Finite-dimensional connected Hopf algebras

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We work over a base field $k$, algebraically closed. A Hopf algebra $H$ is called \textit{connected} if its coradical equals $k$.

Throughout, let $H$ be a finite-dimensional connected Hopf algebra over $k$. Then, $H$ satisfies the following basic facts.

- The characteristic of $k$ is $p > 0$.
- $\dim H = p^n$ for some integer $n \geq 0$.
- The dual Hopf algebra $H^*$ is local.

Since finite-dimensional connected Hopf algebras only appear in positive characteristic, we let $\text{char} k = p > 0$. 

Associated graded Hopf algebras

Let $k = H_0 \subset H_1 \subset H_2 \subset \cdots H_n \subset \cdots$ be the coradical filtration of $H$. The associated graded Hopf algebra of $H$ is defined to be

$$\text{gr}H = \bigoplus_{n \geq 0} H_n/H_{n-1}.$$ 

**Theorem 1** As an algebra,

$$\text{gr}H \cong k[x_1, x_2, \cdots, x_d]/(x_1^p, x_2^p, \cdots, x_d^p),$$

where $\dim H = p^d$.

Idea: we study the dual Hopf algebra $(\text{gr}H)^*$ to show that $\text{gr}H$ is local commutative. Then, we prove inductively by using the result of finite connected group schemes.
Classification

Examples of finite-dimensional connected Hopf algebras:

- dual of $p$-group algebras;
- restricted universal enveloping algebras of finite-dimensional restricted Lie algebras;
- finite unipotent group schemes.

Denote by $K$ the Hopf subalgebra of $H$ generated by all primitives $P(H)$.

- If $K = H$, then $H \cong u(P(H))$.
- If $K \neq H$, we need to recover $H$ by $K$ and other generators and relations.
The cobar construction on $K$

The *cobar construction* on $K$ is the differential graded algebra $\Omega K$ defined as follows:

- As a graded algebra, $\Omega K$ is the tensor algebra $T(K^+)$, where $K^+$ is the augmentation ideal of $K$;
- The differentials are given by

\[
d^n = \sum_{i=0}^{n-1} (-1)^{i+1} 1^i \otimes \Delta \otimes 1^{n-i-1},
\]

where $\overline{\Delta}(a) = \Delta(a) - a \otimes 1 - 1 \otimes a$ for any $a \in K^+$. 
The cohomology group $H^\bullet(\Omega K)$

For the cobar construction on $K$, we have the following complex:

\[ k \xrightarrow{0} K^+ \xrightarrow{d^1} K^+ \otimes K^+ \xrightarrow{d^2} K^+ \otimes K^+ \otimes K^+ \xrightarrow{d^3} \cdots. \]

The differentials are given by

\[ d^1(a) = -\overline{\Delta}(a), \quad d^2(a \otimes b) = -\overline{\Delta}(a) \otimes b + a \otimes \overline{\Delta}(b), \]

for any $a, b \in K^+$. It is easy to see that we can identify $H^1(\Omega K)$ with $P(H)$. 
The cohomology group $H^\bullet(\Omega K)$

Define the map $\omega : H^1(\Omega K) \rightarrow H^2(\Omega K)$ by

$$\omega(a) = \left[ \sum_{1 \leq i \leq p-1} \left(\frac{p}{i}\right) / p \ a^i \otimes a^{p-i} \right],$$

for any $a \in P(H)$.

**Proposition 1** The map $\omega$ is semi-linear, i.e., we have

$$\omega(\alpha a + b) = \alpha^p \ \omega(a) + \omega(b),$$

for any $a, b \in P(H)$ and $\alpha \in k$.

This map is related to Bockstein homomorphism in cohomology of elementary abelian $p$-groups.
The cohomology ring $H^\bullet(\Omega K)$

Moreover, as cohomology ring,

$$H^\bullet(\Omega K) \cong \begin{cases} 
S(P(H)) & p = 2; \\
\Lambda(P(H)) \otimes S(\omega P(H)) & p > 2.
\end{cases}$$

where $\Lambda$ and $S$ are the exterior and symmetric algebra functors.

Stefan and Van Oystaeyen’s idea: $H^\bullet(\Omega K) \cong H^\bullet(K^*, k) \cong H^\bullet(C_p^d, k)$, where $d = \text{dim } P(H)$.

By abuse of notations, we also consider $\omega(a) = \sum_{1 \leq i \leq p-1} \binom{p}{i}/p \ a^i \otimes a^{p-i}$ as an element of $K^+ \otimes K^+$ for any $a \in K^+$. 
Consider $K \subsetneq H$ as an inclusion of connected Hopf algebras. Denote by $n$ the minimal integer such that $K_n \neq H_n$. There is an injection induced by $d^1$ such that

$$d^1 : H_n/K_n \longrightarrow H^2(\Omega K).$$

**Theorem 2** Choose $x_1, x_2, \cdots, x_d$ as a basis of $P(H)$. Then, we can find some $z \in H \setminus K$ such that

$$\Delta(z) = z \otimes 1 + 1 \otimes z + \omega \left( \sum_{i=1}^{d} \alpha_i x_i \right) + \sum_{1 \leq i < j \leq d} \alpha_{ij} x_i \otimes x_j,$$

for some coefficients $\alpha_i, \alpha_{ij} \in k$ not all zero.
K is normal in H

Suppose \( \dim H / \dim K = p^s \), and \( H \) is generated by \( K \) and some \( z \in H \) as an algebra. Also assume that \( \Delta(z) = z \otimes 1 + 1 \otimes z + u \), where \( u \in K \otimes K \) and \([K, z] \subseteq K + Kz\).

**Theorem 3** We have \( H \) is a free (left) \( K \)-module such that

\[
H = \bigoplus_{i=0}^{p^s-1} Kz^i.
\]

Furthermore, if \( K \) is normal in \( H \), i.e., \( K^+H = HK^+ \), then \( z \) satisfies a polynomial equation as follows:

\[
z^{p^s} + \sum_{i=0}^{s-1} \alpha_i z^{p^i} + a = 0,
\]

for some \( \alpha_i \in k \) and \( a \in K \).
When \( \dim P(H) = 1 \)

Suppose \( \dim P(H) = 1 \). Then, we know \( H^* \) is isomorphic to \( k[x]/(x^{p^d}) \) as algebras for some \( d \geq 0 \).

**Proposition 2**

- \( H \) is cocommutative.
- \( P(H) \subset Z(H) \), the center of \( H \).
- There exists an increasing sequence of normal Hopf algebras:

\[
k = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_d = H,
\]

where \( \dim N_{i+1}/ \dim N_i = p \).
Example for \( \dim P(H) = 1 \)

**Example 1** \([W^2]\) Let \( \lambda \in k \), and let \( A(\lambda) \) be the \( k \)-algebra of dimension \( p^3 \) generated by elements \( x, y, z \), subject to the following relations

\[
[x, y] = 0, \quad [x, z] = 0, \quad [y, z] = x, \\
x^p = 0, \quad y^p = 0, \quad z^p + x^{p-1}y = \lambda x.
\]

Then \( A(\lambda) \) becomes a connected Hopf algebra via

\[
\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \Delta(y) = y \otimes 1 + 1 \otimes y + \omega(x), \\
\Delta(z) = z \otimes 1 + 1 \otimes z + \omega(x)(y \otimes 1 + 1 \otimes y)^{p-1} + \omega(y), \\
\epsilon(x) = \epsilon(y) = \epsilon(z) = 0, \quad S(x) = -x, \quad S(y) = -y, \quad S(z) = -z.
\]

When \( p > 2 \), \( A(\lambda) \cong A(\lambda') \iff \lambda = \gamma \lambda' \) for some \( \gamma \in p^2 + p^{-1} \sqrt{1} \).
Let $H$ be cocommutative, and $L = H^*$, which is commutative.

- Define the upper power series of $H$ as:
  \[
  \Gamma^n(H) = k\langle H_{p^n-1}\rangle,
  \]
  for $n \geq 0$, where $k\langle H_{p^n-1}\rangle$ denotes the subalgebra generated by the $(p^n - 1)$-th term of its coradical filtration.

- Define the lower power series of $L$ as:
  \[
  \Gamma_n(L) = \{ h^{p^n} | h \in L \},
  \]
  for all $n \geq 0$. 

Upper and lower power series
Proposition 3 Upper and lower power series are sequences of normal Hopf subalgebras. Moreover, we have

$$\Gamma^n(H) \cong (L/\Gamma_n(L)^+L)^*, \quad \Gamma_n(L) \cong (H/\Gamma^n(H)^+H)^*.$$  

Theorem 4 The following are equivalent:

- $H$ is local.
- $K$ is local.
- All elements in $P(H)$ are nilpotent.

Remark: The locality criteria for finite-dimensional cocommutative connected Hopf algebras parallel the unipotency criteria for finite connected group schemes over $k$. 
Example for Local Noncocommutative Connected Hopf Algebras

Example 2 $[W^2]$ Let $\lambda \in k$, and let $B(\lambda)$ be the $k$-algebra of dimension $p^3$ generated by elements $x, y, z$ subject to the following relations

$$[x, y] = 0, \ [x, z] = 0, \ [y, z] = 0,$$

$$x^p = y, \ y^p = 0, \ z^p = \lambda x.$$

Then $B(\lambda)$ becomes a connected Hopf algebra via

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \ \Delta(y) = y \otimes 1 + 1 \otimes y,$$

$$\Delta(z) = z \otimes 1 + 1 \otimes z + + \omega(y) + x \otimes y,$$

$$\epsilon(x) = \epsilon(y) = \epsilon(z) = 0, \ S(x) = -x, \ S(y) = -y, \ S(z) = -z + xy.$$

When $p > 2$, $B(\lambda) \cong B(\lambda') \Leftrightarrow \lambda = \gamma \lambda'$ for some $\gamma \in \frac{p^2 - p - 1}{\sqrt{1}}$. 
Example for Non-local Noncocommutative Connected Hopf Algebras

Example 3 \([W^2]\) Let \(\lambda \in \mathbb{F}_p\), and let \(C(\lambda)\) be the \(k\)-algebra of dimension \(p^3\) generated by elements \(x, y, z\), subject to the following relations

\[
\begin{align*}
[x, y] &= 0, \quad [x, z] = \lambda x, \quad [y, z] = (1 - \lambda)y, \\
x^p &= 0, \quad y^p = 0, \quad z^p = z.
\end{align*}
\]

Then \(C(\lambda)\) becomes a connected Hopf algebra via

\[
\begin{align*}
\epsilon(x) &= 0, \quad \Delta(x) = x \otimes 1 + 1 \otimes x, \quad S(x) = -x, \\
\epsilon(y) &= 0, \quad \Delta(y) = y \otimes 1 + 1 \otimes y, \quad S(y) = -y, \\
\epsilon(z) &= 0, \quad \Delta(z) = z \otimes 1 + 1 \otimes z + x \otimes y, \quad S(z) = -z + xy.
\end{align*}
\]

We have \(C(\lambda) \cong C(\lambda')\) if and only if \(\lambda = \lambda'\) or \(\lambda + \lambda' = 1\).
1) Recall \( \dim H = p^{d+1} \) for some \( d \geq 1 \). We are interested in the case when \( H \) is almost primitively generated, i.e., \( \dim K = p^d \). There are two cases.

- \( K \) is commutative. We prove that \( H \) is always an extension of restricted universal enveloping algebras. And the classification is accomplished in terms of automorphism groups acting on cohomology groups.

- \( K \) is noncommutative. We classified the case when \( \dim H = p^3 \).
Further Projects

Example 4 [$W^2$] Suppose dim $H = p^3$, and dim $K = p^2$. There are three isomorphism classes for $K$ is noncommutative. As algebras, they are

- $k \langle x, y, z \rangle / ([x, y] - y, [x, z], [y, z], x^p - x, y^p, z^p)$;
- $k \langle x, y, z \rangle / ([x, y] - y, [x, z], [y, z] - yf(x), x^p - x, y^p, z^p - z)$ with $f(x) = \sum_{1 \leq i \leq p-1} (-1)^{i-1}(p - i)^{-1}x^i$;
- $k \langle x, y, z \rangle / ([x, y] - y, [x, z], [y, z] - y^2, x^p - x, y^p, z^p)$.

For the coalgebra structures, $x$ and $y$ are always primitive and the comultiplication of $z$ is given by

- $\Delta(z) = z \otimes 1 + 1 \otimes z + \omega(y)$;
- $\Delta(z) = z \otimes 1 + 1 \otimes z + \omega(x)$;
- $\Delta(z) = z \otimes 1 + 1 \otimes z - 2x \otimes y$, for $p > 2$. 
Further Projects

2) We want to compute the cohomology rings for those connected Hopf algebras (and their duals) of dimension $p$, $p^2$ and $p^3$ we have classified.

It is conjectured that the cohomology ring of a finite-dimensional Hopf algebra is always finitely generated. It is a special case of a conjecture about finite tensor categories by Etingof and Ostrik.

Recently, Witherspoon and Nguyen showed that a series of finite-dimensional Hopf algebras, $A\# kG$ for some Nichols algebra $A$ in positive characteristic, have finitely generated cohomology rings.
3) We like to classify all semismiple connected Hopf algebras. By Masuoka’s result, \( H \) is semisimple if and only if \( H = (k[G])^\ast \) for some \( p \)-group \( G \). Let \( p = 2 \). Then the following are in 1-1 correspondence with each other.

- The isomorphism classes of semisimple connected Hopf algebras of dimension \( p^{d+1} \) with \( \dim P(H) = d \).
- The isomorphism classes of quadratic curves in \( \mathbb{P}^{d-1}_{\mathbb{F}_p} \) up to the automorphisms of the projective space.
- The isomorphism classes of \( p \)-groups of order \( p^{d+1} \), whose Frattini group is isomorphic to \( C_p \).

The Frattini group of a \( p \)-group \( G \) is the smallest normal subgroup \( N \), where \( G/N \) is an elementary abelian \( p \)-group.
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