Classification of small unipotent quantum groups in positive characteristic and its applications

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Algebra Seminar Talk

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Outlines

Part I  Backgrounds and Motivations
Part II  Unipotent Quantum Groups
Part III  General Classification Results
Part IV  Primitive Control Deformations
Part V  Applications and Related Topics
Recall that affine group schemes are representable functors from commutative $k$-algebras to groups.

**Alg**: the category of all commutative $k$-algebras.

**Grp**: the category of all groups.

Let $G$ be a functor from Alg to Grp. So there are natural transformations

- group multiplication: $\mu : G \times G \to G$,
- group identity: $i : e \to G$,
- group inverse: $s : G \to G$,

where the following diagram commutes:
Part I: Backgrounds and Motivations

Associativity:

Identity:

Inverse element:

Moreover, the group multiplication $\mu$ and the group identity $i$ respect the diagonal map $\Gamma : G \to G \times G$. 
Suppose $G$ is representable, or $G = \text{Hom}_{\text{Alg}}(H, -)$ for some commutative $k$-algebra $H$. We say $H$ is the coordinate ring of $G$ and write $H = \mathcal{O}(G)$. By Yoneda’s Lemma, we have

- group mult. $\mu : G \times G \to G$ \begin{equation} \mu \end{equation}
- comult. $\Delta : H \to H \otimes_k H$

- group identity $i : e \to G$

- counit $\epsilon : H \to k$

- group inverse $s : G \to G$

- antipode $S : H \to H$

such that the following diagrams commute:
Part I: Backgrounds and Motivations

Coassociativity:

\[ H \otimes H \otimes H \xrightarrow{\Delta \otimes id} H \otimes H \]

\[ \xrightarrow{id \otimes \Delta} \]

\[ H \otimes H \xleftarrow{\Delta} H \]

County:

\[ H \otimes k \xleftarrow{id \otimes \epsilon} H \otimes H \xrightarrow{\epsilon \otimes id} k \otimes H \]

\[ \xrightarrow{\otimes 1} \]

\[ \xrightarrow{1 \otimes} \]

Antipode Axiom:

\[ H \otimes H \xleftarrow{S \otimes id} H \otimes H \]

\[ \xrightarrow{m} \]

\[ \xrightarrow{\Delta} \]

\[ H \xrightarrow{u} k \xleftarrow{\epsilon} H \]

\[ \xrightarrow{m} \]

\[ \xrightarrow{\Delta} \]

\[ H \otimes H \xleftarrow{id \otimes S} H \otimes H \]

Moreover, comultiplication \( \Delta \) and counit \( \epsilon \) are algebra maps.
We call \((H, m, u, \Delta, \epsilon, S)\) a commutative Hopf \(k\)-algebra. As a consequence, the following two categories are equivalent:

\[
\text{Affine group schemes} \equiv (\text{comm. Hopf algebras})^{op}
\]

\[
G \mapsto \mathcal{O}(G)
\]

\[
G = \text{Hom}_{\text{Alg}}(H, -) \leftrightarrow H
\]

In Drienfeld’s philosophy, quantum groups are defined to be

\[
\text{Quantum groups} \equiv (\text{Hopf algebras})^{op}
\]

In a conclusion, quantum groups correspond to noncommutative Hopf algebras defined by maps and commutative diagrams as in the affine group schemes case.
Part I: Backgrounds and Motivations

We want to understand finite quantum groups (finite-dimensional Hopf algebras) in positive characteristic.

- representation theory in positive characteristic: irreducible representations, gauge equivalence, quiver algebras and Auslander-Reiten quivers, etc.
- cohomology theory: cohomology rings of finite-dimensional Hopf algebras.
- quantum group actions on AS-regular algebras in positive characteristic: invariant theory.
- Yetter-Drinfeld modules and Nichols algebras in positive characteristic.
An affine group scheme $G$ is unipotent if its coordinate ring $\mathcal{O}(G)$ has a filtration $C_0 \subset C_1 \subset C_2 \subset \cdots$ such that

$$C_0 = k, \bigcup_{r \geq 0} C_r = \mathcal{O}(G), \quad \text{and} \quad \Delta(C_r) \subset \sum_{0 \leq i \leq r} C_i \otimes C_{r-i}.$$ 

We also say $\mathcal{O}(G)$ is a connected commutative Hopf algebra.

**Theorem**

Let $G$ be an affine algebraic group over an algebraic closed field $k$ of characteristic zero. Then the following are equivalent:

1. $G$ is unipotent group.
2. $G$ is torsion-free nilpotent group.
3. $G$ is a closed subgroup of $T(n, k)$ for some $n \geq 1$.
4. $\mathcal{O}(G)$ is a polynomial algebra over $k$.
5. $\mathcal{O}(G)$ is a connected Hopf algebra over $k$. 

Part II: Unipotent Quantum Groups
Therefore, unipotent quantum groups correspond to connected Hopf algebras defined as in the unipotent affine group schemes case. For finite-dimensional connected Hopf algebras, facts are:

- They only appear in positive characteristic (let char \( k = p > 0 \)).
- They all have dimension \( p^n \) for some integer \( n > 0 \).
- They can be constructed from \( p \)-groups, finite unipotent group schemes and finite restricted Lie algebras.
- They are in one-to-one correspondence with finite-dimensional local Hopf algebras by Cartier duality.
Throughout, let \((H, m, u, \Delta, \epsilon, S)\) be a finite-dimensional connected Hopf algebra over an algebraic closed field \(k\) of characteristic \(p > 0\). Regarding the coalgebra structure of \((H, \Delta, \epsilon)\), we have

- the coradical of \(H\) is the sum of all simple subcoalgebras denoted by \(H_0\) (\(H\) is connected iff \(\dim H_0 = 1\)).
- the coradical filtration of \(H\): \(H_0 \subseteq H_1 \subseteq \ldots H_n \subseteq \ldots \subseteq H\).
- the primitive space \(P(H) = \{x \in H | \Delta(x) = x \otimes 1 + 1 \otimes x\}\) (It is a restricted Lie algebra, where the Lie bracket is given by the commutator and the restricted map is given by the \(p\)-th power in \(H\)).
Part III: General Classification Results

We will provide new examples of finite unipotent quantum groups by classifying all connected Hopf algebras of dimension $p$, $p^2$ and $p^3$ over $k$. 
A list of isomorphism classes is obtained with explicit generators and relations.

- dimension $p$, there are two isomorphism classes.
- dimension $p^2$, there are eight isomorphism classes.
- dimension $p^3$.
  - $\heartsuit \dim P(H) = 1$, there are four isomorphism classes and one infinite parametric family.
  - ♠ $\dim P(H) = 2$ and nonabelian, there are three isomorphism classes.
  - ♣ (suppose $p > 2$). $\dim P(H) = 2$ and abelian, there are thirty three isomorphism classes, one finite parametric family and eight infinite parametric families.
  - ♦ $\dim P(H) = 3$, there are fifteen isomorphism classes and one finite parametric family.
Part III: General Classification Results

This classification involves:

- groups of order $p$, $p^2$ and $p^3$ (Hölder 1893 and Burnside 1897)
  - order $p$: $C_p$
  - order $p^2$: $C_p \times C_p$ and $C_{p^2}$
  - order $p^3$: Abelian $C_{p^3}$, $C_{p^2} \times C_p$ and $C_p \times C_p \times C_p$; Nonabelian $D_4$, $Q_8$ for $p = 2$ and $(C_p \times C_p) \times C_p$ and $C_{p^2} \times C_p$ for $p > 2$
- finite unipotent group schemes of dimension $p$, $p^2$ and $p^3$
- restricted Lie algebras of dimension 1, 2 and 3 (simple Lie algebras over an algebraically closed field of characteristic $p \geq 5$ is recently classified by Premet, Strade and others)
Part III: General Classification Results

Techniques used in classification:

- Dimension $p$: choose $0 \neq x \in P(H)$. Since $x^p \in P(H)$, we have $x^p = \lambda x$ for some $\lambda \in k$. By rescaling, we can take $\lambda = 0$ or 1.
- Dimension $p^2$: restricted Lie algebra theory and Hochschild cohomology of coalgebras (cobar construction).
- Dimension $p^3$: ♠ Primitive control deformations (PCDs) of restricted universal enveloping algebras.
Part III: General Classification Results

Infinite family in ♦: let $\lambda \in k$, and $A(\lambda)$ the $k$-algebra generated by elements $x, y, z$, subject to the following relations

\[
[x, y] = 0, [x, z] = 0, [y, z] = x, x^p = 0, y^p = 0, z^p + x^{p-1} y = \lambda x.
\]

Take the expression $\omega(t) = \sum_{1 \leq i \leq p-1} \binom{p}{i} \frac{t^{i}}{pt^i} \otimes t^{p-i}$. Therefore $A(\lambda)$ becomes a connected Hopf algebra via

\[
\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \Delta(y) = y \otimes 1 + 1 \otimes y + \omega(x),
\]

\[
\Delta(z) = z \otimes 1 + 1 \otimes z + \omega(x) (y \otimes 1 + 1 \otimes y)^{p-1} + \omega(y),
\]

\[
\epsilon(x) = \epsilon(y) = \epsilon(z) = 0, \quad S(x) = -x, \quad S(y) = -y, \quad S(z) = -z.
\]

When $p > 2$, $A(\lambda) \cong A(\lambda') \iff \lambda = \gamma \lambda'$ for some $\gamma \in p^{2+p-1} \sqrt{1}$, or the isomorphism classes of $A(\lambda)$ are parametrized by $k/ p^{2+p-1} \sqrt{1}$.
Suppose $k$ is of characteristic $p > 2$. Consider the restricted Lie algebra $\mathfrak{sl}_2(k)$: all trace zero $2 \times 2$ matrices. As a $k$-vector space, $\mathfrak{sl}_2(k)$ is generated by:

$$
e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The Lie bracket is given by the commutators in $M_2(k)$:
  $$[h, e] = 2f, \quad [h, f] = -2f, \quad [e, f] = h.$$

- The restricted map is given by the $p$-th power map in $M_2(k)$:
  $$e^p = f^p = 0, \quad h^p = h.$$
Consider an extension of two restricted Lie algebras $\mathfrak{g}$ (abelian) and $\mathfrak{h}$:

\[ 0 \rightarrow \mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{h} \rightarrow 0 \]

The semi-product $\mathfrak{g} \times \mathfrak{h}$ is given by an algebraic representation $\rho : \mathfrak{h} \rightarrow \text{End}_k(\mathfrak{g})$ such that

- the Lie bracket is $\left[ (x, y), (x', y') \right] = (\rho_y(x') - \rho_{y'}(x), [y, y'])$,
- the restricted map is $(x, y)[p] = (x[x], y[p])$,

for any $x, x' \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$. 

Part IV: Extension of restricted Lie algebras
Part IV: Algebraic representation

An algebraic representation of $\mathfrak{h}$ on $\mathfrak{g}$ is a linear map

$$\rho : \mathfrak{h} \to \text{End}_k(\mathfrak{g})$$

such that

1. $\rho_{[x,y]} = \rho_x \rho_y - \rho_y \rho_x$,
2. $\rho_{(xp)} = (\rho_x)^p$,
3. $\rho_x([a,b]) = [\rho_x(a), b] + [a, \rho_x(b)]$,
4. $\rho_x(a^p) = \rho_x(a) (\text{ad } a)^{p-1}$,

for any $x, y \in \mathfrak{h}$ and $a, b \in \mathfrak{g}$. 
Part IV: A concrete example of PCDs

The extension $\mathcal{T}$ of restricted Lie algebras are described by

- Let $\dim \mathfrak{g} = 2$ and $\dim \mathfrak{h} = 1$. So we fix basis $x, y$ for $\mathfrak{g}$ and $z$ for $\mathfrak{h}$.

- By Strade and Farnsteiner's terminology, suppose $\mathfrak{g}$ and $\mathfrak{h}$ are tori such that $x^p = x$, $y^p = y$ and $z^p = z$.

- Let $\rho$ be an algebraic representation of $\mathfrak{h}$ on $\mathfrak{g}$. Note that

$$\rho_z(x^p) = \rho_z(x)(\text{ad} x)^{p-1} = 0, \quad \rho_z(y^p) = \rho_z(y)(\text{ad} y)^{p-1} = 0.$$ 

Hence $\rho = 0$.

The only possible extension $\mathcal{T}$ of restricted Lie algebras is

$$\mathcal{T} : 0 \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g} \oplus \mathfrak{h} \longrightarrow \mathfrak{h} \longrightarrow 0$$

And

$$u(\mathfrak{g} \oplus \mathfrak{h}) = \frac{\mathbb{k}\langle x, y, z \rangle}{\langle [x, y], [y, z], [x, z], x^p - x, y^p - y, z^p - z \rangle}.$$
Part IV: A concrete example of PCDs

Any PCD of the extension $T$ is given by

$$\left[ u(g \oplus h) , \chi , \Theta \right] := k[x, y, z]/J$$

where the relation $J$ is generated by

$$x^p - x, y^p - y, z^p - z + \Theta.$$ 

and

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \Delta(y) = y \otimes 1 + 1 \otimes y,$$

$$\Delta(z) = z \otimes 1 + 1 \otimes z + \chi.$$
Part IV: A concrete example of PCDs

Let $B = u(g)$. The parametric space of $A := [u(g \otimes h), \chi, \Theta]$ is

$$\mathcal{P} = Z^2(\Omega B) \times B^+.$$ 

where $Z^2(\Omega B)$ is the set of all cocycles in the cobar construction on $B$:

$$\begin{array}{cccc}
\mathbf{k} & \rightarrow & B^+ & \mathbf{d}^1 \rightarrow & B^+ \otimes B^+ & \mathbf{d}^2 \rightarrow & B^+ \otimes B^+ \otimes B^+ & \rightarrow & \cdots
\end{array}$$

The size of the parametric space is $\dim \mathcal{P} = 2p^2 - 1$. Now all PCDs of the extension $T$ correspond to the subset of $\mathcal{P}$ satisfying

(1) $\text{gr}A = k[X, Y, Z]/(X^p, Y^p, Z^p)$ with respect to the coradical filtration.

(2) $\text{P}(A) = g$. 
Part IV: A concrete example of PCDs

Moreover in order to classify all PCDs of the extension $T$, we construct a subset quotient $\mathcal{H}^2(T)$ by requiring

1. $\text{gr} A = k[X, Y, Z]/(X^p, Y^p, Z^p)$ with respect to the coradical filtration.
2. $\text{P}(A) = g$.
3. equivalence relation for extensions

$$1 \rightarrow u(g) \rightarrow A \rightarrow u(h) \rightarrow 1.$$ 

We show that there is a bijection

$$\mathcal{H}^2(T) \leftrightarrow S = \mathbb{F}_p \times \mathbb{F}_p \times \mathbb{F}_p \setminus \{(0, 0, 0)\}$$

In details, any point $P = (a, b, c) \in S$ can be represented in the parametric space $\mathcal{P}$ by

$$\chi_P = ax \otimes y + \sum_{i=1}^{p-1} \left( \begin{array}{c} p \\ i \end{array} \right) / p \ (bx + cy)^i \otimes (bx + cy)^{p-i},$$

$$\Theta_P = 0.$$
Part IV: A concrete example of PCDs

We define an automorphism group

\[ \text{Aut}(T) = \text{Aut}(g) \times \text{Aut}(h) = \text{GL}(2, \mathbb{F}_p) \times \text{GL}(1, \mathbb{F}_p). \]

Choose any \( \phi = M \times \gamma \in \text{GL}(2, \mathbb{F}) \times \text{GL}(1, \mathbb{F}_p). \) There is an embedding \( \text{Aut}(T) \hookrightarrow \text{GL}(3, \mathbb{F}_p) \) via

\[ \phi = M \times \gamma \mapsto \begin{pmatrix} \gamma^{-1} \det(M) & 0 \\ 0 & \gamma^{-1}M \end{pmatrix}. \]

Then \( \phi \) acts on any point \( P = (a, b, c) \in S \) as follows:

\[ \phi[a, (b, c)] = [\gamma^{-1}(\det M)a, \gamma^{-1}(b, c)M]. \]

We show that \( \text{Aut}(T) \)-orbits in \( S \) are in 1-1 correspondence with isomorphism classes of all PCDs of the extension \( T \).
Part IV: A concrete example of PCDs

Consider the group action on $S = \mathbb{F}_p \times \mathbb{F}_p \times \mathbb{F}_p \setminus \{(0, 0, 0)\}$ by the following two normal subgroups.

- $\text{Aut}(\mathfrak{h}) = \mathbb{F}_p^\times$ acts on $S$ by inverse multiplication
  
  $$\phi(a, b, c) = (\gamma^{-1} a, \gamma^{-1} b, \gamma^{-1} c).$$

- $\text{Aut}(\mathfrak{g}) = \text{GL}(2, \mathbb{F}_p)$ acts on $S$ via the embedding

  $$M \mapsto \begin{pmatrix} \det(M) & 0 \\ 0 & M \end{pmatrix}.$$ 

Since $S/\mathbb{F}_p^\times = \mathbb{P}^2$,

$$\text{Aut}(\mathfrak{T})\text{-orbits in } S \longleftrightarrow \text{Aut}(\mathfrak{g})\text{-orbits in } \mathbb{P}^2$$

via the previous embedding $\text{GL}(2, \mathbb{F}_p) \hookrightarrow \text{PGL}(3, \mathbb{F}_p)$. 

Part IV: A concrete example of PCD

The Aut(g)-orbits in $\mathbb{P}^2$ contain three points

$$A = [1 : 0 : 0], \quad B = [1 : 1 : 0], \quad C = [0 : 1 : 0]$$

The corresponding PCDs are

$$k[x, y, z] \quad \frac{(x^p - x, y^p - y, z^p - z)}{\sum_{i=1}^{p-1} \binom{p}{i}/p \cdot x^i \otimes x^{p-i}}$$

where $x, y$ are primitive elements and

$$A : \Delta(z) = z \otimes 1 + 1 \otimes z + x \otimes y$$

$$B : \Delta(z) = z \otimes 1 + 1 \otimes z + x \otimes y + \sum_{i=1}^{p-1} \binom{p}{i}/p \cdot x^i \otimes x^{p-i}$$

$$C : \Delta(z) = z \otimes 1 + 1 \otimes z + \sum_{i=1}^{p-1} \binom{p}{i}/p \cdot x^i \otimes x^{p-i}$$
Part IV: A concrete example of PCD

By Masuoka’s result, $A$, $B$, $C$ correspond to the dual of the group algebra $kG$, whose Frattini group is $C_p$. There are only three of them

$$A : \Delta(z) = z \otimes 1 + 1 \otimes z + x \otimes y \leftrightarrow (C_p \times C_p) \rtimes C_p$$

$$B : \Delta(z) = z \otimes 1 + 1 \otimes z + x \otimes y + \sum_{i=1}^{p-1} \frac{p}{p} x^i \otimes x^{p-i} \leftrightarrow C_p^2 \rtimes C_p$$

$$C : \Delta(z) = z \otimes 1 + 1 \otimes z + \sum_{i=1}^{p-1} \frac{p}{p} x^i \otimes x^{p-i} \leftrightarrow C_p^2 \times C_p$$
Part IV: Classification of ♠

♠ Suppose $p > 2$. When $\dim P(H) = 2$ and abelian, there are thirty three isomorphism classes, one finite parametric family and eight infinite parametric families.

All Hopf algebras in ♠ come from PCDs of $u(g \rtimes h)$ satisfying

- $\dim g = 2$ and $g$ is abelian.
- $\dim h = 1$.
- The primitive space of the deformation is isomorphic to $g$. 
Part IV: Classification of ♠

We first classify all possible extensions

\[ T : 0 \rightarrow g \rightarrow g \times h \rightarrow h \rightarrow 0 \]

with data \( T = (g, h, \rho) \). There are sixteen isomorphism classes of such extensions.

Classification of \( g : \)

\[ \begin{align*}
A : x^p &= 0, \quad y^p = 0, \\
B : x^p &= 0, \quad y^p = 0, \\
C : x^p &= y, \quad y^p = 0, \\
D : x^p &= x, \quad y^p = x.
\end{align*} \]

Classification of \( h : \)

\[ \begin{align*}
N : z^p &= 0, \\
S : z^p &= z.
\end{align*} \]
Part IV: Classification of ♣

<table>
<thead>
<tr>
<th>Type</th>
<th>g</th>
<th>h</th>
<th>Algebraic representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T1)</td>
<td>A</td>
<td>N</td>
<td>$\rho_z = 0$</td>
</tr>
<tr>
<td>(T2)</td>
<td>A</td>
<td>N</td>
<td>$\rho_z(x) = y, \rho_z(y) = 0$</td>
</tr>
<tr>
<td>(T3)</td>
<td>A</td>
<td>S</td>
<td>$\rho_z = 0$</td>
</tr>
<tr>
<td>(T4)</td>
<td>A</td>
<td>S</td>
<td>$\rho_z(x) = x, \rho_z(y) = \lambda y$ for $\lambda \in \mathbb{F}_p$ and $\lambda \neq -1$</td>
</tr>
<tr>
<td>(T5)</td>
<td>B</td>
<td>N</td>
<td>$\rho_z = 0$</td>
</tr>
<tr>
<td>(T6)</td>
<td>B</td>
<td>N</td>
<td>$\rho_z(x) = 0, \rho_z(y) = x$</td>
</tr>
<tr>
<td>(T7)</td>
<td>B</td>
<td>S</td>
<td>$\rho_z = 0$</td>
</tr>
<tr>
<td>(T8)</td>
<td>B</td>
<td>S</td>
<td>$\rho_z(x) = 0, \rho_z(y) = y$</td>
</tr>
<tr>
<td>(T9)</td>
<td>B</td>
<td>S</td>
<td>$\rho_z(x) = 0, \rho_z(y) = x + y$</td>
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</table>
Part IV: Classification of ♠

<table>
<thead>
<tr>
<th>Type</th>
<th>$g$</th>
<th>$h$</th>
<th>Algebraic representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T10) $C$ $\mathcal{N}$</td>
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<td>$\rho_z = 0$</td>
<td></td>
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<tr>
<td>(T11) $C$ $\mathcal{N}$</td>
<td></td>
<td>$\rho_z(x) = y, \rho_z(y) = 0$</td>
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<tr>
<td>(T12) $C$ $\mathcal{S}$</td>
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<td>$\rho_z = 0$</td>
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<tr>
<td>(T13) $C$ $\mathcal{S}$</td>
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<td>$\rho_z(x) = x, \rho_z(y) = 0$</td>
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<tr>
<td>(T14) $C$ $\mathcal{S}$</td>
<td></td>
<td>$\rho_z(x) = x + y, \rho_z(y) = 0$</td>
<td></td>
</tr>
<tr>
<td>(T15) $\mathcal{D}$ $\mathcal{N}$</td>
<td></td>
<td>$\rho_z = 0$</td>
<td></td>
</tr>
<tr>
<td>(T16) $\mathcal{D}$ $\mathcal{S}$</td>
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<td>$\rho_z = 0$</td>
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### Part IV: Classification of ♣

<table>
<thead>
<tr>
<th>Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T1)</td>
<td>eight points</td>
</tr>
<tr>
<td>(T2)</td>
<td>six points and $k/{\pm 1}, k$</td>
</tr>
<tr>
<td>(T3)</td>
<td>NONE</td>
</tr>
<tr>
<td>(T4)</td>
<td>one point for each $-1 \neq \lambda \in \mathbb{F}_p$, totally $\frac{p+1}{2}$ points</td>
</tr>
<tr>
<td>(T5)</td>
<td>four points and $k/\left(\frac{p-1}{2}\sqrt{1}\right)$</td>
</tr>
<tr>
<td>(T6)</td>
<td>one point and $k/\left(\frac{p^2-1}{2}\sqrt{1}\right)$</td>
</tr>
<tr>
<td>(T7)</td>
<td>one point</td>
</tr>
<tr>
<td>(T8)</td>
<td>three points</td>
</tr>
<tr>
<td>(T9)</td>
<td>$k/(\mathbb{F}_p^\times)^2$ and $k$</td>
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Part IV: Classification of ♣

<table>
<thead>
<tr>
<th>Type</th>
<th>$\text{Aut}(T)$-orbits in $\mathcal{H}^2(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T10)</td>
<td>four points and $k/\sqrt[p^2-p^1-1]{1}$</td>
</tr>
<tr>
<td>(T11)</td>
<td>one point and $k/\sqrt[p^2-p+1]{1}$</td>
</tr>
<tr>
<td>(T12)</td>
<td>NONE</td>
</tr>
<tr>
<td>(T13)</td>
<td>one point</td>
</tr>
<tr>
<td>(T14)</td>
<td>one point</td>
</tr>
<tr>
<td>(T15)</td>
<td>NONE</td>
</tr>
<tr>
<td>(T16)</td>
<td>three</td>
</tr>
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</table>
In algebraically closed field of characteristic zero, we have the following facts about connected Hopf algebras:

- If a connected Hopf algebra $A$ is cocommutative, then it is isomorphic to the universal enveloping algebra $\mathcal{U}(P(H))$.
- Only regarding algebra structure, up to GK-dimension four, connected Hopf algebras are all isomorphic to some universal enveloping algebras.
- It is proved that if $\dim P(A) = \text{GKdim} H - 1 < \infty$. Then $A$ is isomorphic, only as algebras, to some $\mathcal{U}(L)$.
Part V: Algebra Structure of Finite Unipotent Quantum Groups

In positive characteristic, we want to know when a finite connected Hopf algebras is isomorphic, only as algebras, to some restricted universal enveloping algebra. In ♠ we an anti-example.

\[ H := \frac{k\langle x, y, z \rangle}{([x, y] - y, [x, z], [y, z] - yf(x), x^p - x, y^p, z^p - z)} \]

with \( f(x) = \sum_{i=1}^{p-1} (-1)^{i-1} (p - i)^{-1} x^i \) and

\[ \Delta(x) = x \otimes 1 + 1 \otimes x, \quad \Delta(y) = y \otimes 1 + 1 \otimes y \]

\[ \Delta(z) = z \otimes 1 + 1 \otimes z + \sum_{1 \leq i \leq p-1} \binom{p}{i} / p \ x^i \otimes x^{p-i} \]

We show that it has trivial center and furthermore it is not isomorphic to any one in ♠ as algebras.
Consider the following example:

\[ A = \mathbb{k}[u, v] \]

\[ H = \mathbb{k}[x]/(x^p - x) \text{ with } \Delta(x) = x \otimes 1 + 1 \otimes x. \]

Suppose \( H \) gradedly acts on \( A \) (graded \( H \)-module algebra). Then \( \mathbb{k}u \oplus \mathbb{k}v \) becomes a left \( H \)-module and there is an embedding \( x \leftrightarrow \text{GL}(2, \mathbb{k}) \). Since \( x^p - x = 0 \), we know that \( x \) is diagonalizable. After a linear transformation of \( u, v \) and rescaling of \( x \), we have

\[ x \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \]

for some \( \lambda \in \mathbb{F}_p \).
Part V: Hopf Actions on AS-Regular Algebras in Positive Characteristic

Take $\lambda \in \{1, 2, \ldots, p - 1\}$. We use Hirzebruch-Jung’s continued fraction expansion

$$\frac{p}{p - \lambda} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \cdots}} = [a_1, a_2, \ldots, a_k].$$

The invariant ring $A^H := \{a \in A | ha = \epsilon(h) a, \forall h \in H\}$ is generated by

$$f_0 = u^p, \quad f_1 = u^{p-\lambda}v, \quad f_2, \ldots, \quad f_{k+1} = v^p$$

where $f_{i-1}f_{i+1} = f_i^{a_i}$ for $i = 1, \ldots, k$. In complex case: $V = \mathbb{C}^2$ and $G$ is the cyclic group of order $p$ acting on $\mathbb{C}^2$ by diagonal matrixes; by a slight normalisation, we can assume that

$$G = \begin{pmatrix} \xi & 0 \\ 0 & \xi^\lambda \end{pmatrix}$$

where $\xi = \exp \frac{2\pi i}{p}$. We know $V/G$ are surface cyclic quotient singularities of type $\frac{1}{p}(1, \lambda)$. 
Part V: Questions

• How to tell a unipotent quantum group is isomorphic, only as algebras, to some restricted universal enveloping algebras?
• How to tell two semismiple unipotent quantum groups are gauge equivalent?
• What are the cohomology rings of these finite unipotent quantum groups?
• What kind of regular algebras can have finite unipotent quantum group actions and what are these invariant rings look like?
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