Connected Hopf algebras in positive characteristic

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Outlines

Part I. Background and basic definitions.

Part II. Questions and conjectures.

Part III. Classification of low dim’l connected (pointed) Hopf algebras in positive characteristic.

Part IV Further projects.
Part I

- History.
- Basic definitions and notation.
- Examples.
The history of Hopf algebras.

- 1953, Borel introduced the expression Hopf algebra, honoring the foundational work of Heinz Hopf in algebraic topology.
- 1965, Milnor and Moore published their paper *On the structure of Hopf algebras*.
- 1985, Drinfeld and Jimbo constructed quantum groups $U_q(\mathfrak{g})$.

Nowadays, the progresses obtained in understanding the structure of Hopf algebras and its representations have entwined with different areas of mathematics: knot theory, topology, conformal field theory, ring theory, category theory, combinatorics and etc.
Definitions and Notation

Throughout, we work over a base field $k$. An algebra is a vector space $A$ together with two linear maps

- multiplication $m : A \otimes A \to A$ and
- unit $u : k \to A$,

such that the following diagrams commute:

a) associativity

\[
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{m \otimes id} & A \otimes A \\
\downarrow id \otimes m & & \downarrow m \\
A \otimes A & \xrightarrow{m} & A
\end{array}
\]

b) unit

\[
\begin{array}{ccc}
k \otimes A & \xrightarrow{u \otimes id} & A \otimes A \\
\downarrow id \otimes u & & \downarrow m \\
A \otimes k & \xleftarrow{id \otimes u} & A
\end{array}
\]
A coalgebra is a vector space $C$ together with two linear maps
- comultiplication $\Delta : C \to C \otimes C$ and
- counit $\epsilon : C \to k$,

such that the following diagram commute:

a) coassociativity

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\downarrow{\Delta} & & \downarrow{\Delta \otimes id} \\
C \otimes C & \xrightarrow{id \otimes \Delta} & C \otimes C \otimes C
\end{array}
\]

b) counit

\[
\begin{array}{ccc}
k \otimes C & \xleftarrow{\epsilon \otimes id} & C \otimes C \\
\uparrow{1 \otimes \Delta} & & \uparrow{\Delta} \\
C & \xrightarrow{id \otimes \epsilon} & C \otimes k
\end{array}
\]

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\downarrow{\Delta} & & \downarrow{\Delta \otimes id} \\
C \otimes C & \xrightarrow{id \otimes \Delta} & C \otimes C \otimes C
\end{array}
\]

\[
\begin{array}{ccc}
k \otimes C & \xleftarrow{\epsilon \otimes id} & C \otimes C \\
\uparrow{1 \otimes \Delta} & & \uparrow{\Delta} \\
C & \xrightarrow{id \otimes \epsilon} & C \otimes k
\end{array}
\]
 Definitions and Notation

A vector space $B$ is a bialgebra if $(B, m, u)$ is an algebra, $(B, \Delta, \epsilon)$ is a coalgebra, and either of the following (equivalent) conditions holds:

- $\Delta$ and $\epsilon$ are algebra morphisms;
- $m$ and $u$ are coalgebra morphisms.
Let \((H, m, u, \Delta, \epsilon)\) be a bialgebra. Then \(H\) is a *Hopf algebra* if there exists an element \(S \in \text{Hom}_k(H, H)\), which satisfies the following commutative diagram:

\[
\begin{array}{ccc}
H \otimes H & \xrightarrow{S \otimes \text{id}} & H \otimes H \\
\Delta & \downarrow & \Delta \\
H & \xrightarrow{\epsilon} & k \\
\Delta & \downarrow & \downarrow \quad m \\
H \otimes H & \xrightarrow{\text{id} \otimes S} & H \otimes H \\
\end{array}
\]

We call \(S\) an *antipode* for \(H\).
Examples

1. **Group algebras.** For any group $G$, the group algebra $kG$ is a Hopf algebra, with $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$ and $S(g) = g^{-1}$ for any $g \in G$.

2. **Enveloping algebras.** Let $\mathfrak{g}$ be a Lie algebra. Then the enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra, with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$ and $S(x) = -x$ for any $x \in \mathfrak{g}$.

3. **Coordinate ring of algebraic groups.** Let $G$ be an algebraic group. Then its coordinate ring $\mathcal{O}(G)$ is a Hopf algebra, with $\epsilon(f) = f(1_G)$, $\Delta(f)$ the function in $\mathcal{O}(G) \otimes \mathcal{O}(G) \cong \mathcal{O}(G \times G)$ defined by $\Delta(f)(x \times y) = f(xy)$ for $x, y \in G$, and $S(f)(x) = f(x^{-1})$ for $x \in G$.

4. **Quantum groups.** We have quantized enveloping algebras $U_q(\mathfrak{g})$ and quantized coordinate rings $\mathcal{O}_q(G)$ with parameter $0 \neq q \in k$ (not a root of unity). They are non-commutative and non-cocommutative noetherian Hopf algebras.
Quantum group $U_q(\mathfrak{sl}_2(k))$

Assume $\text{Char} k = 0$, and $0 \neq q \in k$ is not a root of unity. Recall $\mathfrak{sl}_2(k)$ has $k$-basis $\{e, f, h\}$ with relations

$$[e, f] = h, \ [h, e] = 2e, \ [h, f] = -2f.$$

The quantized enveloping algebra $U_q(\mathfrak{sl}_2(k))$ is defined as follows: $U_q(\mathfrak{sl}_2(k)) = k\langle E, F, K, K^{-1}\rangle$ with relations

$$KE = q^2EK, \ KF = q^{-2}FK, \ EF - FE = \frac{K^2 - K^{-2}}{q^2 - q^{-2}}.$$

The Hopf algebra structure is given by

$$\Delta(E) = E \otimes K^{-1} + K \otimes E, \quad S(E) = -q^{-2}E, \quad \epsilon(E) = 0,$$

$$\Delta(F) = F \otimes K^{-1} + K \otimes F, \quad S(F) = -q^2F, \quad \epsilon(F) = 0,$$

$$\Delta(K) = K \otimes K, \quad S(K) = K^{-1}, \quad \epsilon(K) = 1.$$
Part II.

- Noetherian Hopf algebras.
- AS-Gorenstein rings.
- Kaplansky’s sixth conjecture.
- Cohomology rings of finite dim’l Hopf algebras.
Questions and Conjectures

1. **Noetherian property.** When is a Hopf algebra $H$ noetherian?

   - If $kG$ is (left) Noetherian, is $G$ polycyclic-by-finite? (The inverse statement is proved by Hall in 1959.)
   - (Brown-Small), suppose that $U(g)$ is (left) noetherian, is $\dim g < \infty$?

Example. Witt algebra $W$ is defined to be the Lie algebra $W$ with basis $\{e_n\}_{n \in \mathbb{Z}}$ and Lie bracket $[e_n, e_m] = (m - n)e_{n+m}$.

Recently, Walton-Sierra (2013) showed that the enveloping algebra of the Witt algebra is not Noetherian (Dean-Small 1990).
Questions and Conjectures

2. **AS-Gorenstein.** A Hopf algebra $A$ is *AS-Gorenstein* if its right and left injective dimension equals $d < \infty$, and $\text{Ext}^i_A(k, A) = \delta_{id} k$.

Questions and Conjectures

3. **Representations.** (Kaplansky’s sixth conjecture 1975) Any semisimple Hopf algebra $H$ is of Frobenius type, i.e., suppose $V$ is an irreducible representation of $H$, then $\dim V | \dim H$. Suppose $\text{Char } k = 0$.

- Nichols-Richmond (1998) proved the case when $H$ has a simple module of dimension two.
- Etingof-Gelaki (1998) proved the case when $H$ is semisimple quasitriangular.
Questions and Conjectures

4. **Cohomology rings.** Let $A$ be a finite dim’l Hopf algebra. Define the cohomology ring $H^\bullet(A, k) := \text{Ext}_A(k, k)$. (Etingof-Ostrik 2004) It is conjectured that $H^\bullet(A, k)$ is finitely generated.

- (Friedlander-Suslin 1997): finite group schemes.
- (Ginzburg-Kumar 1993, Bendel-Nakano-Parshall-Pillen 2007): finite dim’l (Lusztig’s) small quantum group $u_q(\mathfrak{g})$ over $\mathbb{C}$.
- (Mastnak-Pevtsova-Schauenburg-Witherspoon 2010): finite dim’l pointed Hopf algebra (under some assumptions) over $\mathbb{C}$. 
Part III

- Definition of connected and pointed Hopf algebras.
- Overview for classification of finite dim’l Hopf algebras.
- Low dim’l connected Hopf algebras.
- Pointed $p^2$-dim’l Hopf algebras in positive characteristic.
- Ideas and techniques.
- Examples of parametric families.
More definitions and notation

Let $H$ be a Hopf algebra.

- The **coradical** $H_0$ of $H$ is the sum of all simple subcoalgebras of $H$.

- For each $n \geq 1$, set $H_n := \Delta^{-1}(H \otimes H_{n-1} + H_0 \otimes H)$. Then the chain of subcoalgebras $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{n-1} \subseteq H_n \cdots$ is the **coradical filtration** of $H$.

- All the group-like elements in $H$ is denoted by $G(H) = \{g \in H | \Delta(g) = g \otimes g\}$, and the primitive space of $H$ is denoted by $P(H) = \{x \in H | \Delta(x) = x \otimes 1 + 1 \otimes x\}$.

- $H$ is **pointed** if every simple subcoalgebra of $H$ is one-dim'l, or spanned by a group-like element. Moreover, $H$ is connected if $H_0$ is one-dim'l.
Classification

When \( k = \mathbb{C} \). Finite dim’l Hopf algebras over \( \mathbb{C} \) have been studied by many researchers.

- Low dim’l Hopf algebras: other than dimension 24, all Hopf algebras over \( \mathbb{C} \) of dimension less than 32 are classified.
- Certain Hopf algebras over \( \mathbb{C} \) of products of distinct primes dimensions \( pq, p^2 q, pqr \) have been classified. In general, Hopf algebras of dimension \( 3p \) are not classified.
- Hopf algebras over \( \mathbb{C} \) of dimensions \( p^n \) for small integers \( n \).
When \( \text{Char} k = p > 0 \).

- Henderson (1995) classified cocommutative connected graded Hopf algebras of dimension up to \( p^3 \).

- Etingof-Gelaki (1998) studied finite dim’l semisimple and cosemisimple Hopf algebras in positive characteristic and showed that if \( p > q \), then any \( q \)-dim’l Hopf algebra is isomorphic to \( kC_q \).

- Scherotzke (2008) classified the finite dim’l pointed rank one Hopf algebras.
We are interested in classification of finite dimensional connected (pointed) Hopf algebras in positive characteristic. Examples are from

- dual of $p$-group algebras;
- finite restricted enveloping algebras;
- finite unipotent group schemes.
Finite-dim’l connected Hopf algebras only appear in positive characteristic. Assume that $k = \bar{k}$ and $\text{Char} k = p > 0$. Let $H$ be a finite-dim’l connected Hopf algebra. It is known that $\dim H = p^n$ for some integer $n \geq 1$.

1. $\dim H = p$. $H$ is isomorphic to $k[x]/(x^p)$, or $k[x]/(x^p - x)$, where $\Delta(x) = x \otimes 1 + 1 \otimes x$. 
Classification in low-dimensional connected Hopf algebras

2. $\dim H = p^2$. (i) Further assume that $\dim P(H) = 2$.

(A1) $k[x, y]/(x^p, y^p)$,

(A2) $k[x, y]/(x^p - x, y^p)$,

(A3) $k[x, y]/(x^p - y, y^p)$,

(A4) $k[x, y]/(x^p - x, y^p - y)$,

(A5) $k\langle x, y \rangle/([x, y] - y, x^p - x, y^p)$, where $x$ and $y$ are primitive.

(ii) Further assume that $\dim P(H) = 1$.

(A6) $k[x, y]/(x^p, y^p)$,

(A7) $k[x, y]/(x^p, y^p - x)$,

(A8) $k[x, y]/(x^p - x, y^p - y)$, where $x$ is primitive and

$$\Delta(y) = y \otimes 1 + 1 \otimes y + \sum_{1 \leq i \leq p-1} \binom{p}{i}/p \ x^i \otimes x^{p-i}.$$
Classification in low-dimensional connected Hopf algebras

3. \( \dim H = p^3 \).
   - \( \dim P(H) = 1 \). There are 5 isomorphism classes and one infinite parametric family.
   - \( \dim P(H) = 2 \) and \( P(H) \) is non-abelian. There are 3 isomorphism classes.
   - \( \dim P(H) = 2 \) and \( P(H) \) is abelian. Expecting more infinite parametric families of isomorphism classes.
   - \( \dim P(H) = 3 \). There are 15 isomorphism classes and one finite parametric family.
Classification of $p^2$-dim’l pointed Hopf algebras

In recent joint work [W2], we classify pointed Hopf algebras of dimension $p^2$ over an algebraically closed field of arbitrary characteristic. In this part, we assume that $k$ has arbitrary characteristic.

1. **Char** $k \neq p$, such Hopf algebras can only isomorphic to Taft algebras $T_{p,\omega}$ or group algebras $k[C_{p^2}]$ or $k[C_p \times C_p]$. Let $\omega$ be a $p$th primitive root of unity. Then

$$T_{p,\omega} := k\langle g, x \rangle / (g^p - 1, x^p, gx - \omega xg)$$

with the Hopf algebra structure determined by

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1},$$

$$\Delta(x) = x \otimes 1 + g \otimes x, \quad \epsilon(x) = 0, \quad S(x) = -g^{-1}x.$$
Classification of $p^2$-dim’l pointed Hopf algebras

2. Char$k = p$, there are 14 isomorphism classes of such Hopf algebras, including a unique non-commutative and non-cocommutative one, which is given by

$$k\langle g, x \rangle/(g^p - 1, gx - xg - g(g - 1), x^p - x),$$

with the Hopf algebra structure determined by

$$\Delta(g) = g \otimes g, \epsilon(g) = 1, S(g) = g^{-1},$$

$$\Delta(x) = x \otimes 1 + g \otimes x, \epsilon(x) = 0, S(x) = -g^{-1}x.$$
We conjecture that any $p$ or $p^2$ dim’l Hopf algebra over an algebraically closed field of arbitrary characteristic is of Frobenius type. Or more precisely, let $V$ be any irreducible representation of such Hopf algebra, then $\dim V = 1$.

- (Masuoka-Ng-Zhu) The statement holds true for $\text{Char} k = 0$.
- (Etingof-Gelaki) The statement holds true for dimension $p$ and $\text{Char} k > p$.
- Until now, we do not have any counterexample.
Denote by $K$ the Hopf subalgebra of $H$ generated by all primitives $P(H)$.

- If $K = H$, then $H \cong u(P(H))$, the restricted enveloping algebra of $P(H)$ (Milnor-Moore 1965).
- If $K \neq H$, we try to recover $H$ by $K$ and other non-primitive elements.
The cobar construction on $K$

The cobar construction on $K$ is the differential graded algebra $\Omega K$ defined as follows:

- As a graded algebra, $\Omega K$ is the tensor algebra $T(K^+)$, where $K^+$ is the augmentation ideal of $K$;
- The differentials are given by

$$d^n = \sum_{i=0}^{n-1} (-1)^{i+1} 1^i \otimes \Delta \otimes 1^{n-i-1},$$

where $\Delta(a) = \Delta(a) - a \otimes 1 - 1 \otimes a$ for any $a \in K^+$. 

Denote by $n$ the minimal integer such that $K_n \neq H_n$. (Stefan and Van Oystaeyen 1998) There is an injection induced by $d^1$ such that

$$H_n/K_n \hookrightarrow H^2(\Omega K).$$

For element $z \in H_n \setminus K_n$, the comultiplication of $z$ is given by

$$\Delta(z) = z \otimes 1 + 1 \otimes z + (\text{cocyles of degree two in } \Omega K).$$
Comultiplication of non-primitives

We fix a basis \( \{x_1, x_2, \cdots, x_d\} \) for \( P(H) \). We define the expression

\[
\omega(a) = \sum_{1 \leq i \leq p-1} \left( \frac{p}{i} \right) / p \ a^i \otimes a^{p-i}
\]

as an element of \( K^+ \otimes K^+ \) for any \( a \in K^+ \).

**Theorem 1** We can find some \( z \in H \setminus K \) such that

\[
\Delta(z) = z \otimes 1 + 1 \otimes z + \omega \left( \sum_{i=1}^{d} \alpha_i \ x_i \right) + \sum_{1 \leq i < j \leq d} \alpha_{ij} \ x_i \otimes x_j,
\]

for some coefficients \( \alpha_i, \alpha_{ij} \in k \) not all zero.
The parametric family when $\dim P(H) = 1$

**Example 1**\cite{W2} Let $\lambda \in k$, and let $A(\lambda)$ be the $k$-algebra of dimension $p^3$ generated by elements $x, y, z$, subject to the following relations

$$
[x, y] = 0, \ [x, z] = 0, \ [y, z] = x,
$$
$$
x^p = 0, \ y^p = 0, \ z^p + x^{p-1}y = \lambda x.
$$

Then $A(\lambda)$ becomes a connected Hopf algebra via

$$
\Delta(x) = x \otimes 1 + 1 \otimes x, \ \Delta(y) = y \otimes 1 + 1 \otimes y + \omega(x),
$$
$$
\Delta(z) = z \otimes 1 + 1 \otimes z + \omega(x)(y \otimes 1 + 1 \otimes y)^{p-1} + \omega(y),
$$
$$
\epsilon(x) = \epsilon(y) = \epsilon(z) = 0, \ S(x) = -x, \ S(y) = -y, \ S(z) = -z.
$$

When $p > 2$, $A(\lambda) \cong A(\lambda') \iff \lambda = \gamma \lambda'$ for some $\gamma \in \frac{p^2 + p - 1}{\sqrt{1}}$. 
**Example 2**[W'^2] Let $\lambda \in k$, and let $B(\lambda)$ be the $k$-algebra of dimension $p^3$ generated by elements $x, y, z$ subject to the following relations

\[
[x, y] = 0, \quad [x, z] = 0, \quad [y, z] = 0,
\]

\[
x^p = y, \quad y^p = 0, \quad z^p = \lambda x.
\]

Then $B(\lambda)$ becomes a connected Hopf algebra via

\[
\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \Delta(y) = y \otimes 1 + 1 \otimes y,
\]

\[
\Delta(z) = z \otimes 1 + 1 \otimes z + \omega(y) + x \otimes y,
\]

\[
\epsilon(x) = \epsilon(y) = \epsilon(z) = 0, \quad S(x) = -x, \quad S(y) = -y, \quad S(z) = -z + xy.
\]

When $p > 2$, $B(\lambda) \cong B(\lambda') \iff \lambda = \gamma\lambda'$ for some $\gamma \in \frac{p^2 - p - 1}{\sqrt{1}}$. 
Part V

- Algebra structure of finite-dimensional connected Hopf algebras.
- Cohomology rings of connected Hopf algebras.
- Semisimple connected Hopf algebras.
1. Affine connected Hopf algebras over an algebraically closed field of characteristic 0 are classified up to GKdim 4 by Brown, Goodearl, Zhang and Zhuang. Result shows that any such Hopf algebra $H$ is isomorphic, as an algebra, to some universal enveloping algebra $U(g)$ where $\dim g = \text{GKdim } H$. Furthermore, it is proved that the statement is true for $P(H) = \text{GKdim } H - 1 < \infty$. 
Further Projects

In the classification of connected Hopf algebra $H$, where $\dim H = p^3$, $\dim P(H) = 2$ and $P(H)$ is non-abelian, we obtain 3 isomorphism classes.

**Example 3** $[W^2]$ As algebras, these 3 isomorphism classes are

- $k\langle x, y, z \rangle / ([x, y] - y, [x, z], [y, z], x^p - x, y^p, z^p)$;
- $k\langle x, y, z \rangle / ([x, y] - y, [x, z], [y, z] - yf(x), x^p - x, y^p, z^p - z)$ with $f(x) = \sum_{1 \leq i \leq p-1} (-1)^{i-1}(p - i)^{-1}x^i$;
- $k\langle x, y, z \rangle / ([x, y] - y, [x, z], [y, z] - y^2, x^p - x, y^p, z^p)$, where $p > 2$. 
Further Projects

2. We want to compute the cohomology rings for those connected Hopf algebras (and their duals) of dimensions $p$, $p^2$ and $p^3$ that we have classified.

Recall Taft algebra $T_{p,\omega}$, where $\omega$ is $p$th primitive root of unity.

$$T_{p,\omega} := k \langle g, x \rangle / (g^p - 1, x^p, gx - \omega xg)$$

with the Hopf algebra structure determined by

\[
\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1},
\]

\[
\Delta(x) = x \otimes 1 + g \otimes x, \quad \epsilon(x) = 0, \quad S(x) = -g^{-1}x.
\]

(Nguyen 2012)

$$H^n(T_{p,\omega}, k) \cong \begin{cases} 
0 & n = \text{odd} \\
1 & n = \text{even}.
\end{cases}$$

Moreover,

$$H^\bullet(T_{p,\omega}, k) \cong k[\xi], \text{ where } \deg(\xi) = 2.$$
3) (Masuoka 2009) The following are equivalent:

- $H$ is semisimple;
- $H$ is commutative and semisimple;
- $H \cong (k[G])^\ast$ for some $p$-group $G$;
Let $p = 2$. For semisimple connected Hopf algebras, which are almost primitive generated, the following are in 1-1 correspondence with each other:

- The isomorphism classes of semisimple connected Hopf algebras of dimension $p^{d+1}$ with $\dim \mathbb{P}(H) = d$.
- The isomorphism classes of quadratic curves in $\mathbb{P}^{d-1}_{\mathbb{F}_p}$ up to the automorphisms of the projective space.
- The isomorphism classes of $p$-groups of order $p^{d+1}$, whose Frattini group is isomorphic to $C_p$. 
References

N. Andruskiewitsch,
About finite dimensional Hopf algebras,

N. Andruskiewitsch and H.-J. Schneider,
Lifting of quantum linear spaces and pointed Hopf algebras of order $p^3$,

M. Beattie,
A survey of Hopf algebras of low dimension,

M. Beattie and G. Garcia,
Classifying Hopf algebras of a given dimension,

P. Etingof and S. Gelaki,
On finite-dimensional semisimple and cosemisimple Hopf algebras in positive characteristic.

E. M. Friedlander and B. J. Parshall,
Cohomology of Lie agebras and algebraic groups,

G. Henderson,
Low-dimensional cocommutative connected Hopf algebras,
References

A. Masuoka,  
Self-dual Hopf algebras of dimension $p^3$ obtained by extension,  

S.-H. Ng,  
Non-semisimple Hopf algebras of dimension $p^2$,  

S.-H. Ng,  
Hopf algebras of dimension $pq$ II,  
*J. Alg.* **319** (2008), 2772-2788.

D. Radford,  
Operators on Hopf algebras,  
*Amer. J. Math.* **99** (1977), 139-158.

S. Scherotzke,  
Classification of pointed rank one Hopf algebras,  

D. Ştefan and F. Van Oystaeyen,  
Hochschild cohomology and the coradical filtration of pointed coalgebras: applications,  

W,  
Connected Hopf algebras of dimension $p^2$,  
*J. Algebra** **391** (2013), 93–113.
References

W,
Local Criteria for cocommutative Hopf algebras,

W,
Connected Hopf algebras with large abelian primitive space,
in preparation.

W,
Another proof of Masuoka’s Theorem for semisimple irreducible Hopf algebras,

W²,
Classification of pointed Hopf algebras of dimension $p^2$ over any algebraically closed field,

W²,
Classification of Connected Hopf algebras of dimension $p^3$ I,

W²,
Classification of Connected Hopf algebras of dimension $p^3$ II,
in preparation.

Q.-S. Wu and J. J. Zhang,
Noetherian PI Hopf algebras are Gorenstein,