in \mathbb{R}^3$,

$$\boldsymbol{L}_0\boldsymbol{e} = \boldsymbol{C}_0\boldsymbol{e} + (\boldsymbol{R}_0\boldsymbol{h}) \times \boldsymbol{e}.$$

If

$$\boldsymbol{L}_0 = \begin{bmatrix} \boldsymbol{\Lambda}_0 & \boldsymbol{p}_0 \\ \boldsymbol{q}_0 & \alpha_0 \end{bmatrix},$$

then

$$\boldsymbol{L}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{Q}(\boldsymbol{x})\boldsymbol{\Lambda}_0\boldsymbol{Q}(\boldsymbol{x})^T & \boldsymbol{Q}(\boldsymbol{x})\boldsymbol{p}_0 \\ \boldsymbol{Q}(\boldsymbol{x})\boldsymbol{q}_0 & \alpha_0 \end{bmatrix},$$

Let us first apply the link (2.2) to this situation. The goal is to choose parameters in (2.1) such that $\boldsymbol{\Lambda}'(\boldsymbol{x})$ is symmetric. Suppose that the decomposition of $\boldsymbol{\Lambda}_0$ into symmetric and antisymmetric parts is $\boldsymbol{\Lambda}_0 = \boldsymbol{\sigma}_0 + r_0\boldsymbol{S}$ then choosing in (2.2) $r_0' = -r_0$, $\tau_0 = \mu_0 = \nu_0 = 1$ with remaining parameters set to zero we obtain

$$\boldsymbol{L}^* = \begin{bmatrix} \boldsymbol{\sigma}^* + r_0\boldsymbol{S} & \boldsymbol{p}^* \\ \boldsymbol{q}^* & \alpha^* \end{bmatrix},$$

where the blocks above are computed from the homogenization problem

$$\begin{bmatrix} \boldsymbol{\sigma}(\boldsymbol{x}) & \boldsymbol{p}(\boldsymbol{x}) \\ \boldsymbol{q}(\boldsymbol{x}) & \alpha_0 \end{bmatrix}^* = \begin{bmatrix} \boldsymbol{\sigma}^* & \boldsymbol{p}^* \\ \boldsymbol{q}^* & \alpha^* \end{bmatrix}. \tag{3.1}$$

Combining the link in item 4 on page 5 with links (2.2) and (2.4), we conclude that $\boldsymbol{\sigma}^*$ can be computed from 2D homogenization of $\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{Q}(\boldsymbol{x})\boldsymbol{\sigma}_0\boldsymbol{Q}(\boldsymbol{x})^T$, and also that there exists a linear map $\mathbb{P} : \mathbb{R}^2 \to \mathbb{R}^2$, that depends only on $\boldsymbol{\sigma}_0$ and the texture $\boldsymbol{Q}(\boldsymbol{x})$ of the polycrystal, such that $\boldsymbol{p}^* = \mathbb{P}\boldsymbol{p}_0$. The transpose symmetry (2.3) implies that $\boldsymbol{q}^* = \mathbb{P}\boldsymbol{q}_0$. Finally, the third equation in (1.7) implies that there exists a bilinear form $\mathbb{B}$ depending only on $\boldsymbol{\sigma}_0$ and $\boldsymbol{Q}(\boldsymbol{x})$, such that $\alpha^* = \alpha_0 + (\mathbb{B}\boldsymbol{p}_0, \boldsymbol{q}_0)$. The transpose symmetry (2.3) implies that the matrix $\mathbb{B}$ must be symmetric. It follows that without loss of generality we may study the homogenization of

$$\boldsymbol{\Sigma}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{\sigma}(\boldsymbol{x}) & \boldsymbol{p}(\boldsymbol{x}) \\ \boldsymbol{p}(\boldsymbol{x}) & \alpha_0 \end{bmatrix}.$$

Observe that the link (2.1) with $\tau_0 = \mu_0 = \nu_0 = \det \sigma_0$ and all other constants set to zero transforms $\boldsymbol{\Sigma}(\boldsymbol{x})$ into $\boldsymbol{\Sigma}'(\boldsymbol{x})$ of the same form, where $\boldsymbol{p}_0$ is changed into $\boldsymbol{\sigma}_0\boldsymbol{p}_0^\perp$ and α_0 is changed into

$$\alpha' = (\boldsymbol{\sigma}_0\boldsymbol{p}_0^\perp, \boldsymbol{q}_0^\perp) - \alpha_0 \det \boldsymbol{\sigma}_0.$$

Combining the link (2.1) with the representation

$$\boldsymbol{\Sigma}^* = \begin{bmatrix} \boldsymbol{\sigma}^* & \mathbb{P}\boldsymbol{p}_0 \\ \mathbb{P}\boldsymbol{p}_0 & \alpha_0 + (\mathbb{B}\boldsymbol{p}_0, \boldsymbol{p}_0) \end{bmatrix}$$

we conclude after a bit of manipulation that

$$(\mathbb{B}, \boldsymbol{\sigma}_0) = \det \mathbb{P} - 1, \qquad \boldsymbol{\sigma}^*\mathrm{cof}(\mathbb{P}) = \mathbb{P}\boldsymbol{\sigma}_0, \tag{3.2}$$

where $\mathrm{cof}(\mathbb{P})$ is the cofactor matrix for $\mathbb{P}$. The second equation in (3.2) has a general solution of the form

$$\mathbb{P} = (\boldsymbol{\sigma}^*)^{1/2}\varphi(z)\boldsymbol{\sigma}_0^{-1/2}, \quad z \in \mathbb{C},$$

where $\varphi : \mathbb{C} \to \mathrm{End}(\mathbb{R}^2)$ is given by

$$\varphi(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad z = a + ib. \tag{3.3}$$

Assume now that additionally the polycrystalline texture is statistically isotropic in the transversal plane. In that case $\boldsymbol{\sigma}^*$ must be isotropic. Hence, $\boldsymbol{\sigma}^* = \sqrt{\det \boldsymbol{\sigma}_0}\boldsymbol{I}$. Also, if we rotate each crystallite by a fixed in-plane rotation $\boldsymbol{Q}_0$, it should not change the effective tensor of the composite because the texture $\boldsymbol{Q}_0\boldsymbol{Q}(\boldsymbol{x})$ is also statistically isotropic in the transversal plane. Hence $\mathbb{P}\boldsymbol{Q}_0 = \mathbb{P}$ and $\boldsymbol{Q}_0^T\mathbb{B}\boldsymbol{Q}_0 = \mathbb{B}$ for every $\boldsymbol{Q}_0 \in SO(2)$. Therefore, $\mathbb{P} = \boldsymbol{0}$ and, according to (3.2),

$$\mathbb{B} = -\frac{\boldsymbol{I}}{\mathrm{Tr}\,\boldsymbol{\sigma}_0}.$$

Hence

$$\boldsymbol{L}^* = \begin{bmatrix} \sqrt{\det \boldsymbol{\sigma}_0}\boldsymbol{I} + r_0\boldsymbol{S} & \boldsymbol{0} \\ \boldsymbol{0} & \alpha_0 - \dfrac{(\boldsymbol{p}_0, \boldsymbol{q}_0)}{\mathrm{Tr}\,\boldsymbol{\sigma}_0} \end{bmatrix}.$$

4 Example: a two phase composite

Let us apply our exact relations to a two-phase fiber-reinforced composite. Let

$$\boldsymbol{L}_j = \begin{bmatrix} \boldsymbol{\sigma}_j + r_j \boldsymbol{S} & \boldsymbol{p}_j \\ \boldsymbol{q}_j & \alpha_j \end{bmatrix}, \quad j = 1, 2.$$

Recall that the upper left 2×2 block of $\boldsymbol{L}^*$ is determined by the 2D homogenization problem. Now, we may use the 2D Hall effect link of Milton [20] (who was enlarging upon the work of Dykhne [8]) $\boldsymbol{\Lambda}' = \tau_0 \boldsymbol{\Psi}(\boldsymbol{\Lambda}) - r_0' \boldsymbol{S}$, where $\boldsymbol{\Psi}(\boldsymbol{\Lambda})$ is defined in (2.1). Separating symmetric and antisymmetric parts we get

$$\boldsymbol{\sigma}' = \tau_0 \frac{\boldsymbol{\sigma}}{(r_0 - r)^2 + \det \boldsymbol{\sigma}}, \quad r' = \tau_0 \frac{r_0 - r}{(r_0 - r)^2 + \det \boldsymbol{\sigma}} - r_0'. \tag{4.1}$$

Formula (4.1) with $r_0 = r_0' = 0$ has been derived much earlier by Mendelson [19]. We now use (4.1) to reduce the 2D Hall effect homogenization problem to standard conductivity. We need to choose constants τ_0, r_0 and r_0' such that $r' = 0$ when $r = r_1$ *and* $r = r_2$. We see that without loss of generality, we may choose $\tau_0 = 1$, while

$$\frac{r_0 - r_1}{(r_0 - r_1)^2 + \det \boldsymbol{\sigma}_1} = \frac{r_0 - r_2}{(r_0 - r_2)^2 + \det \boldsymbol{\sigma}_2} = r_0'. \tag{4.2}$$

We easily see that (4.2) reduces to a quadratic equation for r_0 that has two distinct real roots, unless $r_1 = r_2$. We may select either root, since the results are independent of the choice.[3] Then Milton's link (4.1) maps a composite made with conductors $\boldsymbol{\Lambda}_j = \boldsymbol{\sigma}_j + r_j \boldsymbol{S}$ to a composite made with conductors

$$\boldsymbol{\sigma}_j' = \frac{\boldsymbol{\sigma}_j}{(r_0 - r_j)^2 + \det \boldsymbol{\sigma}_j}, \qquad j = 1, 2.$$

Let $\boldsymbol{\Sigma}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ denote the effective conductivity function of the 2D composite made with materials $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$. This function depends on the microstructure of the composite. Let $\boldsymbol{\Sigma}^* = \boldsymbol{\Sigma}(\boldsymbol{\sigma}_1', \boldsymbol{\sigma}_2')$. The upper left 2×2 block $\boldsymbol{\Lambda}^* = \boldsymbol{\sigma}^* + r^* \boldsymbol{S}$ of the effective tensor $\boldsymbol{L}^*$ of the original 3D composite is then determined from the relations

$$\frac{\boldsymbol{\sigma}^*}{(r_0 - r^*)^2 + \det \boldsymbol{\sigma}^*} = \boldsymbol{\Sigma}^*, \qquad \frac{r_0 - r^*}{(r_0 - r^*)^2 + \det \boldsymbol{\sigma}^*} = r_0'.$$

We find that

$$\boldsymbol{\sigma}^* = \frac{\boldsymbol{\Sigma}^*}{(r_0')^2 + \det \boldsymbol{\Sigma}^*}, \qquad r^* = r_0 - \frac{r_0'}{(r_0')^2 + \det \boldsymbol{\Sigma}^*}. \tag{4.3}$$

Our next step is to determine $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$. The idea is to use the link (2.2) to map our original composite to the one with $\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{0}$ and $\boldsymbol{q}(\boldsymbol{x}) = \boldsymbol{0}$. Let us first assume that $\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2$

[3] It is not easy to see, but ultimately, the independence of the choice of root in (4.2) boils down to Mendelson's link [19], or (4.1).

is an invertible matrix. In (2.2) we choose $\tau_0 = 1$, $r'_0 = 0$, $\mu_0 = \nu_0 = 1$ and $\boldsymbol{p}_0$ and $\boldsymbol{q}_0$ such that

$$\boldsymbol{\Lambda}_1\boldsymbol{p}_0 + \boldsymbol{p}_1 = \boldsymbol{\Lambda}_2\boldsymbol{p}_0 + \boldsymbol{p}_2 = \boldsymbol{p}'_0, \qquad \boldsymbol{\Lambda}_1^T\boldsymbol{q}_0 + \boldsymbol{q}_1 = \boldsymbol{\Lambda}_2^T\boldsymbol{q}_0 + \boldsymbol{q}_2 = \boldsymbol{q}'_0.$$

In other words,

$$\boldsymbol{p}_0 = (\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2)^{-1}(\boldsymbol{p}_2 - \boldsymbol{p}_1), \qquad \boldsymbol{q}_0 = (\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2)^{-T}(\boldsymbol{q}_2 - \boldsymbol{q}_1). \tag{4.4}$$

Then

$$\begin{aligned} \boldsymbol{p}^* &= (\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}^*)\boldsymbol{p}_0 + \boldsymbol{p}_1 = (\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}^*)\boldsymbol{p}_0 + \boldsymbol{p}_2, \\ \boldsymbol{q}^* &= (\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}^*)^T\boldsymbol{q}_0 + \boldsymbol{q}_1 = (\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}^*)^T\boldsymbol{q}_0 + \boldsymbol{q}_2, \end{aligned} \tag{4.5}$$

And the first volume fraction relation implies that $(\alpha')^* = \langle\alpha'\rangle$. Therefore,

$$\alpha^* = \langle\alpha\rangle + (\langle\boldsymbol{p}\rangle - \boldsymbol{p}^*, \boldsymbol{q}_0) = \langle\alpha\rangle + (\langle\boldsymbol{q}\rangle - \boldsymbol{q}^*, \boldsymbol{p}_0) = \langle\alpha\rangle - ((\langle\boldsymbol{\Lambda}\rangle - \boldsymbol{\Lambda}^*)\boldsymbol{p}_0, \boldsymbol{q}_0). \tag{4.6}$$

The formulas (4.3)–(4.6) have been derived earlier in a series of papers by Bergman and Strelniker [4, 5, 23] and Bergman, Li and Strelniker [3].

If $\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2$ is singular, then we have two cases. Either $\boldsymbol{\Lambda}_1 = \boldsymbol{\Lambda}_2 = \boldsymbol{\Lambda}_0$ or there exists a unit vector $\boldsymbol{e}_0$ and a non-zero vector $\boldsymbol{d}_0$ such that $\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2 = \boldsymbol{d}_0^\perp \otimes \boldsymbol{e}_0^\perp$, in other words, $\boldsymbol{\Lambda}_1\boldsymbol{e}_0 = \boldsymbol{\Lambda}_2\boldsymbol{e}_0$ (and $\boldsymbol{\Lambda}_1^T\boldsymbol{d}_0 = \boldsymbol{\Lambda}_2^T\boldsymbol{d}_0$). If $\boldsymbol{\Lambda}_1 = \boldsymbol{\Lambda}_2 = \boldsymbol{\Lambda}_0$ then $\boldsymbol{\Lambda}^* = \boldsymbol{\Lambda}_0$ and, according to (2.7), we have $\boldsymbol{p}^* = \langle\boldsymbol{p}\rangle$ and $\boldsymbol{q}^* = \langle\boldsymbol{q}\rangle$. Surprisingly, α^* depends on the microstructure in the essential way (unless of course $\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{p}_0$ or $\boldsymbol{q}(\boldsymbol{x}) = \boldsymbol{q}_0$). It is possible to express α^* in terms of the second derivatives of the 2D effective conductivity function $\boldsymbol{\Sigma}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$:

$$\alpha^* = \langle\alpha\rangle + \frac{1}{2}(\boldsymbol{\sigma}_0^{-1}\boldsymbol{\Sigma}''(1)\boldsymbol{\sigma}_0^{-1}(\boldsymbol{p}_1 - \boldsymbol{p}_2), \boldsymbol{q}_1 - \boldsymbol{q}_2),$$

where $\boldsymbol{\Sigma}(t) = \boldsymbol{\Sigma}^*(\boldsymbol{\sigma}_0, t\boldsymbol{\sigma}_0)$ and $\boldsymbol{\sigma}_0$ is the symmetric part of $\boldsymbol{\Lambda}_0$. We can obtain this formula from (4.3)–(4.6) by considering a composite made with materials $\boldsymbol{L}_1$ and $\boldsymbol{L}_2(t)$, where $\boldsymbol{\Lambda}_2(t) = t\boldsymbol{\Lambda}_0$ and computing a limit, as $t \to 1$ (using that $\boldsymbol{\Sigma}'(1) = \theta_2\boldsymbol{\sigma}_0$).

Consider now the remaining case when $\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2 = \boldsymbol{d}_0^\perp \otimes \boldsymbol{e}_0^\perp$. Then $\boldsymbol{\Lambda}^* - \boldsymbol{\Lambda}_2 = \lambda^*\boldsymbol{d}_0^\perp \otimes \boldsymbol{e}_0^\perp$. Then, according to (2.3) and (2.6), we have

$$(\boldsymbol{p}^*, \boldsymbol{d}_0) = (\langle\boldsymbol{p}\rangle, \boldsymbol{d}_0), \qquad (\boldsymbol{q}^*, \boldsymbol{e}_0) = (\langle\boldsymbol{q}\rangle, \boldsymbol{e}_0).$$

The remaining components of the effective tensor can be expressed only in terms of the derivatives of the function $\boldsymbol{\Sigma}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$, by considering a perturbed composite, where $\boldsymbol{\Lambda}_2$ is replaced with $\boldsymbol{\Lambda}_2 + t\boldsymbol{d}_0 \otimes \boldsymbol{e}_0$ and passing to the limit as $t \to 0$ in the formulas (4.3)–(4.6). The explicit formulas are complicated and we do not list them here.

As an example, consider a periodic array of fibers with volume fraction θ and conductivity $\boldsymbol{L}_f = \sigma_1\boldsymbol{I} + \boldsymbol{B}_f$ embedded in a matrix with conductivity $\boldsymbol{L}_m = \sigma_2\boldsymbol{I} + \boldsymbol{B}_m$, where

$$\boldsymbol{B}_f = \begin{bmatrix} r_1\boldsymbol{S} & -\boldsymbol{a}_1 \\ \boldsymbol{a}_1 & 0 \end{bmatrix}, \qquad \boldsymbol{B}_m = \begin{bmatrix} r_2\boldsymbol{S} & -\boldsymbol{a}_2 \\ \boldsymbol{a}_2 & 0 \end{bmatrix}$$

are 3×3 skew-symmetric matrices of Hall effect-induced conductivity. Assume also that the cross-section of the fibers have the square-symmetric Vigdergauz shape [26, 27, 28] (see also [13]). Then $\boldsymbol{\Sigma}^* = \varsigma(\sigma_1', \sigma_2')\boldsymbol{I}$, where

$$\varsigma(\sigma_1, \sigma_2) = \sigma_2 \frac{\sigma_2 + \sigma_1 + (\sigma_1 - \sigma_2)\theta}{\sigma_2 + \sigma_1 - (\sigma_1 - \sigma_2)\theta} \tag{4.7}$$

and

$$\sigma_j' = \frac{\sigma_j}{(r_0 - r_j)^2 + \sigma_j^2}, \quad j = 1, 2,$$

with r_0—a solution of (4.2).

Then the effective conductivity $\boldsymbol{L}^*$ has the form

$$\boldsymbol{L}^* = \begin{bmatrix} \sigma^* \boldsymbol{I} + r^* \boldsymbol{S} & \boldsymbol{p}^* \\ \boldsymbol{q}^* & \alpha^* \end{bmatrix},$$

where

$$\sigma^* = \sigma_2 + (\sigma_1 - \sigma_2)\mathfrak{Re}\tau + (r_1 - r_2)\mathfrak{Im}\tau, \qquad r^* = r_2 + \frac{r_1 - r_2}{\theta}|\tau|^2,$$

$$\boldsymbol{p}^* = -\boldsymbol{a}_2 + \varphi(\overline{\tau})(\boldsymbol{a}_2 - \boldsymbol{a}_1), \qquad \boldsymbol{q}^* = \boldsymbol{a}_2 - \varphi(\tau)(\boldsymbol{a}_2 - \boldsymbol{a}_1),$$

$$\alpha^* = \theta\sigma_1 + (1 - \theta)\sigma_2 + \frac{1 - \theta}{2\sigma_2}\mathfrak{Re}\tau|\boldsymbol{a}_2 - \boldsymbol{a}_1|^2,$$

$$\tau = \frac{2\theta\sigma_2}{\sigma_2 + \sigma_1 + (\sigma_2 - \sigma_1)\theta + i(1 - \theta)(r_2 - r_1)},$$

where the map φ is given by (3.3). We observe that if σ_j are independent of the magnetic field $\boldsymbol{h}$ and $\boldsymbol{B}_j$ depend on $\boldsymbol{h}$ linearly, the effective ohmic conductivity and effective Hall tensor both depend on $\boldsymbol{h}$ in a non-linear manner. Hence, the assumption that in (1.1) the tensors $\boldsymbol{C}$ and $\boldsymbol{R}$ are independent of $\boldsymbol{h}$ is reasonable only in the weak magnetic field limit.

If the magnetic field is weak, the components of $\boldsymbol{B}_j$, $j = 1, 2$ are small. Hence, neglecting expressions that are quadratic in small quantities we see that the Hall effect does not influence the normal effective conductivity of the composite:

$$\sigma^* = \varsigma(\sigma_1, \sigma_2), \qquad \alpha^* = \theta\sigma_1 + (1 - \theta)\sigma_2.$$

The components of the anti-symmetric part $\boldsymbol{B}^*$ to the effective conductivity are given by

$$r^* = \frac{\tau_0^2}{\theta} r_1 + \left(1 - \frac{\tau_0^2}{\theta}\right) r_2, \qquad \boldsymbol{a}^* = \tau_0 \boldsymbol{a}_1 + (1 - \tau_0)\boldsymbol{a}_2,$$

where

$$\tau_0 = \frac{\varsigma(\sigma_1, \sigma_2) - \sigma_2}{\sigma_1 - \sigma_2} = \frac{2\theta\sigma_2}{\sigma_2 + \sigma_1 + (\sigma_2 - \sigma_1)\theta}.$$

It is easy to check that $0 \le \tau_0 \le \sqrt{\theta}$ with equalities only at $\theta = 0$ and 1.

5 The general theory

We conclude the paper by a brief description of the general theory of exact relations [11, 14] that was used to obtain the results described in Section 2. General theory tells us that exact relations are in one-to-one correspondence with subspaces $\Pi \subset \mathrm{End}(\mathbb{R}^3)$ such that

$$\boldsymbol{P}_1 *_{\boldsymbol{A}} \boldsymbol{P}_2 = \frac{1}{2}(\boldsymbol{P}_1\boldsymbol{A}\boldsymbol{P}_2 + \boldsymbol{P}_2\boldsymbol{A}\boldsymbol{P}_1) \in \Pi \tag{5.1}$$

for all $\{\boldsymbol{P}_1, \boldsymbol{P}_2\} \subset \Pi$ and all $\boldsymbol{A} \in \mathcal{A}$, where

$$\mathcal{A} = \left\{ \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix} : \boldsymbol{A}^T = \boldsymbol{A},\ \mathrm{Tr}\,\boldsymbol{A} = 0 \right\}. \tag{5.2}$$

The multiplication in (5.1) makes Π into a Jordan algebra. The new twist here is that Π is a Jordan algebra with respect to an infinite family of multiplications parameterized by $\boldsymbol{A} \in \mathcal{A}$.

In addition to the exact relations we have a way of recognizing when the relations involving volume averages are also present. These additional relations appear when the derived Jordan ideal Π^2 defined by

$$\Pi^2 = \mathrm{Span}\{\boldsymbol{P}\boldsymbol{A}\boldsymbol{P} : \boldsymbol{P} \in \Pi,\ \boldsymbol{A} \in \mathcal{A}\} \subset \Pi$$

is strictly smaller than Π. For example, the exact relation $\boldsymbol{p}^* = \boldsymbol{p}_0$ corresponds to

$$\Pi = \left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & \rho \end{bmatrix} : \boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2),\ \boldsymbol{v} \in \mathbb{R}^2,\ \rho \in \mathbb{R} \right\}.$$

$$\Pi^2 = \left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{v} & 0 \end{bmatrix} : \boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2),\ \boldsymbol{v} \in \mathbb{R}^2 \right\} \neq \Pi.$$

Therefore this exact relation admits additional relations involving volume fractions. The number of such relations is equal to the co-dimension of Π^2 in Π. In our example, we should have one additional relation. It is $\alpha^* = \langle \alpha \rangle$, resulting in (2.5).

Now let us turn to the links between exact relations. The links are nothing more than the exact relations for the two uncoupled Hall-conductivity problems. As such, they correspond to subspaces $\widehat{\Pi} \subset V = \mathrm{End}(\mathbb{R}^3) \oplus \mathrm{End}(\mathbb{R}^3)$ that satisfy

$$\widehat{\boldsymbol{P}}\widehat{\boldsymbol{A}}\widehat{\boldsymbol{P}} \in \widehat{\Pi} \tag{5.3}$$

for all $\widehat{\boldsymbol{P}} \in \widehat{\Pi}$ and all $\widehat{\boldsymbol{A}} \in \widehat{\mathcal{A}}$, where $\widehat{\mathcal{A}} = \{[\boldsymbol{A}, \boldsymbol{A}] : \boldsymbol{A} \in \mathcal{A}\}$. We can construct all subspaces $\widehat{\Pi}$ satisfying (5.3) from the list of all solutions Π of (5.1), provided we also understand their Jordan algebra structure. For any solution $\widehat{\Pi}$ of (5.3), let Π_1 and Π_2 be its canonical projections onto the first and the second copy of $\mathrm{End}(\mathbb{R}^3)$ in V, respectively, and let the Jordan ideals $\mathcal{I}_1 \subset \Pi_1$ and $\mathcal{I}_2 \subset \Pi_2$ be the kernels of these canonical projections. The pairs $(\mathcal{I}_1, \Pi_1)$

and $(\mathcal{I}_2, \Pi_2)$ arise from a subspace $\widehat{\Pi}$, if and only if the factor algebras $\Pi_1/\mathcal{I}_1$ and $\Pi_2/\mathcal{I}_2$ are isomorphic. Conversely, if we have found an isomorphism $\Phi : \Pi_1/\mathcal{I}_1 \to \Pi_2/\mathcal{I}_2$, then

$$\widehat{\Pi} = \{[\boldsymbol{P}_1, \boldsymbol{P}_2] \in \Pi_1 \oplus \Pi_2 : [\![\boldsymbol{P}_2]\!] = \Phi([\![\boldsymbol{P}_1]\!])\} \tag{5.4}$$

solves (5.3), where $[\![\boldsymbol{P}_i]\!]$ is the equivalence class in $\Pi_i/\mathcal{I}_i$ containing $\boldsymbol{P}_i$, $i = 1, 2$.

The equivalence classes in $\Pi_i/\mathcal{I}_i$ can be labeled by elements of $\mathcal{I}_i^{\perp}$—the orthogonal complement of $\mathcal{I}_i$ in Π_i, $i = 1, 2$. Then the map Φ can be thought of as an isomorphism between $\mathcal{I}_1^{\perp}$ and $\mathcal{I}_2^{\perp}$.

For example, item 4 on page 5 says in part that $\boldsymbol{p}^*$ depends on $\boldsymbol{\Lambda}(\boldsymbol{x})$ and $\boldsymbol{p}(\boldsymbol{x})$, but not on $\boldsymbol{q}(\boldsymbol{x})$ or $\alpha(\boldsymbol{x})$. This link corresponds to the following subspace

$$\widehat{\Pi} = \left\{ \left[\begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{v} & \rho \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{0} & 0 \end{bmatrix} \right] : \boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2),\ \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2,\ \rho \in \mathbb{R} \right\}$$

Here, $\Pi_1 = \mathrm{End}(\mathbb{R}^3)$, $\mathcal{I}_2 = \{\boldsymbol{0}\}$,

$$\mathcal{I}_1 = \left\{ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{v} & \rho \end{bmatrix} : \boldsymbol{v} \in \mathbb{R}^2,\ \rho \in \mathbb{R} \right\}, \qquad \Pi_2 = \left\{ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{u} \\ \boldsymbol{0} & 0 \end{bmatrix} : \boldsymbol{K} \in \mathrm{End}(\mathbb{R}^2),\ \boldsymbol{u} \in \mathbb{R}^2 \right\}.$$

Observe that $\Pi_2 = \mathcal{I}_1^{\perp}$ and Φ here is the resulting natural isomorphism $\Pi_1/\mathcal{I}_1 \cong \mathcal{I}_1^{\perp} \cong \Pi_2 \cong \Pi_2/\mathcal{I}_2$.

Another example with highly non-trivial Φ corresponds to the links (2.1) and (2.2). Here $\Pi_1 = \Pi_2 = \mathrm{End}(\mathbb{R}^3)$ and $\mathcal{I}_1 = \mathcal{I}_2 = \{\boldsymbol{0}\}$. While,

$$\Phi(\boldsymbol{P}) = \boldsymbol{XPY}, \qquad \boldsymbol{X} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ a_1 & a_2 & a_3 \end{bmatrix}, \qquad \boldsymbol{Y} = \begin{bmatrix} \cos\theta & -\sin\theta & b_1 \\ \sin\theta & \cos\theta & b_2 \\ 0 & 0 & b_3 \end{bmatrix},$$

where θ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ are parameters.

In the Summer of 2002 the first group of 5 REU undergraduates (see footnote 1 on page 1) have found *all* solutions of (5.1). These solutions came in 22 infinite families, some of them complicated. In the Summer of 2003 the next group of 4 students translated most solutions into exact relations. They have also computed additional volume fraction relations, corresponding to non-trivial derived ideals, as explained above. At the same time they also made further progress on understanding the algebraic structure of solutions. In the Summer of 2004 the final group of 5 students have identified all the ideals and pairs of isomorphic factor-algebras, in the process bringing in the complete understanding of the algebraic structure of solutions. This new understanding permitted the author to simplify and streamline many tedious steps of the analysis of the previous years. It has also made possible the completion of the analysis: translation of the subspaces $\widehat{\Pi}$ into the physical language of links and removal of the redundant information. The complete description of all calculations is submitted for publication elsewhere and is available as a preprint on the author's web page.

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