The action of Grothendieck-Teichmueller(GT) shadows on child's drawings

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Loosely based on my paper "The Action of GT-Shadows on Child's Drawings" (J. of Algebra, 2025)

Dedication

To my parents...

The key characters of the story

- The absolute Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of rational numbers.
- The (gentle version of) the Grothendieck-Teichmueller group $\widehat{\mathrm{GT}}$. This group was introduced by V. Drinfeld in 1990. $\widehat{\mathrm{GT}}$ is a subset of $\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$, where F_2 is the free group $\langle\, x, y \,\rangle$ and $\widehat{\mathrm{F}}_2$ is its profinite completion.
- The groupoid GTSh of Grothendieck-Teichmueller(GT) shadows. $\mathrm{Ob}(\mathrm{GTSh}) := \{\mathrm{N} \trianglelefteq_{f.i.} \mathrm{F}_2 \mid w(\mathrm{N}) = \mathrm{N},\ \ \forall\, w \in \mathrm{B}_3\}$, where B_3 is the Artin braid group on 3 strands. Morphisms of GTSh are called GT*-shadows*.
- Grothendieck's child's drawings, i.e. isomorphism classes of finite degree coverings of $\mathbb{CP}^1 - \{0, 1, \infty\}$.

The (gentle version of) the group $\widehat{\mathsf{GT}}$

Let $\widehat{\mathsf{F}}_2$ be the profinite completion of $\mathsf{F}_2 := \langle\, x, y \,\rangle$.

As a set, $\widehat{\mathsf{GT}}$ consists of pairs $(\hat{m}, \hat{f})$ in $\widehat{\mathbb{Z}} \times \widehat{\mathsf{F}}_2$ satisfying

$$\hat{f}\,\theta(\hat{f}) = 1_{\widehat{\mathsf{F}}_2}, \qquad \tau^2(y^{\hat{m}}\hat{f})\,\tau(y^{\hat{m}}\hat{f})\,y^{\hat{m}}\hat{f} = 1_{\widehat{\mathsf{F}}_2},$$

$\hat{f} \in [\widehat{\mathsf{F}}_2, \widehat{\mathsf{F}}_2]^{top.clos.}$ and the invertibility condition.

Here θ and τ are the automorphisms of F_2 (and of $\widehat{\mathsf{F}}_2$) defined by the formulas $\theta(x) := y$, $\theta(y) := x$, $\tau(x) := y$, $\tau(y) := y^{-1}x^{-1}$.

The multiplication on $\widehat{\mathsf{GT}}$ is defined using a monoid structure on $\widehat{\mathbb{Z}} \times \widehat{\mathsf{F}}_2$ that is inspired by the action of $G_{\mathbb{Q}}$ on $\widehat{\mathsf{F}}_2 \cong \pi_1^{alg}(\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\})$. The pair $(0_{\widehat{\mathbb{Z}}}, 1_{\widehat{\mathsf{F}}_2})$ is the identity element of $\widehat{\mathsf{GT}}$.

One can easily see that $(-1_{\widehat{\mathbb{Z}}}, 1_{\widehat{\mathsf{F}}_2})$ is also an element of $\widehat{\mathsf{GT}}$.

The Ihara embedding

In his 1994 paper "On the embedding of Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$ into $\widehat{\mathsf{GT}}$", Y. Ihara used two isomorphic field extensions of $\overline{\mathbb{Q}}(t)$ to construct a map

$$Ih : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}} \times \widehat{F}_2$$

of the form $Ih(g) = \big((\chi(g) - 1)/2, f_g\big)$, where χ denotes the cyclotomic character: $\chi : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^{\times}$.

Using the appropriate versions of the fundamental groupoids of the moduli spaces $\mathcal{M}_{0,4}$, $\mathcal{M}_{0,5}$ of curves, Ihara proved (in ICM 1990) that, for every $g \in G_{\mathbb{Q}}$, the pair $((\chi(g) - 1)/2, f_g)$ is an element $\widehat{\mathsf{GT}}$. In particular, $f_g \in [\widehat{\mathsf{F}}_2, \widehat{\mathsf{F}}_2]^{\textit{top. cl.}}$.

Using famous Belyi's theorem, one can prove that the resulting group homomorphism $Ih : G_{\mathbb{Q}} \to \widehat{\mathsf{GT}}$ is injective. We call Ih the *Ihara embedding*. It is known that the pair $(-1_{\widehat{\mathbb{Z}}}, 1_{\widehat{\mathsf{F}}_2}) \in \widehat{\mathsf{GT}}$ equals $Ih(c^*)$, where c^* denotes the complex conjugation.

The virtual cyclotomic character

For every $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}$, $2\hat{m} + 1$ is an invertible element of the ring $\widehat{\mathbb{Z}}$. Moreover, the formula

$$\chi_{vir}(\hat{m}, \hat{f}) := 2\hat{m} + 1$$

defines a group homomorphism $\chi_{vir} : \widehat{\mathsf{GT}} \to \widehat{\mathbb{Z}}^{\times}$ and we call it the *virtual cyclotomic character*.

The following diagram

$$\begin{tikzcd}
G_{\mathbb{Q}} \arrow[rr, "Ih"] \arrow[dr, "\chi"'] & & \widehat{\mathsf{GT}} \arrow[dl, "\chi_{vir}"] \\
& \widehat{\mathbb{Z}}^{\times} &
\end{tikzcd}$$

commutes. Therefore, the surjectivity of $\chi : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^{\times}$ implies the surjectivity of the virtual cyclotomic character $\chi_{vir} : \widehat{\mathsf{GT}} \to \widehat{\mathbb{Z}}^{\times}$.

Is *Ih* surjective?

The following question is probably *very hard:*

Is the homomorphism Ih $: G_{\mathbb{Q}} \to \widehat{\mathsf{GT}}$ *surjective?*

In several remarkable papers, F. Pop gave positive answers to versions of the above question. In these versions, $\widehat{\mathsf{GT}}$ is replaced by subgroups of $\widehat{\mathsf{GT}}$ with infinitely many defining conditions.

For example, the birational version $\widehat{\mathsf{GT}}_{bir}$ of $\widehat{\mathsf{GT}}$ is defined using the etale fundamental group functor from the sub-category of concrete algebraic varieties obtained from $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$. In "*Finite tripod variants of I/OM: ...*", 2019, F. Pop proved that the homomorphism *Ih* lands in $\widehat{\mathsf{GT}}_{bir}$ and the group $\widehat{\mathsf{GT}}_{bir}$ is isomorphic to $G_{\mathbb{Q}}$ via *Ih*.

The Artin braid group B_3 and PB_3

B_3 (resp. PB_3) denotes the Artin braid group (resp. the pure braid group) on 3 strands. σ_1, σ_2 are the standard generators of B_3:

We set $\Delta := \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

PB_3 is generated by

$$x_{12} := \sigma_1^2, \qquad x_{23} := \sigma_2^2, \qquad c := \Delta^2\,.$$

It is known that $\mathcal{Z}(B_3) = \mathcal{Z}(PB_3) = \langle\, c \,\rangle \cong \mathbb{Z}$, the subgroup $\langle\, x_{12}, x_{23} \,\rangle$ is isomorphic to F_2. In fact, $PB_3 \cong F_2 \times \langle\, c \,\rangle$.

A bit more about F_2, PB_3 and B_3

It is natural to identify F_2 with the quotient group $PB_3/\mathcal{Z}(PB_3)$ and set

$$x := x_{12}\,\mathcal{Z}(PB_3), \qquad y := x_{23}\,\mathcal{Z}(PB_3).$$

Since $\mathcal{Z}(B_3) = \mathcal{Z}(PB_3)$, the group B_3 acts naturally on the quotient $F_2 \cong PB_3/\mathcal{Z}(PB_3)$. We denote by θ (resp. τ) the automorphism of F_2 corresponding to $\Delta := \sigma_1\sigma_2\sigma_1$ (resp. to $\sigma_1\sigma_2$).

It is easy to see that

$$\theta(x) := y, \quad \theta(y) := x, \qquad \tau(x) := y, \quad \tau(y) := y^{-1}x^{-1}.$$

Although the elements Δ and $\sigma_1\sigma_2$ are of infinite order, the automorphisms θ and τ have finite orders: $\mathrm{ord}(\theta) = 2$, $\mathrm{ord}(\tau) = 3$.
Our goal is to construct a groupoid GTSh with

$$\mathrm{Ob}(\mathrm{GTSh}) := \{\, N \trianglelefteq F_2 \mid w(N) = N, \ \forall\, w \in B_3, \ |F_2 : N| < \infty \,\}.$$

Preparation

For $\mathsf{N} \in \mathsf{Ob}(\mathsf{GTSh})$, we set

$$N_{\mathrm{ord}} := \mathrm{ord}(x\mathsf{N}) = \mathrm{ord}(y\mathsf{N}).$$

We say that $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$ satisfies the *cocycle conditions* modulo N if

$$f\theta(f) \in \mathsf{N}, \qquad \tau^2(y^m f)\tau(y^m f)y^m f \in \mathsf{N}.$$

For $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$ and $\mathsf{N} \in \mathsf{Ob}(\mathsf{GTSh})$, we denote by $T_{m,f}$ the homomorphism $T_{m,f} : \mathsf{F}_2 \to \mathsf{F}_2/\mathsf{N}$:

$$T_{m,f}(x) := x^{2m+1}\mathsf{N}, \qquad T_{m,f}(y) = f^{-1}y^{2m+1}f\mathsf{N}.$$

If the pair $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$ satisfies the cocycle conditions modulo N, then $\ker(T_{m,f})$ is also B_3-invariant, hence

$$\ker(T_{m,f}) \in \mathsf{Ob}(\mathsf{GTSh}).$$

A GT-shadow is . . .

Definition

Let N *be a* B_3*-invariant finite index normal subgroup of* F_2. *A* GT*-shadow with the target* N *is a pair*

$$[m, f] \; := \; (m + N_{\mathrm{ord}}\mathbb{Z}, f\mathsf{N}) \in \mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathsf{F}_2/\mathsf{N}$$

satisfying the cocycle conditions (modulo N*) and such that*

- $2m + 1$ *represents a unit in the ring* $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z}$,
- $f\mathsf{N} \in [\mathsf{F}_2/\mathsf{N}, \mathsf{F}_2/\mathsf{N}]$, *and*
- *the homomorphism* $T_{m,f} : \mathsf{F}_2 \to \mathsf{F}_2/\mathsf{N}$ *is surjective.*

GT(N) is the set of GT-shadows with the target N.

The groupoid GTSh

Guess what?!.... GTSh is a groupoid!

$$\mathrm{Ob}(\mathsf{GTSh}) := \{\, \mathsf{N} \trianglelefteq \mathsf{F}_2 \mid w(\mathsf{N}) = \mathsf{N}, \ \ \forall\, w \in \mathsf{B}_3, \ \ |\mathsf{F}_2 : \mathsf{N}| < \infty \,\}.$$

For $\mathsf{K}, \mathsf{N} \in \mathrm{Ob}(\mathsf{GTSh})$, we set

$$\mathsf{GTSh}(\mathsf{K}, \mathsf{N}) := \Big\{ [m, f] \in \mathsf{GT}(\mathsf{N}) \mid \ker(T_{m,f}) = \mathsf{K} \Big\}.$$

Let $\mathsf{N}^{(1)}, \mathsf{N}^{(2)}, \mathsf{N}^{(3)} \in \mathrm{Ob}(\mathsf{GTSh})$ and

$$\mathsf{N}^{(3)} \xrightarrow{[m_2, f_2]} \mathsf{N}^{(2)} \xrightarrow{[m_1, f_1]} \mathsf{N}^{(1)}.$$

The composition of morphisms is defined by the formula:

$$[m_1, f_1] \circ [m_2, f_2] := \big[\, 2 m_1 m_2 + m_1 + m_2 \,,\ f_1 E_{m_1, f_1}(f_2) \,\big].$$

$[0, 1_{\mathsf{F}_2}]$ is the identity morphism in $\mathsf{GTSh}(\mathsf{N}, \mathsf{N})$.

$E_{m,f}$ is ...

For $(m, f) \in \mathbb{Z} \times \mathrm{F}_2$, $E_{m,f} \in \mathrm{End}(\mathrm{F}_2)$ defined by the formulas:

$$E_{m,f}(x) := x^{2m+1}, \qquad E_{m,f}(y) := f^{-1} y^{2m+1} f\,.$$

A similar (continuous) endomorphism of $\widehat{\mathrm{F}}_2$

$$E_{\hat{m},\hat{f}}(x) := x^{2\hat{m}+1}, \qquad E_{\hat{m},\hat{f}}(y) := \hat{f}^{-1} y^{2\hat{m}+1} \hat{f}$$

is used in the definition of the multiplication on $\widehat{\mathrm{GT}}$.

Basic facts about GTSh

Since F_2 is residually finite, the groupoid GTSh has infinitely many objects. However, GTSh is highly disconnected. Indeed, if $F_2/K \ncong F_2/N$, then the set $\mathrm{GTSh}(K, N)$ is empty.

For an object N, we denote by $\mathrm{GTSh}_{\mathrm{conn}}(N)$ the connected component of N in the groupoid GTSh. It is not hard to see that $\mathrm{GTSh}_{\mathrm{conn}}(N)$ is a finite groupoid for every N.

If N is the only object of $\mathrm{GTSh}_{\mathrm{conn}}(N)$, then we say that N is *isolated*. Note that N is isolated $\iff$ $\mathrm{GT}(N) = \mathrm{GTSh}(N, N)$. In this case, $\mathrm{GT}(N)$ is a (finite) group.

In the remainder of my talk, let us focus mostly on isolated objects of GTSh.

GT-shadows as approximations of elements of $\widehat{\mathsf{GT}}$

For every $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}$ and $\mathsf{N} \in \mathsf{Ob}(\mathsf{GTSh})$ the pair

$$\mathsf{PR}_{\mathsf{N}}(\hat{m}, \hat{f}) := \big(\mathcal{P}_{N_{\mathrm{ord}}}(\,\hat{m}\,)\,,\, \mathcal{P}_{\mathsf{N}}(\,\hat{f}\,) \big) \;\in\; \mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathsf{F}_2/\mathsf{N}$$

is a GT-shadow with the target N. (For $\mathsf{K} \trianglelefteq_{f.i.} G$, $\mathcal{P}_{\mathsf{K}}$ denotes the standard continuous homomorphism $\widehat{G} \to G/\mathsf{K}$.) $\mathsf{PR}_{\mathsf{N}}(\hat{m}, \hat{f})$ is an *approximation* of the element $(\hat{m}, \hat{f})$.

If $\mathsf{H}, \mathsf{N} \in \mathsf{Ob}(\mathsf{GTSh})$ and $\mathsf{H} \subsetneq \mathsf{N}$, then $\mathsf{PR}_{\mathsf{H}}(\hat{m}, \hat{f})$ is a “better” approximation of $(\hat{m}, \hat{f})$ than $\mathsf{PR}_{\mathsf{N}}(\hat{m}, \hat{f})$.

It is not hard to see that, for every isolated object N of GTSh, the map

$$\mathsf{PR}_{\mathsf{N}} : \widehat{\mathsf{GT}} \to \mathsf{GT}(\mathsf{N})$$

is a group homomorphism. GT-shadows that belong to $\mathsf{PR}_{\mathsf{N}}(\widehat{\mathsf{GT}})$ are called *genuine*. GT-shadows that do not belong to $\mathsf{PR}_{\mathsf{N}}(\widehat{\mathsf{GT}})$ are called *fake*.

$G_{\mathbb{Q}}$, $\widehat{\mathsf{GT}}$, GT(N) and . . .

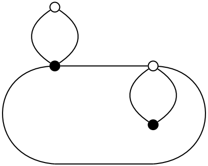
Let N be an isolated object of GTSh. Composing the homomorphism $\mathrm{PR}_{\mathrm{N}} : \widehat{\mathsf{GT}} \to \mathsf{GT}(\mathrm{N})$ with $Ih : G_{\mathbb{Q}} \to \widehat{\mathsf{GT}}$, we get the homomorphism $Ih_{\mathrm{N}} : G_{\mathbb{Q}} \to \mathsf{GT}(\mathrm{N})$.
If H, N are isolated objects of GTSh and $\mathrm{H} \subset \mathrm{N}$, then we have the natural homomorphism $\mathcal{R}_{\mathrm{H},\mathrm{N}} : \mathsf{GT}(\mathrm{H}) \to \mathsf{GT}(\mathrm{N})$. We call $\mathcal{R}_{\mathrm{H},\mathrm{N}}$ the *reduction homomorphism*. In this set-up, the following diagram commutes:

$$\begin{tikzcd}
 & G_{\mathbb{Q}} \arrow[d, "Ih"] \arrow[ddl, bend right, "Ih_{\mathrm{H}}"'] \arrow[ddr, bend left, "Ih_{\mathrm{N}}"] & \\
 & \widehat{\mathsf{GT}} \arrow[dl, "\mathrm{PR}_{\mathrm{H}}"'] \arrow[dr, "\mathrm{PR}_{\mathrm{N}}"] & \\
\mathsf{GT}(\mathrm{H}) \arrow[rr, "\mathcal{R}_{\mathrm{H},\mathrm{N}}"] & & \mathsf{GT}(\mathrm{N})
\end{tikzcd}$$

GT-shadows that belong to the image of Ih_{N} are called *arithmetical*.

A child's drawing of degree d is ...

An isom. class of a connected bipartite ribbon graph with d edges.



An equiv. class of a pair (g_1, g_2) of permutations in S_d for which the group $\langle\, g_1, g_2 \,\rangle$ acts transitively on $\{1, 2, \dots, d\}$.

An S_d-conjugacy class $[\psi]$ of a group homomorphism $\psi : \mathrm{F}_2 \to S_d$ with the subgroup $\psi(\mathrm{F}_2)$ being transitive.

An isom. class of a degree d connected covering of $\mathbb{CP}^1 - \{0, 1, \infty\}$.

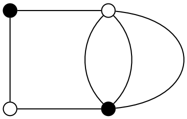
An equiv. class of a (non-constant) holomorphic map $f : \Sigma \to \mathbb{CP}^1$ from a compact connected Riemann surface (without boundary) that does not have branch points above every $w \in \mathbb{CP}^1 - \{0, 1, \infty\}$.

The action of $G_{\mathbb{Q}}$ on child's drawings

Given a child's drawing D, we can find a smooth projective curve X defined over $\overline{\mathbb{Q}}$ and an algebraic map $\beta : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ that does not have branch points above every $w \in \mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}$. (X, β) is called a *Belyi pair* corresponding to D.

The coefficients defining the curve X and the map β lie in some finite Galois extension E of $\mathbb{Q}$. Given any $g \in \mathrm{Gal}(E/\mathbb{Q})$, the child's drawing $g(D)$ is the one corresponding to the new Belyi pair $(g(X), g(\beta))$. We simply act by g on the coefficients defining X and β!

The $G_{\mathbb{Q}}$-orbit of the above child's drawing has two elements. It's 'Galois conjugate' is



Basic invariants of child's drawings

The conjugacy class of the subgroup $\psi(\mathrm{F}_2) \le S_d$ is call the *monodromy group* of $[\psi]$.

For a child's drawing represented by $(g_1, g_2) \in S_d \times S_d$, its *passport* is the triple of partitions $\big(\mathrm{ct}(g_1), \mathrm{ct}(g_2), \mathrm{ct}(g_2^{-1} g_1^{-1})\big)$ of d, where $\mathrm{ct}(h)$ denotes the cycle type of a permutation $h \in S_d$.

The *cartographic group* and more...

Since the braid group B_3 acts on F_2, B_3 acts naturally on the set of homomorphisms $\mathrm{F}_2 \to S_d$. Hence B_3 acts on child's drawings. The pure braid group $\mathrm{PB}_3 \le \mathrm{B}_3$ acts on child's drawings trivially. Thus we get an action of $S_3 \cong \mathrm{B}_3/\mathrm{PB}_3$ on child's drawings. One can prove that the resulting S_3 action on child's drawings coincides with the one induced by the Moebius transformations of $\mathbb{P}^1$:

$$t \mapsto 1 - t, \qquad t \to \frac{1}{1-t}.$$

Thus the action of $G_{\mathbb{Q}}$ commutes with the action of S_3.

The action of $\widehat{\mathsf{GT}}$ on child's drawings

Let $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}$ and D be a child's drawing. It is convenient to represent D by a group homomorphism

$$\psi : \mathsf{F}_2 \to S_d \,,$$

where the subgroup $\psi(\mathsf{F}_2)$ acts transitively on $\{1, 2, \dots, d\}$.

ψ extends uniquely to a (continuous) group homomorphism

$$\hat{\psi} : \widehat{\mathsf{F}}_2 \to S_d \,.$$

The child's drawing $D^{(\hat{m},\hat{f})}$ corresponds to the group homomorphism

$$\hat{\psi} \circ E_{\hat{m},\hat{f}}\big|_{\mathsf{F}_2} : \mathsf{F}_2 \to S_d \,,$$

where

$$E_{\hat{m},\hat{f}}(x) := x^{2\hat{m}+1} \,, \qquad E_{\hat{m},\hat{f}}(y) := \hat{f}^{-1}\, y^{2\hat{m}+1} \hat{f}.$$

The action of GT-shadows on child's drawings

Let N be an isolated object of GTSh and $\psi : \mathsf{F}_2 \to S_d$ be a homomorphism that represents a child's drawing D. We say that D is *subordinate* to N if $\mathsf{N} \subset \ker(\psi)$. Dessin(N) denotes the set of child's drawings subordinate to N.

For every child's drawing D, we denote by J_D the subposet of objects N of GTSh such that $D \in$ Dessin(N). One can show that the poset J_D is infinite and it contains the unique largest element N_D. The element N_D can be constructed explicitly.

Theorem. (V.D., 2021) Let N be an isolated object of GTSh and $\psi : \mathsf{F}_2 \to S_d$ be a homomorphism that represents $D \in$ Dessin(N). Let $[m, f] \in$ GT(N) and $D^{[m,f]}$ be the child's drawing represented by the homomorphism:

$$\tilde{\psi}(x) := \psi(x^{2m+1}), \qquad \tilde{\psi}(y) := \psi(f^{-1} y^{2m+1} f).$$

Then $D^{[m,f]} \in$ Dessin(N) and the assignment $D \mapsto D^{[m,f]}$ gives us a right action of the finite group GT(N) on the set Dessin(N).

All these actions are compatible

Theorem. Let H, N be isolated objects of GTSh such that $\mathsf{H} \subset \mathsf{N}$ and $\mathcal{R}_{\mathsf{H},\mathsf{N}}$ be the reduction homomorphism $\mathsf{GT}(\mathsf{H}) \to \mathsf{GT}(\mathsf{N})$. Then the following diagram commutes:

$$\begin{tikzcd}
G_{\mathbb{Q}} \arrow[r, "Ih"] \arrow[dr] & \widehat{\mathsf{GT}} \arrow[r, "\mathsf{PR}_{\mathsf{H}}"] \arrow[d] & \mathsf{GT}(\mathsf{H}) \arrow[r, "\mathcal{R}_{\mathsf{H},\mathsf{N}}"] \arrow[dl] & \mathsf{GT}(\mathsf{N}) \arrow[dll] \\
 & S_{\mathsf{Dessin}(\mathsf{N})} & &
\end{tikzcd}$$

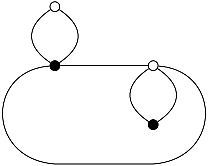
Hence, for every $D \in \mathsf{Dessin}(\mathsf{N})$, we have

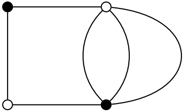
$$\mathsf{GT}(\mathsf{N})(D) \supset \mathsf{GT}(\mathsf{H})(D) \supset \cdots \supset \widehat{\mathsf{GT}}(D) \supset G_{\mathbb{Q}}(D).$$

Properties of the action of GTSh

- The action of GTSh on child's drawings is compatible with the action of S_3. Hence the passport of a child's drawing is invariant with respect to the GTSh-action.
- The GTSh-action is compatible with the partial order on the set of child's drawings. (We say that $[\tilde{H}] \le [H]$ if $\exists\ w \in \mathsf{F}_2$ such that $\tilde{H} \le w\,H\,w^{-1}$, i.e the child's drawing $[\tilde{H}]$ "covers" $[H]$.)
- If a child's drawing $D \in$ Dessin(N) is Galois, then so is $D^{[m,f]}$ for every $[m, f] \in$ GT(N).
- The GTSh-action commutes with the operation of taking the Galois (normal) closure of a child's drawing.
- If a child's drawing $[\psi] \in$ Dessin(N) is abelian (i.e. the monodromy group $\psi(\mathsf{F}_2)$ is abelian), then the orbit GT(N)$([\psi])$ is a singleton.

An example of degree 6 and genus 0





$D_{6,0}$ admits a Belyi pair defined over $\mathbb{Q}[\sqrt{3}]$. Let N be the “largest” object of GTSh that dominates $D_{6,0}$. Then

$$|\mathrm{F}_2 : \mathrm{N}| = 11664 = 2^4 \cdot 3^6 \,, \qquad N_{\mathrm{ord}} = 4.$$

GT(N) has 32 elements and N is an isolated object of GTSh. In particular, GT(N) is a group.
The GT-shadow $[1, yx^2y^2xyx] \in \mathrm{GT}(\mathrm{N})$ transforms the child’s drawing $D_{6,0}$ to its Galois conjugate $D^*_{6,0}$.

An example of degree 15 and genus 4

The child's drawing $D_{15,4}$ is represented by the permutation pair:

$$\Big((1,6,5,4,3,2)(7,12,11,10,9,8)(13,15,14),$$

$$(1,15,9,12,6,2)(3,13,7)(4,10,8,5,14,11)\Big).$$

The monodromy group of $D_{15,4}$ is isomorphic to A_7 and the kernel N of the corresponding homomorphism $\psi : \mathsf{F}_2 \to S_{15}$ is B_3-invariant. Thus N is the "largest" object of GTSh that dominates $D_{15,4}$. $N_{\mathrm{ord}} = 6$.
N is an isolated object of GTSh and the group GT(N) has order 48. The orbit GT(N)($D_{15,4}$) has two elements $D_{15,4}$ and $D^*_{15,4}$ and $D^{[-1,1]}_{15,4} = D^*_{15,4}$. Since $[-1,1]$ is the image of the complex conjugation, $\mathrm{GT}(\mathrm{N})(D_{15,4}) = G_{\mathbb{Q}}(D_{15,4})$.

Note that the size of the passport of $D_{15,4}$ is > 260.

Questions for further exploration

Jointly with I. Bortnovskyi, B. Holikov and V. Pashkovskyi, we found an infinite family of (isolated) objects K of GTSh such that the group GT(K) is non-abelian and every GT-shadow with the target K is arithmetical. It would be good to find other examples N of such objects of GTSh and use them to compute Galois orbits of child's drawings subordinate to N.

Let D be a child's drawing. It is not hard to show that there exists an object N of GTSh such that $D \in \mathrm{Dessin}(\mathsf{N})$ and $\mathrm{GT}(\mathsf{N})(D) = \widehat{\mathrm{GT}}(D)$. Is there a procedure that allows us to construct such N for a given child's drawing? Can we find such a procedure that works for a class of interesting child's drawings?

Using *dynamical Belyi maps*, one can produce many examples of invariants of the Galois action on child's drawings. Are these functions invariants of the GTSh-action on child's drawings?

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More References?!... Sure!

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THANK YOU!