My research interest is **Noncommutative Algebra**, which has motivating contexts in noncommutative geometry, number theory, combinatorics, and mathematical physics. Below is the mindmap of my projects.

**Hopf Algebras** were born in earlier 1950s with two honorable parents–algebraic topology and algebraic geometry. Later until 1980s, it had enjoyed a blooming period where a deluge of interactions with knot theory and topology, conformal field theory, ring theory, category theory, combinatorics, etc. appeared. I learned about Hopf algebra by reading Montgomery’s legendary book “Hopf algebras and their actions on rings”. My current interests are Frobenius-Schur indicators and cohomology rings of Hopf algebras.

**Poisson Algebras** have lately been playing an important role in algebra, geometry, mathematical physics and other subjects. For example, Poisson structures can be obtained through the semiclassical limits of quantized coordinate rings studied by Goodearl [9] and can be used in the study of noncommutative discriminants in the work of Nguyen, Trampel and Yakimov [18]. My interest of Poisson algebras involves applying Poisson geometry to study the representation and cohomology theories of noncommutative algebras.

**Noncommutative Invariant Theory** has been among the most fast growing and influential topics concerning Hopf algebras and their (co)actions on rings. It has deep relations with Calabi-Yau manifold, Chevalley-Shephard-Todd theorem, Auslander theorem, McKay quiver, Dynkin diagram and etc. The main mission is to broaden the classical picture of actions of finite groups on polynomial rings by its quantum counterpart of (co)actions of (semisimple) Hopf algebras on regular graded algebras. My work is to study various properties of generalized quantum symmetry groups associated to Artin-Schelter regular algebras.

**Noncommutative Projective Algebraic Geometry** was initiated by Artin and Schelter [1] in a program of classifying some graded algebras of dimension 3, among which the family of 3 dimensional Sklyanin algebras is the most difficult one to deal with. I study the representation theory of PI Sklyanin algebras of dimension 3 and 4 through Poisson geometry using the symplectic foliation on the maximal spectrum of the center of these PI algebras, which has application in String Theory, namely to the understanding of marginal supersymmetric deformations of the $N = 4$ super-Yang-Mills theory in 4 dimensions [2].
1 Projects in Hopf Algebras

1.1 Classification of pointed Hopf algebras in characteristic $p > 0$

The classification program of finite dimensional Hopf algebras has been going on for almost two decades and most results are regarding certain dimensions over the complex numbers. On the contrary, structural information remains elusive for those in positive characteristic. My first paper [31] provided a complete classification of Hopf algebras of the fixed dimension $p^2$. Later, I associated the structure of certain Hopf algebras with invariants in a cohomology theory [34]. By examining all the possible 2-cocycles, I (with Van Nguyen and Linghong Wang) classified all such Hopf algebras of dimension $p^3$ [20] [21]. In a summary, I proved that there are 2 isoclasses in dimension $p$, 8 isoclasses in dimension $p^2$, and 56 isoclasses, 2 finite and 9 infinite parametric families in dimension $p^3$. I discovered a lot of new examples of finite dimensional noncommutative and noncocommutative Hopf algebras that are of special interests to expects working on cohomology and representation theories.

1.2 General structure theorems

I have made several original contributions to the general structure theorems of finite dimensional Hopf algebras in positive characteristic. One of my contributions is to prove that every semisimple irreducible Hopf algebra is the vector space dual of a $p$-group algebra [32] (first proved by Masuoka by a different method). Consequently, I showed that the problem of classifying all semisimple irreducible Hopf algebras in characteristic $p > 0$ is equivalent to the classification problem for all $p$-groups in group theory. Another of my results asserts the sufficient and necessary conditions for a finite-dimensional cocommutative Hopf algebra to be a local algebra, which can be thought as the counterpart for the famous Milnor-Moore-Cartier-Kostant’s theorem regarding cocommutative Hopf algebras in positive characteristic [33].

1.3 Indicators and cohomology rings

My current interests of Hopf algebras are focusing on Frobenius-Schur indicators [11] [29] and cohomology rings of Hopf algebras [19] [10]. I plan to study Frobenius-Schur indicators for $p$-adic Lie groups in the analytic setting seeking for a particular application in the representation theory of Iwasawa algebras. Another future project of mine is to study the cohomology rings of finite dimensional Hopf algebras in positive characteristic. I am very interested in the finite generation problem of these cohomology rings and I have been working on these cohomology rings using Anick resolution and May spectral sequence. I aim at generalizing the famous result of Friedlander and Suslin [8] about the finite generation of the cohomology ring of any finite group scheme over a field of positive characteristic to a noncocommutative Hopf algebra in positive characteristic.

2 Projects in Poisson Algebras

2.1 Poisson Hopf algebras

Poisson Hopf algebras were introduced by Drinfel’d in 1985 and they have been intensively studied in connection with homological algebra and deformation quantization. By exploring its relation with Lie-Rinehart algebra, I proved a PBW basis theorem for the universal enveloping algebra of a Poisson algebra in terms of its Kähler differentials. Regarding the extended bialgebra structure on the universal enveloping algebra, I further showed that its coradical coincides with the coradical of the Poisson Hopf algebra. Consequently, the universal enveloping algebra is a Hopf algebra as long as the original one is pointed, which corrects Oh’s
The notion of Poisson-Ore extension of a Poisson algebra can be thought as the Poisson version of the Ore extension of an associative algebra via the semiclassical limits. By using its universal property established by Oh, I succeeded to provide a complete answer to the following question: What happens to the universal enveloping algebras regarding Poisson-Ore extensions? I proved that the universal enveloping algebra of a Poisson-Ore extension is a length two iterated Ore extensions of that of the original Poisson algebra [15]. Since many ring-theoretic and homological properties are preserved under Ore extensions, consequently I showed that the universal enveloping algebras of those iterated Poisson-Ore extensions, which are obtained from the semiclassical limits of quantized coordinate rings studied by Goodearl, Letzter and Launois, all possess nice properties similar to the polynomial algebras. Recently, I have also generalized this result to the settings of differential graded Poisson algebras [16] and double Poisson-Ore extensions [12].

2.3 Unimodularity versus Calabi-Yau

The modular class of a smooth real Poisson manifold was first introduced by Weinstein in [35]. Later, Xu [36] proved that any Poisson manifold of zero modular class (called unimodular) satisfies Poincaré duality between Poisson homology and cohomology. On the other side, in the algebraic world, Poincaré duality between Hochschild homology and cohomology was observed by Van den Bergh for some Gorenstein rings, among which are the Calabi-Yau algebras defined by Ginzburg in the inspiration of Calabi-Yau manifold. My contribution is to discover a reformulation of the important unimodularity condition for Poisson algebras/manifolds in pure algebraic terms involving their Poisson Picard groups. This is a key result since the original unimodularity notion plays a very important role but has a different, differential-geometric flavor. Then by using D-module theory, I constructed an explicit formula for the rigid dualizing complex of the universal enveloping algebra of the coordinate ring of every smooth affine Poisson variety. Based on the theory developed by Yekutieli and Van den Bergh, I successfully established the equivalence between the unimodularity condition for the affine Poisson variety and the Calabi-Yau condition for the universal enveloping algebra of its coordinate ring. My paper is published in Letters in Mathematical Physics [17] and I have no doubts that it will play a crucial role in future studies of Poisson algebras.

3 Projects in Noncommutative Invariant Theory

3.1 Quantum symmetry groups of Artin-Schelter regular algebras

I (with Chelsea Walton) established that quantum symmetry groups of AS-regular algebras of dimension 2 enjoy many nice ring-theoretic and homological attributes of the algebras being universally coacted upon. Moreover, I gave a partial answer to one of the major problems in Noncommutative Invariant Theory asking when Hopf (co)actions on an associative algebra always factor through (co)commutative ones—when it happens we usually say that the algebra has quantum symmetry. By using A-infinity Koszul duality, I was able to provide a criterion for quantum symmetry of graded Calabi-Yau algebras in terms of their Nakayama automorphisms regarding graded Hopf coactions. The paper is published in Mathematische Zeitschrift [25]. Continued in a joint work with Alexandru Chirvasitu and Chelsea Walton, I further showed that the quantum symmetry groups of N-Koszul AS-regular algebras of arbitrary dimension all share an
elegant finite presentation. I used the fact that these AS-regular algebras are superpotential algebras and I was able to give the presentations of their quantum symmetry groups in terms of the superpotentials. This work was accepted by Journal of Noncommutative Geometry [7]. It provides a complete description of all quantum symmetry groups of $N$-Koszul AS-regular algebras with explicit generators and relations.

### 3.2 Homological properties of Hopf algebras

My work (with Xianlan Yu and Yinhuo Zhang) focused on finding homological properties that are invariant under comodule tensor equivalence. By using the Yetter-Drinfeld category as a bridge between the algebra and coalgebra categories, I obtained a fundamental theorem saying that Calabi-Yau property is among such invariants. For applications, first I gave an affirmative answer to Julien Bichon's question [3] about whether two Hopf algebras have the same global dimension if they are comodule tensor equivalent in the case of Calabi-Yau Hopf algebras. Next I generalized Radford's famous $S^4$ formula for finite dimensional Hopf algebras to infinite dimensional ones including those noetherian AS-Gorenstein Hopf algebras did by Brown and Zhang earlier [6]. At last, I concluded in [13] that, for noetherian Hopf algebras, Brown's conjecture [4] about AS-Gorensteinness implies Skryabin's conjecture [23] about bijectivity of antipodes. My results are published in Pacific Journal of Mathematics [30], which is definitely among one of the very few pioneer works that help one to understand the homological behaviors of a Hopf algebra directly from its comodule representation category.

### 4 Projects in Noncommutative Projective Algebraic Geometry

#### 4.1 Irreducible representations of 3 dimensional PI Sklyanin algebras

My joint work with Chelsea Walton and Milen Yakimov [26] gave a complete description of all irreducible representations of 3 dimensional PI Sklyanin algebras. Previous research by Walton [24] was only able to give an estimation on the lower and upper bound of these dimensions. Instead of relying on the traditional algebraic methods in noncommutative algebra which unfortunately has been proved to be ineffective in this case, we took advantage of powerful results in Poisson geometry and successfully transformed the original problem into a geometric one about accounting for the numbers of different symplectic leaves occurring in the foliation of a complex Poisson manifold which we were able to construct via “high level specializations” introduced in [26]. This is the first time that Poisson geometry has been interacted with noncommutative algebras which do not possess a PBW basis to incredibly produce fruitful results. Our results indicated for the first time that 3 dimensional PI Sklyanin algebras possess exactly 2 or 3 non-isomorphic classes of central quotient algebras depending on whether their PI degrees are coprime to 3 or not. Consequently, we proved that there are no nontrivial irreducible representations of intermediate dimensions, that is they either have the maximal dimension which equals the PI degree by a general result of Brown and Goodearl [5] or they have dimension equal to the lower bound attained in Walton’s earlier research.

### 5 Undergraduate Research Projects

Starting from 2015, I have been enthusiastically participated in the Undergraduate Student Research Program at Temple University. The goal of the program is to initiate and engage junior and senior math majors in research activities, where students explore the frontier of mathematics and attempt research projects in a supportive environment largely devoted to group work. During 2016-2017, I co-supervised (with Linhong
Wang at the University of Pittsburg) for the undergraduate research project “Hopf algebras and Frobenius-Schur higher indicators” together with two math majored students Hao Hu and Xinyi Hu. During the project, we prepared notes for each weekly meeting, and spent the most of each meeting introducing new topics to the students. There is always a discussion at each meeting which answers students’ questions and have them think about the questions we proposed, as well as having students to present their answers on the board. Our paper was accepted by *Involve. A Journal of Mathematics* [11] and was presented by Hao Hu at MAA Allegheny Mountain Section’s Spring Meeting at Duquesne University April 2017.

References


