A pleasant surprise of the cyclotomic character

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Loosely based on a joint work with Ivan Bortnovskyi, Borys Holikov and Vadym Pashkovskyi

The absolute Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

We denote by $G_{\mathbb{Q}}$ the absolute Galois group of the field $\mathbb{Q}$ of rational numbers, i.e. $G_{\mathbb{Q}}$ is the group of all automorphisms of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$.

One can show that $G_{\mathbb{Q}}$ is the limit of the functor that sends a finite Galois extension $E \supset \mathbb{Q}$ to the corresponding finite group $\mathrm{Gal}(E/\mathbb{Q})$. Thus $G_{\mathbb{Q}}$ is a profinite group. It is uncountable and $G_{\mathbb{Q}}$ is *not* topologically finitely generated.

Only two elements of $G_{\mathbb{Q}}$ *are known explicitly*: the identity element and the complex conjugation $c^*(a + bi) := a - bi$.

Using the action of $G_{\mathbb{Q}}$ on all roots unity, we get the natural group homomorphism

$$\chi : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^{\times}$$

called the *cyclotomic character*. It is not hard to show that the homomorphism χ is surjective.

The (gentle version of) the group $\widehat{\mathsf{GT}}$

Let $\mathsf{F}_2 := \langle\, x, y \,\rangle$ be the free group on two generators and $\widehat{\mathsf{F}}_2$ be its profinite completion.

In 1990, V. Drinfeld introduced a rather mysterious group $\widehat{\mathsf{GT}}$ (the Grothendieck-Teichmuellller group). As a set, $\widehat{\mathsf{GT}}$ consists of pairs $(\hat{m}, \hat{f})$ in $\widehat{\mathbb{Z}} \times \widehat{\mathsf{F}}_2$ satisfying

$$\hat{f}\theta(\hat{f}) = 1_{\widehat{\mathsf{F}}_2}, \qquad \tau^2(y^{\hat{m}}\hat{f})\tau(y^{\hat{m}}\hat{f})y^{\hat{m}}\hat{f} = 1_{\widehat{\mathsf{F}}_2},$$

$\hat{f} \in [\widehat{\mathsf{F}}_2, \widehat{\mathsf{F}}_2]^{top.clos.}$ and the invertibility condition.

Here θ and τ are the automorphisms of F_2 (and of $\widehat{\mathsf{F}}_2$) defined by the formulas $\theta(x) := y$, $\theta(y) := x$, $\tau(x) := y$, $\tau(y) := y^{-1}x^{-1}$.

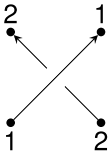
The multiplication on $\widehat{\mathsf{GT}}$ is defined using a monoid structure on $\widehat{\mathbb{Z}} \times \widehat{\mathsf{F}}_2$ that is inspired by the action of $G_{\mathbb{Q}}$ on $\widehat{\mathsf{F}}_2 \cong \pi_1^{alg}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\})$. The pair $(0_{\widehat{\mathbb{Z}}}, 1_{\widehat{\mathsf{F}}_2})$ is the identity element of $\widehat{\mathsf{GT}}$.

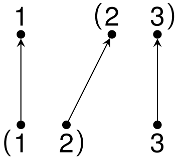
The operad PaB of parenthesized braid

Let B_n be the Artin braid group on n strands and PB_n be the kernel of the standard homomorphism $B_n \to S_n$. PB_n is called the pure braid group on n strands.

One can "assemble" the family $(B_n)_{n \ge 1}$ into an operad PaB in the category of groupoids. PaB is called the operad of parenthesized braids.

Due to MacLane's coherence theorem, PaB is generated by these two morphisms:

$\beta :=$ 

$\alpha :=$ 

Thus every $\varphi \in \mathrm{Aut}(\mathrm{PaB})$ is uniquely determined by $\varphi(\beta)$ and $\varphi(\alpha)$.

$\widehat{\text{GT}}$ is the group of automorphisms of $\widehat{\text{PaB}}$

In his 1990 paper, V. Drinfeld showed that Aut(PaB) $\cong \mathbb{Z}_2$:-(

If we replace PaB by its profinite completion $\widehat{\text{PaB}}$ then the story is much more interesting!

The group Aut($\widehat{\text{PaB}}$) of continuous automorphisms of $\widehat{\text{PaB}}$ is infinite and we have an injective homomorphism $G_{\mathbb{Q}} \to$ Aut($\widehat{\text{PaB}}$). Aut($\widehat{\text{PaB}}$) is (the original version of) the Grothendieck-Teichmueller group.
For every $\varphi \in$ Aut($\widehat{\text{PaB}}$), the value $\varphi(\beta)$ (resp. $\varphi(\alpha)$) is uniquely determined by an element $\hat{m} \in \widehat{\mathbb{Z}} \cong \widehat{\text{PB}}_2$ (resp. an element $\hat{f} \in \text{P}\hat{\text{B}}_3$).

Using the relations of PaB one can show that $\hat{f}$ belongs to a subgroup of $\text{P}\hat{\text{B}}_3$ isomorphic to $\widehat{\text{F}}_2$. Moreover, $2\hat{m} + 1$ must be invertible in the ring $\widehat{\mathbb{Z}}$. This is how we get a bijection between pairs $(\hat{m}, \hat{f}) \in \widehat{\mathbb{Z}} \times \widehat{\text{F}}_2$ satisfying various conditions and automorphisms of $\widehat{\text{PaB}}$.

Another interpretation of the defining relations of $\widehat{\mathsf{GT}}$

Let C be a group acting on a nonabelian group G. One can define the sets

$$H^0(C, G) \qquad \text{and} \qquad H^1(C, G).$$

$H^0(C, G)$ is simply the set of C-invariant elements in G and $H^1(C, G)$ is the set of equivalence classes of splittings of the exact sequence

$$1 \to G \to G \rtimes C \to C \to 1$$

In their 1997 paper, P. Lochak and L. Schneps suggested an interpretation of the defining relations of $\widehat{\mathsf{GT}}$ in terms of $H^1(C, \widehat{F}_2)$ with $C = \mathbb{Z}_2$ and $C = \mathbb{Z}_3$.

They successfully used this idea to prove several interesting statements about $\widehat{\mathsf{GT}}$ and about the image of $G_{\mathbb{Q}}$ in $\widehat{\mathsf{GT}}$.
In my opinion, natural version of this idea for GT-shadows is not fully explored.

The Ihara embedding

In his 1994 paper "On the embedding of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) into $\widehat{\mathrm{GT}}$" ($+$ the appendix by M. Emsalem and P. Lochak), Y. Ihara used the algebraic fundamental groups of $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ and $\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}$ to construct a map

$$Ih : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^{\times} \times \widehat{F}_2$$

of the form $Ih(g) = (\chi(g), f_g)$, where χ denotes the cyclotomic character.

Using the appropriate versions of the fundamental groupoids of the moduli spaces $\mathcal{M}_{0,4}$, $\mathcal{M}_{0,5}$ of curves, Ihara proved (in ICM 1990) that, for every $g \in G_{\mathbb{Q}}$, the pair $((\chi(g) - 1)/2, f_g)$ is an element $\widehat{\mathrm{GT}}$. In particular, $f_g \in [\widehat{\mathrm{F}}_2, \widehat{\mathrm{F}}_2]^{top.\ cl.}$.

Using famous Belyi's theorem, one can prove that the resulting group homomorphism $Ih : G_{\mathbb{Q}} \to \widehat{\mathrm{GT}}$ is injective. We call Ih the *Ihara embedding*. It is known that the pair $(-1, 1) \in \widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$ equals $Ih(c^*)$, where c^* denotes the complex conjugation.

Is Ih surjective?

The following question is probably *very hard:*

$$\textit{Is the homomorphism } Ih : G_{\mathbb{Q}} \to \widehat{\mathsf{GT}} \textit{ surjective?}$$

In several remarkable papers, F. Pop gave positive answers to versions of the above question. In these versions, $\widehat{\mathsf{GT}}$ is replaced by subgroups of $\widehat{\mathsf{GT}}$ with infinitely many defining conditions.

For example, the birational version $\widehat{\mathsf{GT}}_{bir}$ of $\widehat{\mathsf{GT}}$ is defined using the etale fundamental group functor from the sub-category of concrete algebraic varieties obtained from $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$. In "*Finite tripod variants of I/OM: ...*", 2019, F. Pop proved that the homomorphism Ih lands in $\widehat{\mathsf{GT}}_{bir}$ and the group $\widehat{\mathsf{GT}}_{bir}$ is isomorphic to $G_{\mathbb{Q}}$ via Ih.

A bit more about (the gentle version of) $\widehat{\mathrm{GT}}$

For $(\hat{m}, \hat{f}) \in \widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$, the formulas

$$E_{\hat{m},\hat{f}}(x) := x^{2\hat{m}+1}, \qquad E_{\hat{m},\hat{f}}(y) := \hat{f}^{-1} y^{2\hat{m}+1} \hat{f}$$

define a continuous endomorphism $E_{\hat{m},\hat{f}}$ of $\widehat{\mathrm{F}}_2$.

$\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$ is a monoid with the binary operation

$$(\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2) := \big(2\hat{m}_1 \hat{m}_2 + \hat{m}_1 + \hat{m}_2, \ \hat{f}_1 E_{\hat{m}_1, \hat{f}_1}(\hat{f}_2) \big)$$

and the identity element $(0, 1)$.

Let $\widehat{\mathrm{GT}}_{mon}$ be the submonoid of $\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$ that consists of pairs $(\hat{m}, \hat{f})$ satisfying the cocycle conditions:

$$\hat{f}\theta(\hat{f}) = 1_{\widehat{\mathrm{F}}_2}, \qquad \tau^2(y^{\hat{m}}\hat{f})\tau(y^{\hat{m}}\hat{f})y^{\hat{m}}\hat{f} = 1_{\widehat{\mathrm{F}}_2},$$

and $\hat{f} \in [\widehat{\mathrm{F}}_2, \widehat{\mathrm{F}}_2]^{top.clos.}$.

$\widehat{\mathsf{GT}}_{gen}$ is the group of invertible elements of the monoid $\widehat{\mathsf{GT}}_{mon}$. The formula $\chi_{vir}(\hat{m}, \hat{f}) := 2\hat{m} + 1$ defines a (continuous) group homomorphism $\chi_{vir} : \widehat{\mathsf{GT}}_{gen} \to \widehat{\mathbb{Z}}^{\times}$. Since the diagram

$$\begin{array}{ccc} G_{\mathbb{Q}} & \xrightarrow{Ih} & \widehat{\mathsf{GT}}_{gen} \\ & {\scriptstyle \chi} \searrow \quad \swarrow {\scriptstyle \chi_{vir}} & \\ & \widehat{\mathbb{Z}}^{\times} & \end{array}$$

commutes, we call χ_{vir} the *virtual cyclotomic character*.

For every $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}_{gen}$, the endomorphism $E_{\hat{m},\hat{f}}$ of $\widehat{\mathsf{F}}_2$ is invertible and the assignment

$$(\hat{m}, \hat{f}) \mapsto E_{\hat{m},\hat{f}}$$

defines a group homomorphism from $\widehat{\mathsf{GT}}_{gen}$ to the group $\mathrm{Aut}(\widehat{\mathsf{F}}_2)$ of continuous automorphisms of $\widehat{\mathsf{F}}_2$.

The Artin braid group B_3 and PB_3

B_3 (resp. PB_3) denotes the Artin braid group (resp. the pure braid group) on 3 strands. σ_1, σ_2 are the standard generators of B_3

We set $\Delta := \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

PB_3 is generated by

$$x_{12} := \sigma_1^2, \qquad x_{23} := \sigma_2^2, \qquad c := \Delta^2 .$$

It is known that $\mathcal{Z}(B_3) = \mathcal{Z}(PB_3) = \langle\, c \,\rangle \cong \mathbb{Z}$, the subgroup $\langle\, x_{12}, x_{23} \,\rangle$ is isomorphic to F_2. In fact, $PB_3 \cong F_2 \times \langle\, c \,\rangle$.

A bit more about F_2, PB_3 and B_3

It is natural to identify F_2 with the quotient group $PB_3/\mathcal{Z}(PB_3)$ and set

$$x := x_{12}\,\mathcal{Z}(PB_3), \qquad y := x_{23}\,\mathcal{Z}(PB_3).$$

Since $\mathcal{Z}(B_3) = \mathcal{Z}(PB_3)$, the group B_3 acts on $F_2 \cong PB_3/\mathcal{Z}(PB_3)$ by conjugation. We denote by θ (resp. τ) the automorphism of F_2 corresponding to $\Delta := \sigma_1\sigma_2\sigma_1$ (resp. to $\sigma_1\sigma_2$).

It is easy to see that

$$\theta(x) := y, \quad \theta(y) := x, \qquad \tau(x) := y, \quad \tau(y) := y^{-1}x^{-1}.$$

Although the elements Δ and $\sigma_1\sigma_2$ are of infinite order, the automorphisms θ and τ have finite orders: $\mathrm{ord}(\theta) = 2$, $\mathrm{ord}(\tau) = 3$. We set

$$\mathrm{NFI}^{B_3}(F_2) := \{\, N \trianglelefteq F_2 \mid g(N) = N, \ \ \forall\, g \in B_3 \ \ |F_2 : N| < \infty \,\}$$

and we often abbreviate $\mathrm{NFI} := \mathrm{NFI}^{B_3}(F_2)$.

Preparation

For $\mathsf{N} \in \mathsf{NFI}$, we set

$$N_{\text{ord}} := \text{lcm}\big(\text{ord}(x\mathsf{N}), \text{ord}(y\mathsf{N})\big).$$

We say that $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$ satisfies the *cocycle conditions* modulo N if

$$f\theta(f) \in \mathsf{N}, \qquad \tau^2(y^m f)\tau(y^m f)y^m f \in \mathsf{N}.$$

For $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$ and $\mathsf{N} \in \mathsf{NFI}$, we denote by $T_{m,f}$ the following homomorphism

$$T_{m,f} : \mathsf{F}_2 \to \mathsf{F}_2/\mathsf{N}$$

If the pair $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$ satisfies the cocycle conditions modulo N, then $\ker(T_{m,f})$ is also B_3-invariant, hence

$$\ker(T_{m,f}) \in \mathsf{NFI}\,.$$

A GT-shadow is . . .

Definition

Let N $\in$ NFI. *A* GT*-shadow with the target* N *is a pair*

$$[m, f] := (m + N_{\text{ord}}\mathbb{Z}, f\mathrm{N}) \in \mathbb{Z}/N_{\text{ord}}\mathbb{Z} \times \mathrm{F}_2/\mathrm{N}$$

satisfying the cocycle conditions (modulo N*) and such that*

- $2m + 1$ *represents a unit in the ring* $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$*,*
- $f\mathrm{N} \in [\mathrm{F}_2/\mathrm{N}, \mathrm{F}_2/\mathrm{N}]$*, and*
- *the homomorphism* $T_{m,f} : \mathrm{F}_2 \to \mathrm{F}_2/\mathrm{N}$ *is surjective.*

GT(N) is the set of GT-shadows with the target N.

The groupoid GTSh

Guess what?!.... GT-shadows form a groupoid GTSh.

$$\mathrm{Ob(GTSh)} := \mathrm{NFI}; \quad \text{for} \quad \mathrm{K}, \mathrm{N} \in \mathrm{NFI},$$

$$\mathrm{GTSh(K,N)} := \Big\{ [m, f] \in \mathrm{GT(N)} \mid \ker(T_{m,f}) = \mathrm{K} \Big\}.$$

Let $\mathrm{N}^{(1)}, \mathrm{N}^{(2)}, \mathrm{N}^{(3)} \in \mathrm{NFI}$ and

$$\mathrm{N}^{(3)} \xrightarrow{[m_2, f_2]} \mathrm{N}^{(2)} \xrightarrow{[m_1, f_1]} \mathrm{N}^{(1)}.$$

The composition of morphisms is defined by the formula:

$$[m_1, f_1] \circ [m_2, f_2] := \big[\, 2m_1 m_2 + m_1 + m_2 \,,\ f_1 E_{m_1, f_1}(f_2) \,\big]$$

$\forall\, \mathrm{N} \in \mathrm{NFI}$, $[0, 1_{\mathrm{F}_2}]$ is the identity morphism in $\mathrm{GTSh(N,N)}$.

A comment

For $(m, f) \in \mathbb{Z} \times \mathrm{F}_2$, the formulas

$$E_{m,f}(x) := x^{2m+1}, \qquad E_{m,f}(y) := f^{-1} y^{2m+1} f$$

define an endomorphism of F_2.

Moreover, for all $(m_1, f_1), (m_2, f_2) \in \mathbb{Z} \times \mathrm{F}_2$,

$$E_{m_1,f_1} \circ E_{m_2,f_2} = E_{m,f},$$

where $m := 2m_1 m_2 + m_1 + m_2$ and $f := f_1 E_{m_1,f_1}(f_2)$.

One can show that the set $\mathbb{Z} \times \mathrm{F}_2$ is a monoid with respect to the binary operation

$$(m_1, f_1) \bullet (m_2, f_2) := \big(2m_1 m_2 + m_1 + m_2\,,\ f_1 E_{m_1,f_1}(f_2)\big)$$

with $(0, 1_{\mathrm{F}_2})$ being the identity element.

Basic facts about GTSh

- GTSh has infinitely many objects. (NFI is infinite because F_2 is residually finite.)
- GTSh is highly disconnected. However, for every $N \in \mathrm{NFI}$, the connected component $\mathrm{GTSh}_{\mathrm{conn}}(N)$ of N is a finite groupoid.
- If $\mathrm{GTSh}_{\mathrm{conn}}(N)$ has only one object, then $\mathrm{GT}(N) = \mathrm{GTSh}(N, N)$, i.e. GT(N) is a (finite) group. In this case, we say that N is an *isolated* object of GTSh.
- For every $N \in \mathrm{NFI}$, the object

$$N^{\diamond} := \bigcap_{K \in \mathrm{Ob}(\mathrm{GTSh}_{\mathrm{conn}}(N))} K$$

is isolated. In particular, the subposet $\mathrm{NFI}^{isol.} \subset \mathrm{NFI}$ of isolated objects is coinitial.

The reduction map

Let $\mathsf{N}, \mathsf{H} \in \mathsf{NFI}$ with $\mathsf{N} \le \mathsf{H}$. Then $H_{\mathrm{ord}} \mid N_{\mathrm{ord}}$.

If a pair $(m, f) \in \mathbb{Z} \times \mathrm{F}_2$ represents a GT-shadow with the target N, then *the same pair* also represents a GT-shadow with the target H.

Hence we have a natural map

$$\mathcal{R}_{\mathsf{N},\mathsf{H}} : \mathsf{GT}(\mathsf{N}) \to \mathsf{GT}(\mathsf{H})$$

If N, H are isolated (i.e. $\mathsf{GT}(\mathsf{N}), \mathsf{GT}(\mathsf{H})$ are groups) then $\mathcal{R}_{\mathsf{N},\mathsf{H}}$ is a group homomorphism.

$\widehat{\mathrm{GT}}$ versus GTSh

For every $(\hat{m}, \hat{f}) \in \widehat{\mathrm{GT}}$ and $\mathsf{N} \in \mathsf{NFI}$ the pair

$$\mathsf{PR}_{\mathsf{N}}(\hat{m}, \hat{f}) := \big(\mathcal{P}_{N_{\mathrm{ord}}}(\, \hat{m}\,) \,,\, \mathcal{P}_{\mathsf{N}}(\, \hat{f}\,) \big) \;\in\; \mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}$$

is a GT-shadow with the target N. (For $\mathsf{K} \in \mathsf{NFI}(G)$, $\mathcal{P}_{\mathsf{K}}$ denotes the standard continuous homomorphism $\widehat{G} \to G/\mathsf{K}$.) $\mathsf{PR}_{\mathsf{N}}(\hat{m}, \hat{f})$ is an *approximation* of the element $(\hat{m}, \hat{f})$.

A GT-shadow $[m, f] \in \mathsf{GT}(\mathsf{N})$ is called *genuine* if $\exists\ (\hat{m}, \hat{f}) \in \widehat{\mathrm{GT}}$ such that $\mathsf{PR}_{\mathsf{N}}(\hat{m}, \hat{f}) = [m, f]$. Otherwise, it is called *fake*.

A GT-shadow $[m, f] \in \mathsf{GT}(\mathsf{N})$ *survives into* $\mathsf{K} \in \mathsf{NFI}$ (with $\mathsf{K} \le \mathsf{N}$) if $[m, f] \in \mathcal{R}_{\mathsf{K},\mathsf{N}}(\mathsf{GT}(\mathsf{K}))$.

Proposition. A GT-shadow $[m, f] \in \mathsf{GT}(\mathsf{N})$ is genuine $\iff$ $[m, f]$ survives into K for every $\mathsf{K} \in \mathsf{NFI}$ such that $\mathsf{K} \le \mathsf{N}$.

The approximation functor PR

If a GT-shadow $[m, f] \in \mathsf{GT}(\mathsf{N})$ comes from an element $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}$, then we denote by $\mathsf{N}^{(\hat{m},\hat{f})}$ the source of $[m, f]$. One can show that the assignment $\mathsf{N} \mapsto \mathsf{N}^{(\hat{m},\hat{f})}$ defines a *right action* of $\widehat{\mathsf{GT}}$ on the poset NFI. We denote by

$$\widehat{\mathsf{GT}}_{\mathsf{NFI}}$$

the corresponding transformation groupoid.

One can show that "passing from elements of $\widehat{\mathsf{GT}}$ to GT-shadows" gives us a functor

$$\mathsf{PR} : \widehat{\mathsf{GT}}_{\mathsf{NFI}} \to \mathsf{GTSh}\,.$$

Informally, we may call it the *approximation functor*.

The Main Line functor

Let $\mathsf{K}, \mathsf{N} \in \mathsf{NFI}$ be isolated objects of the groupoid GTSh and $\mathsf{K} \le \mathsf{N}$. Since $\mathcal{R}_{\mathsf{K},\mathsf{N}}$ is a group homomorphism

$$\mathsf{GT}(\mathsf{K}) \to \mathsf{GT}(\mathsf{N}),$$

the assignments

$$\mathsf{ML}(\mathsf{N}) := \mathsf{GT}(\mathsf{N}), \qquad \mathsf{ML}(\,\mathsf{K} \le \mathsf{N}\,) := \mathcal{R}_{\mathsf{K},\mathsf{N}}$$

define a functor from the poset $\mathsf{NFI}^{isol.}$ to the category of finite groups.

Theorem. (J. Guynee, V.D.) The limit of ML is isomorphic to (the gentle version of) $\widehat{\mathsf{GT}}$.

Proposition. (I. Bortnovskyi) For every $\mathsf{N} \in \mathsf{NFI}$, there exists $\mathsf{K} \in \mathsf{NFI}$ such that $\mathsf{K} \le \mathsf{N}$ with the following property: *if a* GT*-shadow* $[m, f] \in \mathsf{GT}(\mathsf{N})$ *survives into* K *then* $[m, f]$ *is genuine.*

The version of *Ih* for GT-shadows

Let N be an isolated object of the groupoid GTSh, i.e. N is the only objects of its connected component in GTSh. In particular, GT(N) is naturally a group.

Using the approximation functor, we get a natural group homomorphism

$$\mathrm{PR}_{\mathrm{N}} : \widehat{\mathrm{GT}} \to \mathrm{GT}(\mathrm{N}).$$

Precomposing PR_{N} with the Ihara embedding $Ih : G_{\mathbb{Q}} \to \widehat{\mathrm{GT}}$, we get the group homomorphism

$$Ih_{\mathrm{N}} : G_{\mathbb{Q}} \to \mathrm{GT}(\mathrm{N})$$

We say that a GT-shadow $[m, f] \in \mathrm{GT}(\mathrm{N})$ is *arithmetical* if $[m, f]$ belongs to the image of Ih_{N}. Clearly, every arithmetical GT-shadow is genuine. If there are genuine GT-shadows that are not arithmetical, then the Ihara embedding $Ih : G_{\mathbb{Q}} \to \widehat{\mathrm{GT}}$ is not surjective.

GT-shadows for the dihedral poset Dih

Let $n \in \mathbb{Z}_{\geq 3}$ and $D_n := \langle\, r, s \mid r^n, s^2, rsrs \,\rangle$ be the dihedral group of order $2n$. Let ψ_n be the following homomorphism $\mathsf{F}_2 \to D_n^3$

$$\psi_n(x) := (r, s, s), \qquad \psi_n(y) := (rs, r, rs)$$

and

$$\mathsf{K}^{(n)} := \ker(\mathsf{F}_2 \xrightarrow{\psi_n} D_n^3).$$

One can show that $\mathsf{K}^{(n)}$ is B_3-invariant, i.e. $\mathsf{K}^{(n)} \in \mathsf{NFI}$.

We call

$$\big\{\mathsf{K}^{(n)} \;:\; n \in \mathbb{Z}_{\geq 3}\big\} \;\subset\; \mathsf{NFI}$$

the *dihedral poset* of NFI. We denote this poset by Dih.

Results. Part 1

Jointly with I. Bortnovskyi, B. Holikov and V. Pashkovskyi, we proved the following:

Every $\mathsf{K} \in \mathsf{Dih}$ is an isolated object of GTSh, i.e. the connected component $\mathsf{GTSh}_{\mathrm{conn}}(\mathsf{K})$ is essentially the (finite) group $\mathsf{GT}(\mathsf{K})$.

If $\mathsf{K} \subset \mathsf{H}$ (for $\mathsf{K}, \mathsf{H} \in \mathsf{Dih}$), then the reduction homomorphism

$$\mathcal{R}_{\mathsf{K},\mathsf{H}} : \mathsf{GT}(\mathsf{K}) \to \mathsf{GT}(\mathsf{H})$$

is *surjective*.

For every $\mathsf{K} \in \mathsf{Dih}$, we gave a description of the finite group $\mathsf{GT}(\mathsf{K})$. For example, if $n = n_0 2^a$ (with n_0 odd and $a \geq 2$), then $\mathsf{GT}(\mathsf{K}^{(n)})$ is isomorphism to a concrete index 2 subgroup of the group:

$$\left(\mathbb{Z}/n_0\mathbb{Z} \rtimes (\mathbb{Z}/n_0\mathbb{Z})^{\times}\right) \times \left(\mathbb{Z}/2^{a-1}\mathbb{Z} \rtimes (\mathbb{Z}/2^{a+1}\mathbb{Z})^{\times}\right).$$

Results. Part 2

For every $\mathsf{K} \in \mathsf{Dih}$, we established a lower bound on the number of arithmetical GT-shadows with the target K. For $n = 2^a n_0 \geq 3$ with n_0 being odd, the number of arithmetical elements in $\mathsf{GT}(\mathsf{K}^{(n)})$ is greater or equal than

$$\begin{cases} 2\phi(n_0) & \text{if } \; a = 0 \text{ or } a = 1, \\ 2^{2a-2}\phi(n_0) & \text{if } \; a \geq 2. \end{cases}$$

In particular, for every $a \in \mathbb{Z}_{\geq 2}$, the group homomorphism $Ih_{\mathsf{K}^{(2^a)}} : G_{\mathbb{Q}} \to \mathsf{GT}(\mathsf{K}^{(2^a)})$ is surjective.

We considered the subposet $\mathsf{Dih}_2 := \{\mathsf{K}^{(2^a)} \mid a \geq 2\} \subset \mathsf{Dih}$ and described the limit

$$\mathsf{ML}\,\big|_{\mathsf{Dih}_2}$$

as a concrete index 2 subgroup of $\mathbb{Z}_2 \rtimes \mathbb{Z}_2^{\times}$.

Finally, ...

we proved that the composition

$$G_{\mathbb{Q}} \xrightarrow{Ih} \widehat{\mathsf{GT}} \to \lim \left(\mathsf{ML}\,\big|_{\mathsf{Dih}_2} \right)$$

is surjective. This way we produced the first example of a nonabelian (infinite) profinite quotient of $\widehat{\mathsf{GT}}$ that receives a surjective homomorphism from $G_{\mathbb{Q}}$.

Our proofs involve relatively elementary tools:

- basic properties of group homomorphisms;
- the surjectivity of the cyclotomic character $\chi : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^{\times}$;
- the image of the complex conjugation in $\widehat{\mathsf{GT}}$ is $(-1_{\widehat{\mathbb{Z}}}, 1_{\widehat{\mathsf{F}}_2})$;
- the fundamental theorem of arithmetic.

Further questions for exploration

It makes sense to explore other subposets $\mathcal{J}$ of the poset of isolated objects of GTSh. For $\mathcal{J} \subset \mathsf{NFI}^{B_3}(F_2)^{isol}$, we could try to...

- Give an explicit description of finite groups $\mathsf{GT}(\mathsf{N})$ for $\mathsf{N} \in \mathcal{J}$.
- Give an explicit description of the profinite group $\lim \left(\mathsf{ML}\big|_{\mathcal{J}} \right)$.
- Use the reduction maps or other tools (e.g. consequences of the Lochak-Schneps results from "A cohomological interpretation of ...", 1997) to find examples (if any) of fake GT-shadows.
- Find a lower bound on the number of arithmetical GT-shadows with a target N for $\mathsf{N} \in \mathcal{J}$.

Let $\mathsf{N} \in \mathsf{NFI}^{B_3}(F_2)$ such that F_2/N is metabelian. William Chen: *Is this true that every* GT*-shadow with the target* N *is arithmetical?*

It also makes sense to write a software package (e.g. using SageMath) for working with GT-shadows and their action on child's drawings.

Selected References

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What?!... Even more references?!

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THANK YOU!