# A Proof of Tsygan's Formality Conjecture for an Arbitrary Smooth Manifold

by

Vasiliy A. Dolgushev

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Author . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
Department of Mathematics
April 12, 2005

Certified by . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
Pavel Etingof
Associate Professor of Mathematics, MIT
Thesis Supervisor

Certified by . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
Dmitry Tamarkin
Assistant Professor of Mathematics, Northwestern University
Thesis Supervisor

Accepted by . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
Pavel Etingof
Chairman, Department Committee on Graduate Students

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## Abstract

Proofs of Tsygan's formality conjectures for chains would unlock important algebraic tools which might lead to new generalizations of the Atiyah-Patodi-Singer index theorem and the Riemann-Roch-Hirzebruch theorem. Despite this pivotal role in the traditional investigations and the efforts of various people the most general version of Tsygan's formality conjecture has not yet been proven. In my thesis I propose Fedosov resolutions for the Hochschild cohomological and homological complexes of the algebra of functions on an arbitrary smooth manifold. Using these resolutions together with Kontsevich's formality quasi-isomorphism for Hochschild cochains of $\mathbb{R}[[y^1, \dots, y^d]]$ and Shoikhet's formality quasi-isomorphism for Hochschild chains of $\mathbb{R}[[y^1, \dots, y^d]]$ I prove Tsygan's formality conjecture for Hochschild chains of the algebra of functions on an arbitrary smooth manifold. The construction of the formality quasi-isomorphism for Hochschild chains is manifestly functorial for isomorphisms of the pairs $(M, \nabla)$, where $M$ is the manifold and $\nabla$ is an affine connection on the tangent bundle. In my thesis I apply these results to equivariant quantization, computation of Hochschild homology of quantum algebras and description of traces in deformation quantization.

Thesis Supervisor: Pavel Etingof
Title: Associate Professor of Mathematics, MIT

Thesis Supervisor: Dmitry Tamarkin
Title: Assistant Professor of Mathematics, Northwestern University

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*To my brother Nickolay Dolgushev*

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# Chapter 1

# Introduction

Proofs of Tsygan's formality conjectures for chains [49, 50, 53] would unlock important algebraic tools which might lead to new generalizations of the Atiyah-Patodi-Singer index theorem and the Riemann-Roch-Hirzebruch theorem [1, 6, 23, 25, 35, 42, 43, 49]. Despite this pivotal role in traditional investigations and the efforts of various people [26, 46, 47, 49, 50] the most general version of Tsygan's formality conjecture [49] has resisted proof.

In my thesis I prove Tsygan's conjecture for Hochschild chains of the algebra of functions on an arbitrary smooth manifold $M$ using the globalization technique proposed in [13] and [19] and the formality quasi-isomorphism for Hochschild chains of $\mathbb{R}[[y^1, \dots y^d]]$ constructed by Shoikhet [46]. This result allows me to prove Tsygan's conjecture [53] about Hochschild homology of the quantum algebra of functions on an arbitrary Poisson manifold and to describe traces on this algebra.

The most general version of Tsygan's formality conjecture for chains says that a pair of spaces of Hochschild cochains and Hochschild chains of any associative algebra is endowed with the so-called $Calc_\infty$-structure and if the algebra in question is the algebra of functions on a smooth manifold then the corresponding $Calc_\infty$-structure is formal. This statement was announced in [17] and [49] but the proof has not yet been formulated.

In this context I would like to mention paper [26], in which the authors prove a statement closely related to the cyclic formality theorem. In particular, this assertion allows them to prove a generalization of Connes-Flato-Sternheimer conjecture [16] in the Poisson framework.

The structure of my thesis is as follows. In the next chapter I recall basic notions related to $L_\infty$- or the so-called homotopy Lie algebras. I introduce a notion of partial homotopy between $L_\infty$-morphisms and describe a useful technical tool that allows me to utilize Maurer-Cartan elements of differential graded Lie algebras (DGLA). In the third chapter I recall algebraic structures on Hochschild complexes of associative algebra and introduce the respective versions of these complexes for the algebra of functions on a smooth manifold. In this section I formulate the main result of my thesis (see theorem 1 on page 54) and recall Kontsevich's and Shoikhet's formality theorems for $\mathbb{R}^d$. The main part of this work concerns the construction of Fedosov resolutions of the algebras of polydifferential operators and polyvector fields, as well as the modules of Hochschild chains and exterior forms. These resolutions are constructed in chapter 4. Using Fedosov's resolutions in chapter 5, I prove theorem 1. In this chapter I also show that the Fedosov resolutions provide me with a simple functorial construction of Kontsevich's quasi-isomorphism from the DGLA of polyvector fields to the DGLA of polydifferential operators (see theorem 6 on page 87). At the end of chapter 5 I apply theorems 1 and 6 to equivariant quantization, computation of Hochschild homology of quantum algebras and description of traces in deformation quantization. In the concluding chapter I discuss recent works related to generalizations and applications of the formality theorems for Hochschild (co)chains.

My thesis is based on papers [18, 19].

**Notation.** Throughout this work I assume the summation over repeated indices. $M$ is a smooth real manifold of dimension $d$. The definition of antisymmetrization goes without any auxiliary factors. Thus,

$$v_1 \wedge v_2 = v_1 \otimes v_2 - (-)^{|v_1||v_2|} v_2 \otimes v_1 .$$

I assume the Koszul rule of signs which says that a transposition of any two vectors

$v_1$ and $v_2$ of degrees $k_1$ and $k_2$, respectively, yields the sign

$$(-1)^{k_1 k_2}\,.$$

"DGLA" always means a differential graded Lie algebra, while "DGA" means a differential graded associative algebra. The arrow $\succ\!\rightarrow$ denotes an $L_\infty$-morphism of $L_\infty$-algebras, the arrow $\succ\!\succ\!\rightarrow$ denotes a morphism of $L_\infty$-modules, and the notation

$$\mathcal{L}$$

$$\downarrow_{mod}$$

$$\mathcal{M}$$

means that $\mathcal{M}$ is an $L_\infty$-module over the $L_\infty$-algebra $\mathcal{L}$. $S_n$ denotes the symmetric group of permutations of $n$ elements and for natural numbers $k_1, \dots, k_q$, $k_1 + \dots + k_q = n$ $Sh(k_1, \dots, k_q) \subset S_n$ is the subset of $(k_1, \dots, k_q)$-shuffles. Namely,

$$Sh(k_1, \dots, k_q) =$$

$$\{\varepsilon \in S_n \,|\, \varepsilon(1) < \varepsilon(2) < \dots < \varepsilon(k_1)\,, \dots, \varepsilon(n-k_q+1) < \varepsilon(n-k_q+2) < \dots < \varepsilon(n)\}\,.$$

I omit the symbol $\wedge$ referring to a local basis of exterior forms, as if one thought of $dx^i$'s as anti-commuting variables. The symbol $\circ$ always stands for a composition of morphisms. I denote by $\mathrm{cxp}\,(x)$ the following function

$$\mathrm{cxp}\,(x) = e^x - 1\,.$$

Finally, I denote by $\Gamma(M, \mathcal{G})$ the vector space of smooth sections of the bundle $\mathcal{G}$ and by $\Omega^\bullet(M, \mathcal{G})$ the vector space of exterior forms with values in $\mathcal{G}$.

# Chapter 2

# $L_\infty$-structures

In this chapter I recall the notions of $L_\infty$-algebras, $L_\infty$-morphisms, $L_\infty$-modules and morphisms between $L_\infty$-modules. I introduce a notion of partial homotopy between $L_\infty$-morphisms and describe an important technical tool, which allows me to modify $L_\infty$-structures with the help of a Maurer-Cartan element. A more detailed discussion of this theory and its applications can be found in papers [24, 34, 39].

In this chapter all the vector spaces, $L_\infty$-algebras, and $L_\infty$-modules are considered over a field of characteristic zero.

## 2.1 $L_\infty$-algebras and $L_\infty$-morphisms

Let $\mathcal{L}$ be a $\mathbb{Z}$-graded vector space

$$\mathcal{L} = \bigoplus_{k \in \mathbb{Z}} \mathcal{L}^k \,. \tag{2.1}$$

I assume that the direct sum in the right hand side of (2.1) is bounded below. To the space $\mathcal{L}$ I assign a coassociative cocommutative coalgebra (without counit) $C(\mathcal{L})$ cofreely cogenerated by $\mathcal{L}$ with a shifted parity.

The vector space of $C(\mathcal{L})$ is the exterior algebra of $\mathcal{L}$

$$C(\mathcal{L}) = \bigwedge \mathcal{L} \,, \tag{2.2}$$

where the antisymmetrization is graded, that is for any $\gamma_1 \in \mathcal{L}^{k_1}$ and $\gamma_2 \in \mathcal{L}^{k_2}$

$$\gamma_1 \wedge \gamma_2 = -(-)^{k_1 k_2} \gamma_2 \wedge \gamma_1 \,.$$

The comultiplication

$$\Delta \,:\, C(\mathcal{L}) \mapsto C(\mathcal{L}) \bigwedge C(\mathcal{L}) \tag{2.3}$$

is defined by the formulas ($n > 1$)

$$\Delta(\gamma_1) = 0 \,,$$

$$\Delta(\gamma_1 \wedge \cdots \wedge \gamma_n) = \sum_{k=1}^{n-1} \sum_{\varepsilon \in Sh(k,n-k)} \pm \gamma_{\varepsilon(1)} \wedge \cdots \wedge \gamma_{\varepsilon(k)} \bigotimes \gamma_{\varepsilon(k+1)} \wedge \cdots \wedge \gamma_{\varepsilon(n)} \,, \tag{2.4}$$

where $\gamma_1, \ldots, \gamma_n$ are homogeneous elements of $\mathcal{L}$.

**Remark.** I would like to mention that although I use the Koszul rule the definition of the signs in (2.4) is delicate. In fact one has to define $C(\mathcal{L})$ as the cofree coalgebra of the suspended cooperad of cocommutative coalgebras in the category of graded vector spaces. To determine the correct signs in (2.4) it is also helpful to use the fact that the functor $\mathcal{L} \mapsto C(\mathcal{L})$ should give a cotriple. In the setting of commutative algebras the reader can see the remark of E. Getzler on p. 217 in [30].

I now give the definition of $L_\infty$-algebra.

**Definition 1** *A graded vector space $\mathcal{L}$ is said to be endowed with a structure of an $L_\infty$-algebra if the cocommutative coassociative coalgebra $C(\mathcal{L})$ cofreely cogenerated by the vector space $\mathcal{L}$ with a shifted parity is equipped with a 2-nilpotent coderivation $Q$ of degree* 1 *.*

To unfold this definition I first mention that the kernel of $\Delta$ coincides with the subspace $\mathcal{L} \subset C(\mathcal{L})$.

$$ker \Delta = \mathcal{L} \,. \tag{2.5}$$

Next, I recall that a map $Q$ is a coderivation of $C(\mathcal{L})$ if and only if for any $X \in C(\mathcal{L})$

$$\Delta QX = -(Q \otimes I \pm I \otimes Q)\Delta X \,. \tag{2.6}$$

Substituting $X = \gamma_1 \wedge \cdots \wedge \gamma_n$ in (2.6), using (2.5), and performing the induction on $n$ I get that equation (2.6) has the following general solution

$$Q\,\gamma_1 \wedge \cdots \wedge \gamma_n = Q_n(\gamma_1, \ldots, \gamma_n) +$$

$$\sum_{k=1}^{n-1} \sum_{\varepsilon \in Sh(k,n-k)} \pm Q_k(\gamma_{\varepsilon(1)}, \ldots, \gamma_{\varepsilon(k)}) \wedge \gamma_{\varepsilon(k+1)} \wedge \cdots \wedge \gamma_{\varepsilon(n)} \,, \tag{2.7}$$

where $\gamma_1 \ldots \gamma_n$ are homogeneous elements of $\mathcal{L}$ and $Q_n$ for $n \geq 1$ are arbitrary polylinear antisymmetric graded maps

$$Q_n \;:\; \wedge^n \mathcal{L} \mapsto \mathcal{L}[2-n] \,, \qquad n \geq 1 \,. \tag{2.8}$$

It is not hard to see that $Q$ can be expressed inductively in terms of the structure maps (2.8) and vice-versa.

Similarly, one can show that the nilpotency condition $Q^2 = 0$ is equivalent to a semi-infinite collection of quadratic relations on (2.8). The lowest of these relations are

$$(Q_1)^2 \gamma = 0 \,, \qquad \forall\, \gamma \in \mathcal{L} \,, \tag{2.9}$$

$$Q_1(Q_2(\gamma_1, \gamma_2)) - Q_2(Q_1(\gamma_1), \gamma_2) - (-)^{k_1} Q_2(\gamma_1, Q_1(\gamma_2)) = 0 \,, \tag{2.10}$$

and

$$(-)^{k_1 k_3} Q_2(Q_2(\gamma_1, \gamma_2), \gamma_3) + \; c.p.(1,2,3) =$$

$$= Q_1 Q_3(\gamma_1, \gamma_2, \gamma_3) + Q_3(Q_1\gamma_1, \gamma_2, \gamma_3) + (-)^{k_1} Q_3(\gamma_1, Q_1\gamma_2, \gamma_3) \tag{2.11}$$

$$+ (-)^{k_1+k_2} Q_3(\gamma_1, \gamma_2, Q_1\gamma_3) \,,$$

where $\gamma_i \in \mathcal{L}^{k_i}$ .

Thus (2.9) says that $Q_1$ is a differential in $\mathcal{L}$ , (2.10) says that $Q_2$ satisfies Leibniz rule with respect to $Q_1$, and (2.11) implies that $Q_2$ satisfies Jacobi identity up to $Q_1$-cohomologically trivial terms.

**Example.** Any differential graded Lie algebra (DGLA) $(\mathcal{L}, \mathfrak{d}, [,])$ is an example of an $L_\infty$-algebra with the only two nonvanishing structure maps

$$Q_1 = \mathfrak{d}\,, \qquad Q_2 = [\,,\,]\,,$$

$$Q_3 = Q_4 = Q_5 = \dots = 0\,.$$

**Definition 2** *An $L_\infty$-morphism $F$ from the $L_\infty$-algebra $(\mathcal{L}, Q)$ to the $L_\infty$-algebra $(\mathcal{L}^\diamond, Q^\diamond)$ is a homomorphism of the cocommutative coassociative coalgebras*

$$F \,:\, C(\mathcal{L}) \mapsto C(\mathcal{L}^\diamond)\,,$$

$$\Delta F(X) = F \otimes F(\Delta X)\,, \qquad X \in C(\mathcal{L}) \tag{2.12}$$

*compatible with the coderivations $Q$ and $Q^\diamond$*

$$Q^\diamond F(X) = F(QX)\,, \qquad \forall\ X \in C(\mathcal{L})\,. \tag{2.13}$$

In what follows the notation

$$F \,:\, (\mathcal{L}, Q) \succ\!\!\!\rightarrow (\mathcal{L}^\diamond, Q^\diamond)$$

means that $F$ is an $L_\infty$-morphism from the $L_\infty$-algebra $(\mathcal{L}, Q)$ to the $L_\infty$-algebra $(\mathcal{L}^\diamond, Q^\diamond)$.

The compatibility of the map (2.12) with coproducts in $C(\mathcal{L})$ and $C(\mathcal{L}^\diamond)$ means that $F$ is uniquely determined by the semi-infinite collection of polylinear graded maps

$$F_n : \wedge^n \mathcal{L} \mapsto \mathcal{L}^\diamond[1-n], \qquad n \geq 1 \tag{2.14}$$

via the equations $(n \geq 1)$

$$F(\gamma_1 \wedge \cdots \wedge \gamma_n) = F_n(\gamma_1, \ldots, \gamma_n) + \tag{2.15}$$

$$\sum_{p>1} \sum_{k_1,\ldots,k_p \geq 1}^{k_1+\cdots+k_p=n} \sum_{\varepsilon \in Sh(k_1,\ldots,k_p)} \pm F_{k_1}(\gamma_{\varepsilon(1)}, \ldots, \gamma_{\varepsilon(k_1)}) \wedge \ldots$$

$$\cdots \wedge F_{k_p}(\gamma_{\varepsilon(n-k_p+1)}, \ldots, \gamma_{\varepsilon(n)}),$$

where $\gamma_1, \ldots, \gamma_n$ are homogeneous elements of $\mathcal{L}$.

The compatibility of $F$ with coderivations (2.13) is a rather complicated condition for general $L_\infty$-algebras. However it is not hard to see that if (2.13) holds then

$$F_1(Q_1\gamma) = Q_1^{\diamond} F_1(\gamma), \qquad \forall\, \gamma \in \mathcal{L},$$

that is the first structure map $F_1$ is always a morphism of complexes $(\mathcal{L}, Q_1)$ and $(\mathcal{L}^{\diamond}, Q_1^{\diamond})$. This observation motivates the following natural definition:

**Definition 3** *A quasi-isomorphism $F$ from the $L_\infty$-algebra $(\mathcal{L}, Q)$ to the $L_\infty$-algebra $(\mathcal{L}^{\diamond}, Q^{\diamond})$ is an $L_\infty$-morphism from $\mathcal{L}$ to $\mathcal{L}^{\diamond}$, the first structure map $F_1$ of which induces a quasi-isomorphism of complexes*

$$F_1 \,:\, (\mathcal{L}, Q_1) \mapsto (\mathcal{L}^{\diamond}, Q_1^{\diamond}). \tag{2.16}$$

Let us suppose that our $L_\infty$-algebras $(\mathcal{L}, Q)$ and $(\mathcal{L}^{\diamond}, Q^{\diamond})$ are just DGLAs $(\mathcal{L}, \mathfrak{d}, [,])$ and $(\mathcal{L}^{\diamond}, \mathfrak{d}^{\diamond}, [,]^{\diamond})$. Then if $F$ is an $L_\infty$-morphism from $\mathcal{L}$ to $\mathcal{L}^{\diamond}$ the compatibility of $F$ with the respective coderivations $Q$ and $Q^{\diamond}$ is equivalent to the following semi-infinite collection of equations $(n \geq 1)$

$$\mathfrak{d}^{\diamond} F_n(\gamma_1, \gamma_2, \ldots, \gamma_n) - \sum_{i=1}^{n} (-)^{k_1+\ldots+k_{i-1}+1-n} F_n(\gamma_1, \ldots, \mathfrak{d}\gamma_i, \ldots, \gamma_n) =$$

$$= \frac{1}{2} \sum_{k,l \geq 1}^{k+l=n} \sum_{\varepsilon \in Sh(k,l)} \pm [F_k(\gamma_{\varepsilon_1}, \ldots, \gamma_{\varepsilon_k}), F_l(\gamma_{\varepsilon_{k+1}}, \ldots, \gamma_{\varepsilon_{k+l}})]^{\diamond} - \tag{2.17}$$

$$-\sum_{i\neq j} \pm F_{n-1}([\gamma_i,\gamma_j],\gamma_1,\dots,\hat{\gamma}_i,\dots,\hat{\gamma}_j,\dots\gamma_n), \qquad \gamma_i \in \mathcal{L}^{k_i}\,,$$

where $\hat{\gamma}_i$ means that the polyvector $\gamma_i$ is missing.

**Example.** An important example of a quasi-isomorphism from a DGLA $\mathcal{L}$ to a DGLA $\mathcal{L}^\diamond$ is provided by a DGLA-homomorphism

$$\lambda\,:\,\mathcal{L}\mapsto\mathcal{L}^\diamond\,,$$

which induces an isomorphism on the spaces of cohomology $H^\bullet(\mathcal{L},\mathfrak{d})$ and $H^\bullet(\mathcal{L}^\diamond,\mathfrak{d}^\diamond)$. In this case the quasi-isomorphism has the only nonvanishing structure map $F_1$

$$F_1=\lambda\,, \qquad F_2=F_3=\dots=0\,.$$

## 2.2 $L_\infty$-modules and their morphisms

Another important object of the "$L_\infty$-world" I am going to deal with is an $L_\infty$-module over an $L_\infty$-algebra. Namely,

**Definition 4** *Let $\mathcal{L}$ be an $L_\infty$-algebra. Then a graded vector space $\mathcal{M}$ is endowed with a structure of an $L_\infty$-module over $\mathcal{L}$ if the cofreely cogenerated comodule $C(\mathcal{L})\otimes\mathcal{M}$ over the coalgebra $C(\mathcal{L})$ is endowed with a 2-nilpotent coderivation $\varphi$ of degree $1$.*

To unfold the definition I first mention that the total space of the comodule $C(\mathcal{L})\otimes\mathcal{M}$ is

$$C(\mathcal{L})\otimes\mathcal{M}=\bigwedge(\mathcal{L})\otimes\mathcal{M}\,, \tag{2.18}$$

and the coaction

$$\mathfrak{a}\,:\,C(\mathcal{L})\otimes\mathcal{M}\mapsto C(\mathcal{L})\bigotimes(C(\mathcal{L})\otimes\mathcal{M})$$

is defined on homogeneous elements as follows

$$\mathfrak{a}(\gamma_1\wedge\dots\wedge\gamma_n\otimes v)=$$

$$\sum_{k=1}^{n-1} \sum_{\varepsilon \in Sh(k,n-k)} \pm \gamma_{\varepsilon(1)} \wedge \cdots \wedge \gamma_{\varepsilon(k)} \bigotimes \gamma_{\varepsilon(k+1)} \wedge \cdots \wedge \gamma_{\varepsilon(n)} \otimes v \tag{2.19}$$

$$+\gamma_1 \wedge \cdots \wedge \gamma_n \bigotimes v \,,$$

where $\gamma_1, \dots \gamma_n$ are homogeneous elements of $\mathcal{L}$, $v \in \mathcal{M}$, $S_n$ is the group of permutations of $n$ elements and the signs are determined with the help of the Koszul rule. For example,

$$\mathfrak{a}(v) = 0 \,, \qquad \forall\; v \in \mathcal{M} \,,$$

$$\mathfrak{a}(\gamma \otimes v) = \gamma \bigotimes v \,, \qquad \forall\; v \in \mathcal{M} \,,\; \gamma \in \mathcal{L} \,,$$

and

$$\mathfrak{a}(\gamma_1 \wedge \gamma_2 \otimes v) = \gamma_1 \wedge \gamma_2 \bigotimes v + \gamma_1 \bigotimes (\gamma_2 \otimes v) - (-)^{k_1 k_2} \gamma_2 \bigotimes (\gamma_1 \otimes v)$$

for any $v \in \mathcal{M}$ and for any pair $\gamma_1 \in \mathcal{L}^{k_1}$, $\gamma_2 \in \mathcal{L}^{k_2}$.

A direct computation shows that the coaction (2.19) satisfies the required axiom

$$(I \otimes \mathfrak{a})\mathfrak{a}(X) = (\Delta \otimes I)\mathfrak{a}(X) \,, \qquad \forall\; X \in C(\mathcal{L}) \otimes \mathcal{M} \,,$$

where $\Delta$ is the comultiplication (2.3) in the coalgebra $C(\mathcal{L})$. It is also easy to see that

$$\ker \mathfrak{a} = \mathcal{M} \subset C(\mathcal{L}) \otimes \mathcal{M} \,. \tag{2.20}$$

By definition $\varphi$ is a coderivation of $C(\mathcal{L}) \otimes \mathcal{M}$. This means that for any $X \in C(\mathcal{L}) \otimes \mathcal{M}$

$$\mathfrak{a}\, \varphi X = -Q \otimes I\, (\mathfrak{a} X) \pm I \otimes \varphi\, (\mathfrak{a} X) \,, \tag{2.21}$$

where $Q$ is the $L_\infty$-algebra structure on $\mathcal{L}$ (that is a 2-nilpotent coderivation of $C(\mathcal{L})$).

Substituting $X = \gamma_1 \wedge \cdots \wedge \gamma_n$ in (2.21), using (2.20), and performing the induction on $n$ I get that equation (2.21) has the following general solution

$$\varphi(\gamma_1 \wedge \cdots \wedge \gamma_n \otimes v) = \varphi_n(\gamma_1, \dots, \gamma_n, v) +$$

$$\sum_{k=1}^{n-1} \sum_{\varepsilon \in Sh(k,n-k)} \pm \gamma_{\varepsilon(1)} \wedge \cdots \wedge \gamma_{\varepsilon(k)} \wedge \varphi_{n-k}(\gamma_{\varepsilon(k+1)}, \ldots, \gamma_{\varepsilon(n)}, v) \tag{2.22}$$

$$+(-)^{k_1+\cdots+k_n} \gamma_1 \wedge \cdots \wedge \gamma_n \otimes \varphi_0(v) +$$

$$\sum_{k=1}^{n} \sum_{\varepsilon \in Sh(k,n-k)} \pm Q_k(\gamma_{\varepsilon(1)}, \ldots, \gamma_{\varepsilon(k)}) \otimes \gamma_{\varepsilon(k+1)} \wedge \cdots \wedge \gamma_{\varepsilon(n)} \otimes v\,,$$

where $\gamma_i \in \mathcal{L}^{k_i}$, $v \in \mathcal{M}$, $Q_k$'s represent the $L_\infty$-algebra structure on $\mathcal{L}$ and $\{\varphi_n\}$ for $n \geq 0$ are arbitrary polylinear antisymmetric graded maps

$$\varphi_n \,:\, \wedge^n \mathcal{L} \otimes \mathcal{M} \mapsto \mathcal{M}[1-n]\,. \tag{2.23}$$

Equation (2.22) allows us to express $\varphi$ inductively in terms of its structure maps (2.23) and vice-versa.

Similarly, one can show that the nilpotency condition $\varphi^2 = 0$ is equivalent to the following semi-infinite collection of quadratic relations in $\varphi_k$ and $Q_l$ ($n \geq 0$)

$$\varphi_0(\varphi_n(\gamma_1, \ldots, \gamma_n, v)) - (-)^{1-n} \varphi_n(Q_1(\gamma_1), \ldots, \gamma_n, v) - \ldots$$

$$\ldots - (-)^{k_1+\cdots+k_{n-1}+1-n} \varphi_n(\gamma_1, \ldots, Q_1(\gamma_n), v) -$$

$$(-)^{k_1+\cdots+k_n+1-n} \varphi_n(\gamma_1, \ldots, \gamma_n, \varphi_0(v)) =$$

$$\frac{1}{2} \sum_{k=1}^{n-1} \sum_{\varepsilon \in Sh(k,n-k)} \pm \varphi_k(\gamma_{\varepsilon(1)}, \ldots, \gamma_{\varepsilon(k)}, \varphi_{n-k}(\gamma_{\varepsilon(k+1)}, \ldots, \gamma_{\varepsilon(n)}, v)) + \tag{2.24}$$

$$\frac{1}{2} \sum_{k=1}^{n-1} \sum_{\varepsilon \in Sh(k+1,n-k-1)} \pm \varphi_{n-k}(Q_{k+1}(\gamma_{\varepsilon(1)}, \ldots, \gamma_{\varepsilon(k+1)}), \gamma_{\varepsilon(k+2)}, \ldots, \gamma_{\varepsilon(n)}, v)\,,$$

$$\gamma_i \in \mathcal{L}^{k_i}, \qquad v \in \mathcal{M}\,.$$

The signs in the above equations are defined with the help of the Koszul rule.

For $n = 0$ equation (2.24) says that $\varphi_0$ is a differential on $\mathcal{M}$

$$(\varphi_0)^2 = 0\,,$$

and for $n = 1$ it says that $\varphi_1$ is closed with respect to the natural differential acting on the vector space $Hom(\mathcal{L} \otimes \mathcal{M}, \mathcal{M})$

$$\varphi_0 \varphi_1(\gamma, v) - \varphi_1(Q_1 \gamma, v) - (-)^k \varphi_1(\gamma, \varphi_0(v)) = 0\,,$$

$$\forall\, \gamma \in \mathcal{L}^k\,, \qquad v \in \mathcal{M}\,.$$

For an $L_\infty$-module structure I reserve the following notation

$$\begin{array}{c} \mathcal{L} \\ \downarrow^{\varphi}_{mod} \\ (\mathcal{M}, \varphi_0) \end{array}$$

where $\mathcal{L}$ stands for the $L_\infty$-algebra and $\mathcal{M}$ stands for the respective graded vector space.

**Example.** The simplest example of an $L_\infty$-module is a DG module $(\mathcal{M}, \mathbf{b})$ over a DGLA $(\mathcal{L}, \mathfrak{d}, [\,,\,])$. In this case the only nonvanishing structure maps of $\varphi$ are

$$\varphi_0(v) = \mathbf{b}(v)\,, \qquad v \in \mathcal{M}\,,$$

and

$$\varphi_1(\gamma, v) = \rho(\gamma)\, v\,, \qquad \gamma \in \mathcal{L}\,,\ v \in \mathcal{M}\,,$$

where $\rho$ is the action of $\mathcal{L}$ on $\mathcal{M}$. The axioms of DGLA module

$$\mathbf{b}^2 = 0\,,$$

$$\mathbf{b}(\rho(\gamma)\, v) = \rho(\mathfrak{d}\gamma)\, v + (-)^k \rho(\gamma)\, \mathbf{b}(v)\,, \qquad \gamma \in \mathcal{L}^k\,,$$

$$\rho(\gamma_1)\rho(\gamma_2)\, v - (-)^{k_1 k_2} \rho(\gamma_2)\rho(\gamma_1)\, v = \rho([\gamma_1, \gamma_2]) v\,,$$

$$\gamma_1 \in \mathcal{L}^{k_1}\,, \qquad \gamma_2 \in \mathcal{L}^{k_2}$$

are exactly the axioms of $L_\infty$-module.

**Definition 5** *Let $\mathcal{L}$ be an $L_\infty$-algebra and $(\mathcal{M}, \varphi^{\mathcal{M}})$, $(\mathcal{N}, \varphi^{\mathcal{N}})$ be $L_\infty$-modules over $\mathcal{L}$. Then a morphism $\kappa$ from the comodule $C(\mathcal{L}) \otimes \mathcal{M}$ to the comodule $C(\mathcal{L}) \otimes \mathcal{N}$ compatible with the coderivations $\varphi^{\mathcal{M}}$ and $\varphi^{\mathcal{N}}$*

$$\kappa(\varphi^{\mathcal{M}} X) = \varphi^{\mathcal{N}}(\kappa X)\,, \qquad \forall\; X \in C(\mathcal{L}) \otimes \mathcal{M} \tag{2.25}$$

*is called an morphism between $L_\infty$-modules $(\mathcal{M}, \varphi^{\mathcal{M}})$ and $(\mathcal{N}, \varphi^{\mathcal{N}})$.*

Unfolding this definition one can easily show that the morphism $\kappa$ is uniquely determined by its structure maps

$$\kappa_n \;:\; \wedge^n \mathcal{L} \otimes \mathcal{M} \mapsto \mathcal{N}[-n]\,, \qquad n \geq 0 \tag{2.26}$$

via the following equations

$$\kappa(\gamma_1 \wedge \dots \wedge \gamma_n \otimes v) = \kappa_n(\gamma_1, \dots, \gamma_n, v) +$$

$$\sum_{k=1}^{n-1} \sum_{\varepsilon \in Sh(k,n-k)} \pm\, \gamma_{\varepsilon(1)} \wedge \dots \wedge \gamma_{\varepsilon(k)} \otimes \kappa_{n-k}(\gamma_{\varepsilon(k+1)}, \dots, \gamma_{\varepsilon(n)}, v) \tag{2.27}$$

$$+\gamma_1 \wedge \dots \wedge \gamma_n \otimes \kappa_0(v)\,.$$

Relation (2.25) is equivalent to the following semi-infinite collection of equations $(n \geq 0)$

$$\varphi_0^{\mathcal{N}} \kappa_n(\gamma_1, \dots, \gamma_n, v) - (-)^n \kappa_n(Q_1\gamma_1, \gamma_2, \dots, \gamma_n) - \dots$$

$$\dots - (-)^{k_1 + \dots + k_n + n} \kappa_n(\gamma_1, \dots, \gamma_n, \varphi_0^{\mathcal{M}} v) =$$

$$\sum_{p=0}^{n-1} \sum_{\varepsilon \in Sh(p,n-p)} \pm \kappa_p(\gamma_{\varepsilon(1)}, \dots, \gamma_{\varepsilon(p)}, \varphi_{n-p}^{\mathcal{M}}(\gamma_{\varepsilon(p+1)}, \dots, \gamma_{\varepsilon(n)}, v))$$

$$-\sum_{p=1}^{n} \sum_{\varepsilon \in Sh(p,n-p)} \pm \varphi_p^{\mathcal{N}}(\gamma_{\varepsilon(1)}, \dots, \gamma_{\varepsilon(p)}, \kappa_{n-p}(\gamma_{\varepsilon(p+1)}, \dots, \gamma_{\varepsilon(n)}), v) \tag{2.28}$$

$$+\sum_{p=2}^{n} \sum_{\varepsilon \in Sh(p,n-p)} \pm \kappa_{n-p+1}(Q_p(\gamma_{\varepsilon(1)}, \dots, \gamma_{\varepsilon(p)}), \gamma_{\varepsilon(p+1)}, \dots, \gamma_{\varepsilon(n)}, v) \,,$$

$$\gamma_i \in \mathcal{L}^{k_i} \,, \qquad v \in \mathcal{M} \,.$$

It is not hard to check that an ordinary morphism of DG modules over an ordinary DGLA provides us with the simplest example of the morphism between $L_\infty$-modules.

For $n = 0$ equation (2.28) reduces to

$$\kappa_0(\varphi_0^{\mathcal{M}} v) = \varphi_0^{\mathcal{N}} \kappa_0(v) \,, \qquad v \in \mathcal{M}$$

and hence the zero-th structure map of $\kappa$ is always a morphism of complexes $(\mathcal{M}, \varphi_0^{\mathcal{M}})$ and $(\mathcal{N}, \varphi_0^{\mathcal{N}})$. This motivates the following

**Definition 6** *A quasi-isomorphism $\kappa$ of $L_\infty$-modules $(\mathcal{M}, \varphi^{\mathcal{M}})$ and $(\mathcal{N}, \varphi^{\mathcal{N}})$ is a morphism between these $L_\infty$-modules with the zero-th structure map $\kappa_0$ being a quasi-isomorphism of complexes $(\mathcal{M}, \varphi_0^{\mathcal{M}})$ and $(\mathcal{N}, \varphi_0^{\mathcal{N}})$.*

In what follows the notation

$$\mathcal{M} \succ\!\xrightarrow{\kappa} \mathcal{N}$$

means that $\kappa$ is a morphism from the $L_\infty$-module $\mathcal{M}$ to the $L_\infty$-module $\mathcal{N}$.

To this end I mention that there is another definition of an $L_\infty$-module over an $L_\infty$-algebra which is known [53] to be equivalent to the definition I gave above.

**Definition 7** *Let $\mathcal{L}$ be an $L_\infty$-algebra. Then a complex $(\mathcal{M}, \mathbf{b})$ is called an $L_\infty$-module over $\mathcal{L}$ if there is an $L_\infty$-morphism $\eta$ from $\mathcal{L}$ to $Hom(\mathcal{M}, \mathcal{M})$, where $Hom(\mathcal{M}, \mathcal{M})$ is naturally viewed as a DGLA with the differential induced by $\mathbf{b}$.*

The structure maps $\varphi_n$ of the respective coderivation of the comodule $C(\mathcal{L}) \otimes \mathcal{M}$ are related to $\mathbf{b}$ and the structure maps of the $L_\infty$-morphism $\eta$ in the following simple way

$$\mathbf{b} = \varphi_0 \,, \qquad \eta_n(\gamma_1, \dots, \gamma_n)(v) = \varphi_n(\gamma_1, \dots, \gamma_n, v) \quad (n \geq 1) \,, \tag{2.29}$$

$$\gamma_i \in \mathcal{L}, \; v \in \mathcal{M} \,.$$

## 2.3 Partial homotopies between $L_\infty$-morphisms

In this section I introduce a notion of partial homotopy between two $L_\infty$-morphisms. I will use this notion in section 5.2.

Let $(\mathcal{L}, Q)$ and $(\mathcal{L}^\diamond, Q^\diamond)$ be two $L_\infty$-algebras. As above, I denote by $Q$ and $Q^\diamond$ the corresponding codifferentials of the cocommutative coassociative coalgebras $C(\mathcal{L})$ and $C(\mathcal{L}^\diamond)$ . Let

$$F : C(\mathcal{L}) \mapsto C(\mathcal{L}^\diamond)$$

be an $L_\infty$-morphism from $\mathcal{L}$ to $\mathcal{L}^\diamond$.

One can observe that if a map

$$H : C(\mathcal{L}) \mapsto C(\mathcal{L}^\diamond)$$

is of degree $-1$ then the map

$$\widetilde{F} = F + Q^\diamond H + H\, Q \;:\; C(\mathcal{L}) \mapsto C(\mathcal{L}^\diamond)$$

is of degree zero, and moreover it is compatible with the coderivations $Q$ and $Q^\diamond$

$$Q^\diamond \widetilde{F} = \widetilde{F} Q \,. \tag{2.30}$$

A compatibility of $\widetilde{F}$ with the coproducts (2.4) in $C(\mathcal{L})$ and $C(\mathcal{L}^\diamond)$ is equivalent to a rather complicated equation for the map $H$

$$\Delta\, H\, Q - (Q^\diamond \otimes I \pm I \otimes Q^\diamond)\Delta\, H =$$

$$F \otimes (Q^\diamond H + HQ) + (Q^\diamond H + HQ) \otimes F + (Q^\diamond H + HQ) \otimes (Q^\diamond H + HQ) \,. \tag{2.31}$$

However, if $H$ satisfies the following equation

$$\Delta H = -\Big( F \otimes H + H \otimes F + \frac{1}{2}(H \otimes Q^\diamond H + Q^\diamond H \otimes H) + \frac{1}{2}(HQ \otimes H + H \otimes HQ) \Big) \tag{2.32}$$

then due to (2.6) and (2.12) $H$ satisfies (2.31) as well.

Using (2.5) it is not hard to get the most general solution of equation (2.32). Namely, any solution $H$ of (2.32) is uniquely determined by a semi-infinite collection of polylinear graded maps

$$H_n : \wedge^n \mathcal{L} \mapsto \mathcal{L}^\diamond[-n]\,, \qquad H_n = pr \circ H \Big|_{\wedge^n \mathcal{L}}\,, \tag{2.33}$$

where $pr$ is the canonical projection

$$pr : \bigwedge(\mathcal{L}^\diamond) \to \mathcal{L}^\diamond\,. \tag{2.34}$$

In order to restore the map $H$ from the collection (2.33) one solves (2.32) iteratively from $\wedge^{<n}\mathcal{L}$ to $\wedge^n\mathcal{L}$ starting with

$$H(\gamma) = H_1(\gamma)\,, \qquad \forall\, \gamma \in \mathcal{L}\,. \tag{2.35}$$

I refer to (2.33) as structure maps of $H$.

It is immediate from (2.15) and (2.35) that for any $\gamma \in \mathcal{L}$

$$\widetilde{F}_1(\gamma) = F_1(\gamma) + Q_1^\diamond\, H_1(\gamma) + H_1\, Q_1(\gamma)\,, \tag{2.36}$$

where $F_1$ and $\widetilde{F}_1$ are the first structure maps of $F$ and $\widetilde{F}$, respectively. This observation motivates the following definition

**Definition 8** *A map*

$$H : C(\mathcal{L}) \mapsto C(\mathcal{L}^\diamond)[-1]$$

*is called a partial homotopy between $L_\infty$-morphisms*

$$F,\, \widetilde{F} : (C(\mathcal{L}), Q) \mapsto (C(\mathcal{L}^\diamond), Q^\diamond)$$

*if it satisfies (2.32) and*

$$\widetilde{F} = F + Q^\diamond\, H + H\, Q\,.$$

*Two $L_\infty$-morphisms are called partially homotopic if they are connected by a finite chain of partial homotopies.*

**Remark 1.** It is easy to see that equation (2.32) still holds if I replace $H$ by $-H$ and $F$ by $F + Q^\diamond H + HQ$. However a composition of two partial homotopies is not in general a partial homotopy[1]. That is why I extend the relation of partial homotopy to an equivalence relation by transitivity.

**Remark 2.** The correct notion of homotopy between $L_\infty$-morphisms is based on the structure of the closed model category on the category of $L_\infty$-algebras [33], [44]. Unfortunately, I do not know how to relate the above notion of the partial homotopy to the correct notion of homotopy based on the closed model category structure. For my purposes the above *ad hoc* notion will be sufficient.

Let me prove the following auxiliary statement:

**Lemma 1** *Let*

$$F : C(\mathcal{L}) \mapsto C(\mathcal{L}^\diamond)$$

*be a quasi-isomorphism from an $L_\infty$-algebra $(\mathcal{L}, Q)$ to an $L_\infty$-algebra $(\mathcal{L}^\diamond, Q^\diamond)$. For $n \geq 1$ and any map*

$$\widetilde{H} : \wedge^n \mathcal{L} \mapsto \mathcal{L}^\diamond[-n]$$

*one can construct a quasi-isomorphism*

$$\widetilde{F} : C(\mathcal{L}) \mapsto C(\mathcal{L}^\diamond)$$

*such that for any $m < n$*

$$\widetilde{F}_m = F_m \; : \; \wedge^m \mathcal{L} \mapsto \mathcal{L}^\diamond \tag{2.37}$$

*and*

$$\widetilde{F}_n(\gamma_1, \dots, \gamma_n) = F_n(\gamma_1, \dots, \gamma_n) +$$

$$Q_1^\diamond \widetilde{H}(\gamma_1, \dots, \gamma_n) - (-)^n \widetilde{H}(Q_1(\gamma_1), \gamma_2, \dots, \gamma_n) - \dots \tag{2.38}$$

[1] I am thankful to G. Felder for this observation.

$$\cdots - (-)^{n+k_1+\cdots+k_{n-1}} \widetilde{H}(\gamma_1, \ldots, \gamma_{n-1}, Q_1(\gamma_n)) \,,$$

*where* $\gamma_i \in \mathcal{L}^{k_i}$ .

**Proof.** It is obvious that if a partial homotopy $H$ has the following structure maps:

$$H_m = \begin{cases} \widetilde{H} & \text{if} \quad m = n \,, \\ 0 & \text{otherwise} \end{cases}$$

then $\widetilde{F} = F + Q^{\diamond} H + H Q$ satisfies the desired properties (2.37), (2.38). Since $F$ is a quasi-isomorphism so is $\widetilde{F}$. $\square$

**REMARK.** From now on **all** $L_\infty$-algebras are DGLAs. "Weird" things I still borrow from the "$L_\infty$-world" are $L_\infty$-morphisms, $L_\infty$-modules, and morphisms between such modules.

## 2.4 Maurer-Cartan elements and twisting procedures

Motivated by deformation theory I consider DGLAs $\mathcal{L}$ equipped with a complete descending filtration

$$\mathcal{L} = \mathcal{F}^0 \mathcal{L} \supset \mathcal{F}^1 \mathcal{L} \supset \ldots \,, \qquad \mathcal{L} = \lim_n \mathcal{L} / \mathcal{F}^n \mathcal{L} \,. \tag{2.39}$$

In this section I assume that all DGLAs and $L_\infty$-modules are equipped with complete descending filtrations and all $L_\infty$-morphisms as well as morphisms of $L_\infty$-modules are compatible with these filtrations. Furthermore, I require that all quasi-isomorphisms of the corresponding complexes are *strongly compatible* with the filtrations. Namely,

**Condition 1** *Let* $\lambda$ *be a quasi-isomorphism*

$$\lambda : \mathcal{L}^\bullet \mapsto \widetilde{\mathcal{L}}^\bullet$$

*of filtered complexes $\mathcal{L}^\bullet$, $\widetilde{\mathcal{L}}^\bullet$. I say that $\lambda$ is compatible with the filtrations if for any filtration subcomplex $\mathcal{F}^k\mathcal{L}^\bullet \subset \mathcal{L}^\bullet$*

$$\lambda\Big|_{\mathcal{F}^k\mathcal{L}^\bullet} : \mathcal{F}^k\mathcal{L}^\bullet \mapsto \mathcal{F}^k\widetilde{\mathcal{L}}^\bullet$$

*is a quasi-isomorphism.*

I will assume this compatibility condition throughout my thesis.

If $\mathcal{L}$ is such a filtered DGLA then $\mathcal{F}^1\mathcal{L}^0$ is a projective limit of nilpotent Lie algebras. Therefore, $\mathcal{F}^1\mathcal{L}^0$ can be "integrated" to a prounipotent group. I denote this group by $\mathfrak{G}(\mathcal{L})$.

Let me recall the following definition:

**Definition 9** *Let $(\mathcal{L}, \mathfrak{d}, [,])$ be a filtered DGLA. Then $\pi \in \mathcal{F}^1\mathcal{L}^1$ is called a Maurer-Cartan element if*

$$\mathfrak{d}\pi + \frac{1}{2}[\pi, \pi] = 0\,. \tag{2.40}$$

The Lie algebra $\mathcal{F}^1\mathcal{L}^0$ acts naturally on the cone (2.40) of Maurer-Cartan elements

$$\rho(\xi)\pi = \mathfrak{d}\xi + [\pi, \xi]\,, \qquad \xi \in \mathcal{F}^1\mathcal{L}^0\,, \tag{2.41}$$

and the action (2.41) obviously lifts to the action of the corresponding prounipotent group $\mathfrak{G}(\mathcal{L})$. The quotient space of the cone (2.40) with respect to the $\mathfrak{G}(\mathcal{L})$-action is called the *moduli space* of the DGLA $\mathcal{L}$.

It turns out that a quasi-isomorphism (see definition 3) between DGLAs gives a bijective correspondence between their moduli spaces. A weaker version of this statement is proved in proposition 1 (see claim 4). This version says that if $F$ is an $L_\infty$-morphism from a DGLA $(\mathcal{L}, \mathfrak{d}, [,])$ to a DGLA $(\mathcal{L}^\diamond, \mathfrak{d}^\diamond, [,]^\diamond)$ and $\pi \in \mathcal{F}^1\mathcal{L}^1$ is a Maurer-Cartan element of $\mathcal{L}$ then

$$S = \sum_{n\ge 1} \frac{1}{n!} F_n(\pi, \dots, \pi) \tag{2.42}$$

is a Maurer-Cartan element of $\mathcal{L}^\diamond$.

Notice that the infinite sum in (2.42) is well-defined because $\mathcal{L}^{\diamond}$ is assumed to be complete with the respect to the corresponding filtration. All elements of this sum are of degree 1 since for any $n$ $F_n$ shifts the degree by $1-n$ (see (2.14)).

Using a Maurer-Cartan element $\pi \in \mathcal{F}^1\mathcal{L}^1$ one can naturally modify the structure of the DGLA on $\mathcal{L}$ by adding the inner derivation $[\pi, \cdot\,]$ to the initial differential $\mathfrak{d}$. Thanks to Maurer-Cartan equation (2.40) this new derivation $\mathfrak{d} + [\pi, \cdot\,]$ is 2-nilpotent. This modification can be described in terms of the respective $L_{\infty}$-structure . Namely, the coderivation $Q^{\pi}$ on the coassociative cocommutative coalgebra $C(\mathcal{L})$ corresponding to the new DGLA structure $(\mathcal{L}, \mathfrak{d} + [\pi, \cdot\,], [\,,\,])$ is related to the initial coderivation $Q$ by the equation

$$Q^{\pi}(X) = \exp((-\pi)\wedge) Q(\exp(\pi\wedge)X)\,, \qquad X \in C(\mathcal{L})\,, \tag{2.43}$$

where the sum

$$\exp(\pi\wedge)\underbrace{\phantom{xx}} = \underbrace{\phantom{xx}} + \pi\wedge\underbrace{\phantom{xx}} + \frac{1}{2!}\pi\wedge\pi\wedge\underbrace{\phantom{xx}} + \dots$$

is well-defined since $\pi \in \mathcal{F}^1\mathcal{L}^1$ .

I call this procedure of changing the initial DGLA structure on $\mathcal{L}$ *twisting* of the DGLA $\mathcal{L}$ by the Maurer-Cartan element $\pi$. This terminology is borrowed from Quillen's paper [45] (see App. B 5.3). This twisting procedure is also extensively used in paper [56] by A. Yekutieli on deformation quantization in algebraic geometry setting.

Similar twisting procedures by a Maurer-Cartan element can be defined for an $L_{\infty}$-morphism, for an $L_{\infty}$-module, and for a morphism of $L_{\infty}$-modules. In the following propositions I describe these procedures.

**Proposition 1 (See also theorem 0.1 in [55])** *If $F$ is an $L_{\infty}$-morphism*

$$F\,:\,(\mathcal{L}, Q) \succ\!\!\rightarrow (\mathcal{L}^{\diamond}, Q^{\diamond})$$

*of DGLAs, $\pi \in \mathcal{F}^1\mathcal{L}^1$ and an element $S \in \mathcal{F}^1(\mathcal{L}^{\diamond})^1$ is given by equation (2.42) then*

1. *For any homogeneous element $X \in C(\mathcal{L})$*

$$\Delta(\exp(\pi\wedge)X) = \exp(\pi\wedge) \bigotimes \exp(\pi\wedge)(\Delta X) + \operatorname{cxp}(\pi) \bigotimes \exp(\pi\wedge)X - \tag{2.44}$$

$$(-)^{|X|} \exp(\pi\wedge)X \bigotimes \operatorname{cxp}(\pi),$$

*where*

$$\operatorname{cxp}(\pi) = \sum_{k=1}^{\infty} \frac{1}{k!} \underbrace{\pi \wedge \cdots \wedge \pi}_{k}. \tag{2.45}$$

2. *Equation (2.40) is equivalent to*

$$Q(\operatorname{cxp}(\pi)) = 0. \tag{2.46}$$

3. 
$$F(\operatorname{cxp}(\pi)) = \operatorname{cxp}(S). \tag{2.47}$$

4. *If $\pi$ a Maurer-Cartan element then so is $S$ and the map*

$$F^{\pi} = \exp(-S\wedge)F \exp(\pi\wedge) \, : \, C(\mathcal{L}) \mapsto C(\mathcal{L}^{\diamond}) \tag{2.48}$$

*defines an $L_\infty$-morphism between the DGLAs $\mathcal{L}^{\pi}$ and $\mathcal{L}^{\diamond S}$, obtained via twisting by the Maurer-Cartan elements $\pi$ and $S$, respectively.*

5. *Let $\pi$ be a Maurer-Cartan element. If $F$ is a quasi-isomorphism satisfying condition 1 (on page 31 ) then so is $F^{\pi}$ .*

In what follows I refer to $F^{\pi}$ in (2.48) as an $L_\infty$-morphism (or a quasi-isomorphism) *twisted by the Maurer-Cartan element* $\pi$. It is not hard to see that the structure maps of the twisted $L_\infty$-morphism $F^{\pi}$ are given by

$$F^{\pi}_n(\gamma_1, \ldots, \gamma_n) = \sum_{k=0}^{\infty} \frac{1}{k!} F_{n+k}(\pi, \ldots, \pi, \gamma_1, \ldots, \gamma_n), \qquad \gamma_i \in \mathcal{L}. \tag{2.49}$$

**Proof.** In order to prove claim 1 I introduce an auxiliary variable $t$ and analyze a slightly stronger statement

$$\Delta(\exp(t\pi\wedge)X) \stackrel{?}{=} \exp(t\pi\wedge)\bigotimes\exp(t\pi\wedge)(\Delta X) + \mathrm{cxp}\,(t\pi)\bigotimes\exp(t\pi\wedge)X - \tag{2.50}$$

$$(-)^{|X|}\exp(t\pi\wedge)X\bigotimes\mathrm{cxp}\,(t\pi)\,.$$

It is clear that (2.50) holds for $t = 0$. On the other hand a direct computation shows that both sides of (2.50) satisfies the following differential equation:

$$\frac{d}{dt}W(t) = \Big(\pi\wedge\bigotimes 1 + 1\bigotimes\pi\wedge\Big)W(t) + \pi\bigotimes\exp(t\pi\wedge)X - (-)^{|X|}\exp(t\pi\wedge)X\bigotimes\pi\,.$$

Thus equation (2.50) holds and claim 1 follows.

It is obvious that (2.46) implies (2.40). Let me prove the converse statement. First, I observe that if $(\mathcal{L}, \mathfrak{d}, [,])$ is a DGLA then the collection

$$Q_1^t = t\mathfrak{d}, \quad Q_2^t = [\,,\,], \quad Q_3^t = Q_4^t = \cdots = 0$$

defines a DGLA on $\mathcal{L}[t]$. Second, $t\pi$ is a Maurer-Cartan element in $(\mathcal{L}[t], t\mathfrak{d}, [,])$ and the equation

$$Q^t(\mathrm{cxp}\,(t\pi)) \stackrel{?}{=} 0 \tag{2.51}$$

obviously holds for $t = 0$. Using the Maurer-Cartan equation (2.40) it is not hard to prove that the left hand side $Z(t) = Q^t(\mathrm{cxp}\,(t\pi))$ of (2.51) satisfies the following differential equation:

$$\frac{d}{dt}Z(t) = Z(t)\wedge\pi\,.$$

Since this equation is homogeneous (2.51) holds for any $t$ and claim 2 follows.

To prove that the element

$$Y = F(\mathrm{cxp}\,(\pi)) - \mathrm{cxp}\,(S) \in C(\mathcal{L}^{\diamond})$$

is vanishing I observe that for any $\pi \in \mathcal{L}^1$

$$\Delta(\mathrm{cxp}\,(\pi)) = \mathrm{cxp}\,(\pi) \bigotimes \mathrm{cxp}\,(\pi)\,. \tag{2.52}$$

Furthermore, due to (2.42) $Y$ lies in the kernel of the natural projection

$$pr : \bigwedge(\mathcal{L}^\diamond) \to \mathcal{L}^\diamond\,. \tag{2.53}$$

Let us prove by induction that

$$Y \in \mathcal{F}^m(C(\mathcal{L}^\diamond)) \tag{2.54}$$

for all $m$.

By definition of the Maurer-Cartan element $\pi \in \mathcal{F}^1\mathcal{L}$. Therefore the element $S$ (2.42) belongs to $\mathcal{F}^1\mathcal{L}^\diamond$ and hence

$$Y \in \mathcal{F}^1(C(\mathcal{L}^\diamond))\,.$$

Let me take it as base of the induction and suppose that (2.54) holds for some $m$.

Equation (2.52) and the compatibility of the map $F$ with the coproducts (2.4) in $C(\mathcal{L})$ and $C(\mathcal{L}^\diamond)$ implies that

$$\Delta Y \in \mathcal{F}^{m+1}(\wedge^2 C(\mathcal{L}^\diamond))\,.$$

Therefore due to (2.5) the image of $Y$ in $\mathcal{F}^m(C(\mathcal{L}^\diamond))/\mathcal{F}^{m+1}(C(\mathcal{L}^\diamond))$ belongs to

$$\mathcal{F}^m\mathcal{L}^\diamond/\mathcal{F}^{m+1}\mathcal{L}^\diamond\,.$$

But the image of $Y$ vanishes under the projection $pr$ (2.53). Hence,

$$Y \in \mathcal{F}^{m+1}(C(\mathcal{L}^\diamond))\,,$$

and therefore (2.54) holds for all $m$. Since $\mathcal{L}^\diamond$ is complete with respect to the filtration equation (2.47) is proved.

Let me now turn to claim 4. While the compatibility of $F^\pi$ with the coderivations $Q^\pi$ and $Q^{\diamond\, S}$ follows directly from the definitions the compatibility of $F^\pi$ with the coproducts in $C(\mathcal{L})$ and $C(\mathcal{L}^\diamond)$ requires some work. Using claim 1 and 3 I get that for any homogeneous $X \in C(\mathcal{L})$

$$\Delta \exp(-S\wedge) F \exp(\pi\wedge) X = \exp(-S\wedge) \bigotimes \exp(-S\wedge)(F \bigotimes F)\Delta \exp(\pi\wedge)X +$$

$$\mathrm{cxp}\,(-S) \bigotimes \exp(-S\wedge) F \exp(\pi\wedge) X - (-)^{|X|} \exp(-S\wedge) F \exp(\pi\wedge) X \bigotimes \mathrm{cxp}\,(-S) =$$

$$\mathrm{cxp}\,(-S) \bigotimes F^\pi X + F^\pi X \bigotimes \mathrm{cxp}\,(-S) +$$

$$(F^\pi \bigotimes F^\pi)(\Delta X) +$$

$$\exp(-S\wedge) \bigotimes \exp(-S\wedge)(F \bigotimes F)(\mathrm{cxp}\,(\pi) \bigotimes \exp(\pi\wedge) X) -$$

$$(-)^{|X|} \exp(-S\wedge) \bigotimes \exp(-S\wedge)(F \bigotimes F)(\exp(\pi\wedge) X \bigotimes \mathrm{cxp}\,(\pi))\,.$$

The first and the second terms in the latter expression cancel with the forth and the fifth terms, respectively, due to claim 3 and the following obvious identity between Taylor series

$$e^{-S}\mathrm{cxp}\,(S) = -\mathrm{cxp}\,(-S)\,. \tag{2.55}$$

Thus, I get the desired relation

$$\Delta F^\pi(X) = (F^\pi \bigotimes F^\pi)(\Delta X)\,.$$

Claim 5 is proved by the standard argument of the spectral sequence. We have to prove that the first structure map $F^\pi_1$ of the twisted $L_\infty$-morphism (2.48) is a quasi-isomorphism from the complex $(\mathcal{L}, \mathfrak{d} + [\pi, \cdot])$ to the complex $(\mathcal{L}^\diamond, \mathfrak{d}^\diamond + [S, \cdot]^\diamond)$. These complexes are filtered and $F^\pi_1$ is compatible with the filtration. Since $F_1$ is a quasi-isomorphism between the complexes $(\mathcal{L}, \mathfrak{d})$ and $(\mathcal{L}^\diamond, \mathfrak{d}^\diamond)$ and $F_1$ satisfies condition 1 (on page 31) the map $F^\pi_1$ induces a quasi-isomorphism on the zeroth level of the corresponding spectral sequences. Therefore $F_1^\pi$ gives a quasi-isomorphism on the terminal $E_\infty$-level. Hence, due to the standard snake-lemma argument of homological algebra $F_1^\pi$ is also a quasi-isomorphism.

Proposition 1 is proved. $\square$

**Proposition 2** *If* $(\mathcal{L}, \mathfrak{d}, [,])$ *is a DGLA,* $(\mathcal{M}, \varphi)$ *is an* $L_\infty$*-module over* $\mathcal{L}$ *and* $\pi \in \mathcal{F}^1\mathcal{L}^1$ *is a Maurer-Cartan element then*

1. *For any*[2] $X \in C(\mathcal{L}) \otimes \mathcal{M}$

$$\mathfrak{a}(\exp(\pi\wedge)X) = \exp(\pi\wedge) \bigotimes \exp(\pi\wedge)(\mathfrak{a}X) + \operatorname{cxp}(\pi) \bigotimes \exp(\pi\wedge)X\,, \quad (2.56)$$

*where* $\mathfrak{a}$ *is the coaction (2.19) and* $\operatorname{cxp}(\pi)$ *is defined in the previous proposition.*

2. *The following map*

$$\varphi^\pi = \exp(-\pi\wedge)\varphi \exp(\pi\wedge) \,:\, C(\mathcal{L}) \otimes \mathcal{M} \mapsto C(\mathcal{L}) \otimes \mathcal{M} \quad (2.57)$$

*is a 2-nilpotent coderivation of the comodule* $C(\mathcal{L}) \otimes \mathcal{M}$ *.*

3. *If* $\tilde{\varphi} \,:\, \mathcal{L} \succ\!\rightarrow (Hom(\mathcal{M}, \mathcal{M}), \varphi_0)$ *is the* $L_\infty$*-morphism induced by the module structure* $\varphi$ *then the twisted* $L_\infty$*-morphism* $\tilde{\varphi}^\pi$ *defines the* $L_\infty$*-module structure given in (2.57).*

4. *If* $\kappa \,:\, \mathcal{M} \succ\succ\!\rightarrow \mathcal{N}$ *is a morphism of* $L_\infty$*-modules* $(\mathcal{M}, \varphi)$ *and* $(\mathcal{N}, \psi)$ *over* $\mathcal{L}$ *then the map*

$$\kappa^\pi = \exp(-\pi\wedge)\kappa \exp(\pi\wedge) \,:\, C(\mathcal{L}) \otimes \mathcal{M} \mapsto C(\mathcal{L}) \otimes \mathcal{N} \quad (2.58)$$

*is a morphism between* $L_\infty$*-modules* $(\mathcal{M}, \varphi^\pi)$ *and* $(\mathcal{N}, \psi^\pi)$ *over* $(\mathcal{L}, \mathfrak{d} + [\pi, \cdot], [,])$

5. *If* $\kappa$ *is a quasi-isomorphism of* $L_\infty$*-modules* $\mathcal{M}$ *and* $\mathcal{N}$ *and* $\kappa$ *satisfies condition 1 (on page 31) then* $\kappa^\pi$ *(2.58) is also a quasi-isomorphism.*

[2] if $X = v \in \mathcal{M}$ I set "$\pi \wedge X = \pi \otimes X$"

In what follows I refer to $\varphi^\pi$ in (2.57) and $\kappa^\pi$ in (2.58), respectively, as an $L_\infty$-module structure and a morphism of $L_\infty$-modules twisted by the Maurer-Cartan element $\pi$. It is not hard to see that the structure maps of the twisted coderivation $\varphi^\pi$ and the twisted morphism $\kappa^\pi$ are given by

$$\varphi^\pi_n(\gamma_1, \dots, \gamma_n, v) = \sum_{m=0}^{\infty} \frac{1}{m!} \varphi_{n+m}(\pi, \dots, \pi, \gamma_1, \dots, \gamma_n, v)\,, \tag{2.59}$$

$$\kappa^\pi_n(\gamma_1, \dots, \gamma_n, v) = \sum_{m=0}^{\infty} \frac{1}{m!} \kappa_{n+m}(\pi, \dots, \pi, \gamma_1, \dots, \gamma_n, v)\,, \tag{2.60}$$

where

$$\gamma_i \in \mathcal{L}\,,\ v \in \mathcal{M}\,.$$

**Proof.** Claim 1 is proved with the help of the similar scale trick ($\pi \to t\pi$) I used in the proof of the previous proposition. Claim 2 follows from claim 1 of this proposition and claim 2 of the previous proposition. Claim 4 essentially follows from claim 1 of this proposition and claim 3 is proved by comparing the corresponding structure maps.

Claim 5 is proved by the standard argument of the spectral sequence. We have to prove that the zeroth structure map $\kappa^\pi_0$ of the twisted morphism (2.58) is a quasi-isomorphism from the complex $(\mathcal{M}, \varphi^\pi_0)$ to the complex $(\mathcal{N}, \psi^\pi_0)$. These complexes are filtered and $\kappa^\pi_0$ is compatible with the filtration. Since $\kappa_0$ is a quasi-isomorphism between the complexes $(\mathcal{M}, \varphi_0)$ and $(\mathcal{N}, \psi_0)$ and $\kappa_0$ satisfies condition 1 (on page 31) the map $\kappa^\pi_0$ induces a quasi-isomorphism on the zeroth level of the corresponding spectral sequences. Therefore $\kappa^\pi_0$ gives a quasi-isomorphism on the terminal $E_\infty$-level. Hence, due to the standard snake-lemma argument of homological algebra $\kappa^\pi_0$ is also a quasi-isomorphism. $\square$

From the definitions of the above twisting procedures, it is not hard to see that these procedures are functorial. Namely,

**Proposition 3** *If $F : \mathcal{L} \succ\!\!\rightarrow \mathcal{L}^\diamond$ and $F^\diamond : \mathcal{L}^\diamond \succ\!\!\rightarrow \mathcal{L}^\clubsuit$ are $L_\infty$-morphisms of*

*DGLAs, $\pi$ is a Maurer-Cartan element of $\mathcal{L}$ and $S$ is the corresponding Maurer-Cartan element (2.42) of $\mathcal{L}^\diamond$ then*

$$(F^\diamond \circ F)^\pi = F^{\diamond S} \circ F^\pi \,,$$

*where $\circ$ stands for the composition of $L_\infty$-morphisms. Furthermore, the twisting procedure assigns to any Maurer-Cartan element of a DGLA $\mathcal{L}$ a functor from the category of $L_\infty$-modules to itself.* $\square$

Let us turn to the moduli functor of Maurer-Cartan elements and prove that this functor provides us with a homotopy invariant of a DGLA.

**Proposition 4 (K. Fukaya, [24], theorem 2.2.2)** *Let $(\mathcal{L}, \mathfrak{d}, [,])$ and $(\mathcal{L}^\diamond, \mathfrak{d}^\diamond, [,]^\diamond)$ be two completely filtered DGLAs and let $F$ be a quasi-isomorphism (see definition 3) from $\mathcal{L}$ to $\mathcal{L}^\diamond$ compatible with the filtrations in the sense of condition 1. Then (2.42) gives a bijective correspondence between the moduli spaces of $\mathcal{L}$ and $\mathcal{L}^\diamond$.*

**Remark.** The case of the ordinary (not $L_\infty$) quasi-isomorphism is treated by Goldman and Millson [31, 32]. Its generalization to $L_\infty$ setting has been a folklore[3] and was quoted by several authors (without proofs). In principle, using the "nonsense" of the homotopy theory [33], [44] it is possible to reduce the statement of the above proposition to the result of Goldman and Millson [31, 32]. In [24] K. Fukaya gives a rigorous proof of this statement both in $L_\infty$ and $A_\infty$ settings. However, since his proof is based on other results which appear elsewhere, I decided to give my own proof.

**Proof.** First I have to prove that (2.42) gives a well defined map from the moduli space of $\mathcal{L}$ to the moduli space of $\mathcal{L}^\diamond$. Due to claim 4 of proposition 1 it suffices to check that the map (2.42) of cones of Maurer-Cartan elements is compatible with the action (2.41) of $\mathcal{F}^1\mathcal{L}^0$ and $\mathcal{F}^1(\mathcal{L}^\diamond)^0$, respectively.

[3] I learnt this statement from B. Shoikhet.

If $\pi$ is a Maurer-Cartan element of $\mathcal{L}$ and $\xi \in \mathcal{F}^1\mathcal{L}^0$ then

$$\rho(\xi)(\mathrm{cxp}\,(\pi)) = \exp(\pi\wedge)Q^{\pi}(\xi)\,,$$

where $Q^{\pi}$ the DGLA structure on $\mathcal{L}$ twisted by the Maurer-Cartan element $\pi$. Hence, due to claim 3 of proposition 1

$$\rho(\xi)(\mathrm{cxp}\,(S_{\pi})) = \rho(\xi)F\mathrm{cxp}\,(\pi) = F\exp(\pi\wedge)Q^{\pi}(\xi)\,,$$

where

$$S_{\pi} = \sum_{k=1}^{\infty}\frac{1}{k!}F_k(\pi,\dots,\pi)\,.$$

Or equivalently,

$$\rho(\xi)(\mathrm{cxp}\,(S_{\pi})) = Q^{\diamond}F(\exp(\pi\wedge)\xi) = \exp(S\wedge)(Q^{\diamond})^{S_{\pi}}(F^{\pi}(\xi))\,,$$

where $F^{\pi}$ is the twisted quasi-isomorphism. Thus,

$$\rho(\xi)S_{\pi} = \rho(F^{\pi}(\xi))S_{\pi}$$

and hence (2.42) gives a well-defined map

$$F_{MC} : MC(\mathcal{L}) \mapsto MC(\mathcal{L}^{\diamond}) \tag{2.61}$$

from the moduli space $MC(\mathcal{L})$ of the DGLA $\mathcal{L}$ to the moduli space $MC(\mathcal{L}^{\diamond})$ of the DGLA $\mathcal{L}^{\diamond}$.

Let $S \in \mathcal{F}^1(\mathcal{L}^{\diamond})^1$ be a Maurer-Cartan element of $\mathcal{L}^{\diamond}$. I denote by $\mathfrak{G}$ the prounipotent group corresponding to the Lie algebra $\mathcal{F}^1\mathcal{L}^0$ and by $\mathfrak{G}[S]$ the $\mathfrak{G}$-orbit that passes through $S$. To prove surjectivity of the map (2.61) I show by induction that there exists a collection of pairs $(S_m, \pi_m)$, $m \geq 1$ where $S_m$ are Maurer-Cartan elements of

$\mathcal{L}^\diamond$ belonging to the orbit $\mathfrak{G}[S]$, $\pi_m \in \mathcal{F}^1\mathcal{L}^1$,

$$S_{m+1} - S_m \in \mathcal{F}^m\mathcal{L}^\diamond\,, \qquad \pi_{m+1} - \pi_m \in \mathcal{F}^m\mathcal{L}\,, \tag{2.62}$$

$$\mathfrak{d}\pi_m + \frac{1}{2}[\pi_m, \pi_m] \in \mathcal{F}^m\mathcal{L}\,, \tag{2.63}$$

and

$$S_m - \sum_{k=1}^{\infty} \frac{1}{k!} F_k(\pi_m, \dots, \pi_m) \in \mathcal{F}^m\mathcal{L}^\diamond\,. \tag{2.64}$$

For $m = 1$ I set $S_1 = S$, $\pi_1 = 0$. Then equations (2.63) and (2.64) obviously hold. Let me take it as a base of the induction and assume that (2.62), (2.63), and (2.64) hold up to $m$ but $S_{m+1}$ and $\pi_{m+1}$ are not chosen. It suffices to prove that there exists a pair $(S_{m+1}, \pi_{m+1})$ such that $S_{m+1} \in \mathfrak{G}[S_m]$, (2.62) is satisfied and equations (2.63), (2.64) hold for $m$ replaced by $m+1$.

Due to assumption (2.64) and the Maurer-Cartan equation $\mathfrak{d}^\diamond S_m + \frac{1}{2}[S_m, S_m]^\diamond$ I get that

$$\mathfrak{d}^\diamond(S_m - S_{\pi_m}) + (\mathfrak{d}^\diamond S_{\pi_m} + \frac{1}{2}[S_{\pi_m}, S_{\pi_m}]^\diamond) \in \mathcal{F}^{m+1}\mathcal{L}^\diamond\,, \tag{2.65}$$

where I denoted by $S_{\pi_m}$ the sum

$$S_{\pi_m} = \sum_{k=1}^{\infty} \frac{1}{k!} F_k(\pi_m, \dots, \pi_m)\,.$$

Hence due to claim 2 of proposition 1

$$\mathfrak{d}^\diamond(S_m - S_{\pi_m}) + Q^\diamond \mathrm{cxp}\,(S_{\pi_m}) \in \mathcal{F}^{m+1}(C(\mathcal{L}^\diamond))\,.$$

Therefore using claim 3 of proposition 1 one gets

$$\mathfrak{d}^\diamond(S_m - S_{\pi_m}) + FQ\mathrm{cxp}\,(\pi_m) \in \mathcal{F}^{m+1}(C(\mathcal{L}^\diamond))\,.$$

Applying assumption (2.63) and claim 2 of proposition 1 once again I get

$$\mathfrak{d}^\diamond(S_m - S_{\pi_m}) + F_1(\mathfrak{d}\pi_m + \frac{1}{2}[\pi_m, \pi_m]) \in \mathcal{F}^{m+1}\mathcal{L}^\diamond\,.$$

Since $F_1$ is a quasi-isomorphism of complexes $(\mathcal{L}, \mathfrak{d})$ and $(\mathcal{L}^\diamond, \mathfrak{d}^\diamond)$ compatible with the filtrations in the sense of condition 1 (on page 31) there exist an element $\pi_{\text{add}} \in \mathcal{F}^m\mathcal{L}^1$ and an element $\xi \in \mathcal{F}^m(\mathcal{L}^\diamond)^0$ such that

$$\mathfrak{d}\pi_{\text{add}} + \mathfrak{d}\pi_m + \frac{1}{2}[\pi_m, \pi_m] \in \mathcal{F}^{m+1}\mathcal{L}\,,$$

and

$$S_m - S_{\pi_m} + F_1(\pi_{\text{add}}) + \mathfrak{d}^\diamond\xi \in \mathcal{F}^{m+1}\mathcal{L}^\diamond\,.$$

Thus, if I set $S_{m+1} = \exp(\rho(\xi))S_m$ and $\pi_{m+1} = \pi_m + \pi_{\text{add}}$ then $S_{m+1}$ and $\pi_{m+1}$ satisfy condition (2.62) and, moreover, equations (2.63), (2.64) hold with $m$ replaced by $m+1$.

Since the DGLAs $\mathcal{L}$ and $\mathcal{L}^\diamond$ are complete with respect to the filtrations the surjectivity of the map (2.61) follows from the existence of the desired collection $(S_m, \pi_m)$.

The injectivity is proved by analyzing the differential of the map (2.61).

Indeed, let $\pi$ be a Maurer-Cartan element of $\mathcal{L}$. Then the tangent space to the cone (2.40) is cut in $\mathcal{L}^1$ by the equation

$$\mathfrak{d}\pi^t + [\pi, \pi^t] = 0\,, \qquad \pi^t \in \mathcal{L}^1\,. \tag{2.66}$$

Therefore by definition of the action (2.41) of $\mathcal{F}^1\mathcal{L}^0$ on the cone (2.40) the tangent space of the moduli space $MC(\mathcal{L})$ of Maurer-Cartan elements to the orbit $[\pi]$ passing through $\pi$ is the first cohomology group of the complex $(\mathcal{L}, \mathfrak{d} + [\pi, \cdot])$

$$T_{[\pi]}(MC(\mathcal{L})) = H^1(\mathcal{L}, \mathfrak{d} + [\pi, \cdot])\,.$$

By the assumption of the proposition $F$ is a quasi-isomorphism between $\mathcal{L}$ and $\mathcal{L}^\diamond$. Hence, due to claim 5 $F^\pi$ is a quasi-isomorphism of the twisted DGLAs $(\mathcal{L}, \mathfrak{d} + [\pi, \cdot\,], [,])$ and $(\mathcal{L}^\diamond, \mathfrak{d}^\diamond + [S_\pi, \cdot\,]^\diamond, [,])$. Therefore the differential of the map (2.61) is an isomorphism. Thus, it is injective and the proposition follows. $\square$

# Chapter 3

# Mosaic

In this chapter I recall the basic algebraic structures on Hochschild (co)chains. I formulate the main result of this thesis, the formality theorem for Hochschild chains of the algebra of functions on a smooth manifold, and state Kontsevich's and Shoikhet's formality theorems for Hochschild (co)chains of the algebra of functions on $\mathbb{R}^d$.

## 3.1 Algebraic structures on Hochschild (co)chains

For a unital associative algebra $\mathbb{A}$ (over a field of characteristic zero) I denote by $C^\bullet(\mathbb{A})$ the vector space of Hochschild cochains with a shifted grading

$$C^n(\mathbb{A}) = Hom(\mathbb{A}^{\otimes(n+1)}, \mathbb{A})\,,\ (n \geq 0)\,, \qquad C^{-1}(\mathbb{A}) = \mathbb{A}\,. \tag{3.1}$$

The space $C^\bullet(\mathbb{A})$ can be endowed with the Gerstenhaber bracket [29], defined between homogeneous elements $P_1 \in C^{k_1}(\mathbb{A})$ and $P_2 \in C^{k_2}(\mathbb{A})$ as follows

$$[P_1, P_2]_G = P_1 \bullet P_2 - (-)^{k_1 k_2} P_2 \bullet P_1\,, \tag{3.2}$$

where

$$(P_1 \bullet P_2)(a_0,\, \dots, a_{k_1+k_2}) =$$

$$\sum_{i=0}^{k_1}(-)^{ik_2}P_1(a_0,\,\ldots,P_2(a_i,\,\ldots,a_{i+k_2}),\,\ldots,a_{k_1+k_2})\,. \tag{3.3}$$

Direct computation shows that (3.2) is a Lie bracket and therefore $C^\bullet(\mathbb{A})$ is a graded Lie algebra.

For the same unital algebra $\mathbb{A}$, I denote by $C_\bullet(\mathbb{A})$ the vector space of Hochschild chains

$$C_n(\mathbb{A}) = \mathbb{A}\otimes\mathbb{A}^{\otimes n}\,,\ (n\geq 1), \qquad C_0(\mathbb{A}) = \mathbb{A}\,. \tag{3.4}$$

The space $C_\bullet(\mathbb{A})$ can be endowed with the structure of a graded module over the Lie algebra $C^\bullet(\mathbb{A})$ of Hochschild cochains. For homogeneous elements the action of $C^\bullet(\mathbb{A})$ on $C_\bullet(\mathbb{A})$ is defined as follows:

$$R: C^k(\mathbb{A})\otimes C_n(\mathbb{A})\to C_{n-k}(\mathbb{A})\,, \quad P\otimes(a_0\otimes a_1\otimes\cdots\otimes a_n)\mapsto R_P(a_0\otimes a_1\otimes\cdots\otimes a_n)$$

$$R_P(a_0\otimes a_1\otimes\cdots\otimes a_n) = \sum_{i=0}^{n-k}(-)^{ki}a_0\otimes\cdots\otimes P(a_i,\ldots,a_{i+k})\otimes\cdots\otimes a_n + \tag{3.5}$$

$$\sum_{j=n-k}^{n-1}(-)^{n(j+1)}P(a_{j+1},\ldots,a_n,a_0,\ldots,a_{k+j-n})\otimes a_{k+j+1-n}\otimes\cdots\otimes a_j\,, \qquad a_i\in\mathbb{A}\,.$$

The proof of the required axiom of the Lie algebra module

$$R_{[P_1,P_2]_G} = R_{P_1}R_{P_2} - (-)^{|P_1||P_2|}R_{P_2}R_{P_1} \tag{3.6}$$

can be found in paper [21], in which it is discussed in a more general $A_\infty$ setting (see lemma 2.3 in [21]).

The multiplication $\mu_0$ in the algebra $\mathbb{A}$ can be naturally viewed as an element of $C^1(\mathbb{A})$ and the associativity condition for $\mu_0$ can be rewritten in terms of bracket (3.2) as

$$[\mu_0,\mu_0]_G = 0\,. \tag{3.7}$$

Thus, on the one hand $\mu_0$ defines a 2-nilpotent interior derivation of the graded

Lie algebra $C^\bullet(\mathbb{A})$

$$\partial P = [\mu_0, P]_G : C^k(\mathbb{A}) \mapsto C^{k+1}(\mathbb{A})\,, \qquad \partial^2 = 0\,, \tag{3.8}$$

and on the other hand $\mu_0$ endows the graded vector space $C_\bullet(\mathbb{A})$ with the differential

$$\mathfrak{b} = R_{\mu_0} : C_k(\mathbb{A}) \mapsto C_{k-1}(\mathbb{A})\,, \qquad \mathfrak{b}^2 = 0\,. \tag{3.9}$$

Equation (3.6) implies that

$$R_{\partial P} = \mathfrak{b} R_P - (-)^k R_P \mathfrak{b}\,, \qquad P \in C^k(\mathbb{A})$$

and therefore the vector spaces $C^\bullet(\mathbb{A})$ and $C_\bullet(\mathbb{A})$ become a pair of a DGLA and a DG module over this DGLA.

**Remark 1.** Notice that the differentials (3.8) and (3.9) are exactly the Hochschild coboundary and boundary operators on $C^\bullet(\mathbb{A})$ and $C_\bullet(\mathbb{A})$, respectively. Thus, the Hochschild (co)homology groups $HH^\bullet(\mathbb{A})$ and $HH_\bullet(\mathbb{A})$ form a pair of graded Lie algebra and a graded module of this Lie algebra.

**Remark 2.** Notice that the action $R$ (3.5) is not compatible with the grading on $C_\bullet(\mathbb{A})$ and the differential (3.9) have degree $-1$ (not $+1$). In order to get the DGLA module in the sense of the previous chapter one has to use the converted grading on $C_\bullet(\mathbb{A})$. However, I prefer to restrain the conventional grading on the space of Hochschild chains keeping in mind the above remark.

Let me also recall that the graded vector space $C^{\bullet-1}(\mathbb{A})$ is endowed with the obvious associative product

$$\cup : C^{k_1-1}(\mathbb{A}) \otimes C^{k_2-1}(\mathbb{A}) \mapsto C^{k_1+k_2-1}(\mathbb{A})\,,$$

$$P_1 \cup P_2(a_1, \ldots, a_{k_1+k_2}) = P_1(a_1, \ldots, a_{k_1}) \cdot P_2(a_{k_1+1}, \ldots, a_{k_1+k_2})\,, \tag{3.10}$$

$$P_i \in C^{k_i-1}(\mathbb{A})\,, \qquad a_j \in \mathbb{A}\,,$$

where $\cdot$ denotes the product in the algebra $\mathbb{A}$.

The product (3.10) is compatible with the Hochschild differential (3.8) in the sense of the following equation

$$\partial(P_1 \cup P_2) = P_1 \cup \partial(P_2) + (-)^{k_2}\partial(P_1) \cup P_2\,, \qquad P_2 \in C^{k_2-1}(\mathbb{A})\,. \tag{3.11}$$

Thus Hochschild chains also form a DGA.

I will refer to the product (3.10) as *the cup-product* and I will use it in the proof of proposition 15.

## 3.2 Formality theorems

I will be mainly interested in the algebra $\mathbb{A}_0 = C^\infty(M)$ where $M$ is a smooth manifold of dimension $d$. A natural analogue of the complex of Hochschild cochains for this algebra is the complex $D^\bullet_{poly}(M)$ of polydifferential operators with the same differential as in $C^\bullet(C^\infty(M))$

$$D^\bullet_{poly}(M) = \bigoplus_{k=-1}^{\infty} D^k_{poly}(M)\,, \qquad D^{-1}_{poly}(M) = C^\infty(M)\,, \tag{3.12}$$

where $D^k_{poly}(M)$ consists of polydifferential operators of rank $k+1$

$$P\,:\, C^\infty(M)^{\otimes(k+1)} \mapsto C^\infty(M)\,.$$

Similarly, instead of the complex $C_\bullet(C^\infty(M))$ I consider three versions of the vector space $C^{poly}_\bullet(M)$ of Hochschild chains for $C^\infty(M)$

1. $$C^{poly}_{function}(M) = \bigoplus_{n\geq 0} C^\infty(M^{n+1})\,, \tag{3.13}$$

2.

$$C^{poly}_{germ}(M) = \bigoplus_{n \ge 0} germs_{\Delta(M^{n+1})} C^{\infty}(M^{n+1}) \,, \tag{3.14}$$

3.

$$C^{poly}_{jet}(M) = \bigoplus_{n \ge 0} jets^{\infty}_{\Delta(M^{n+1})} C^{\infty}(M^{n+1}) \,, \tag{3.15}$$

where $\Delta(M^{n+1})$ is the diagonal in $M^{n+1}$.

It is not hard to see that the Gerstenhaber bracket (3.2), the action (3.5), the differentials (3.8), (3.9), and the cup-product (3.10) still make sense if I replace $C^{\bullet}(C^{\infty}(M))$ by $D^{\bullet}_{poly}(M)$ and $C_{\bullet}(C^{\infty}(M))$ by either of versions (3.13), (3.14), (3.15) of $C^{poly}_{\bullet}(M)$. Thus, $D^{\bullet}_{poly}(M)$ and $C^{poly}_{\bullet}(M)$ are DGLA and a DG module over this DGLA, respectively, and, moreover, $D^{\bullet-1}_{poly}(M)$ is a DGA. I use the same notations for all the operations $[,]_G$, $R_P$, $\partial$, $\mathfrak{b}$, and $\cup$ when I speak of $D^{\bullet}_{poly}(M)$ and $C^{poly}_{\bullet}(M)$.

The cohomology of $D^{\bullet}_{poly}(M)$ and of $C^{poly}_{\bullet}(M)$ is described by Hochschild-Kostant-Rosenberg type theorems. The original version of the theorem [36] by Hochschild, Kostant, and Rosenberg says that the module of Hochschild homology of a smooth affine algebra is isomorphic to the module of exterior differential forms on the respective affine algebraic variety. A dual version of this theorem was proved in [55] (see corollary 4.12). In the $C^{\infty}$ setting we have

**Proposition 5 (J. Vey, [54])** *Let*

$$T^{\bullet}_{poly}(M) = \bigoplus_{k=-1}^{\infty} T^{k}_{poly}(M) \,, \qquad T^{k}_{poly}(M) = \Gamma(M, \wedge^{k+1} TM) \tag{3.16}$$

*be a vector space of the polyvector fields on $M$ with shifted grading. If $T^{\bullet}_{poly}(M)$ is regarded as a complex with a vanishing differential then the natural map*

$$\mathcal{V}(\gamma)(a_0, \dots, a_k) = i_{\gamma}(da_0 \wedge \dots \wedge da_k) \, : \, T^{k}_{poly}(M) \mapsto D^{k}_{poly}(M) \,, \qquad k \ge -1 \tag{3.17}$$

*defines a quasi-isomorphism of complexes $(T^{\bullet}_{poly}(M), 0)$ and $(D^{\bullet}_{poly}(M), \partial)$. Here $d$ stands for the De Rham differential and $i_{\gamma}$ denotes the contraction of the polyvector* *field $\gamma$ with an exterior form.*

The most general $C^\infty$-manifold version of the Hochschild-Kostant-Rosenberg theorem is due to N. Teleman [52][1]

**Proposition 6 (Teleman, [52])** *Let*

$$\mathcal{A}^\bullet(M) = \bigoplus_{k\geq 0} \mathcal{A}^k(M)\,, \qquad \mathcal{A}^k(M) = \Gamma(M, \wedge^k T^* M) \tag{3.18}$$

*be a vector space of the exterior forms on $M$. If $\mathcal{A}^\bullet(M)$ is regarded as a complex with a vanishing differential then the natural map*

$$\mathfrak{C}(a_0 \otimes \cdots \otimes a_k) = a_0 da_1 \wedge \cdots \wedge da_k \,:\, C^{poly}_k(M) \mapsto A^k(M)\,, \qquad k \geq 0 \tag{3.19}$$

*defines a quasi-isomorphism of complexes $(C^{poly}_\bullet(M), \mathfrak{b})$ and $(\mathcal{A}^\bullet(M), 0)$ for either of versions (3.13), (3.14), (3.15) of $C^{poly}_\bullet(M)$.*

One can easily check that the Lie algebra structure induced on cohomology

$$H^\bullet(D^\bullet_{poly}(M), \partial) = T^\bullet_{poly}(M)$$

coincides with the one given by the so-called Schouten-Nijenhuis bracket

$$[,]_{SN} \,:\, T^\bullet_{poly}(M) \bigwedge T^\bullet_{poly}(M) \mapsto T^\bullet_{poly}(M)\,.$$

This bracket is defined as an ordinary Lie bracket between vector fields and then extended by Leibniz rule

$$[\gamma_1, \gamma_2 \wedge \gamma_3]_{SN} = [\gamma_1, \gamma_2]_{SN} \wedge \gamma_3 + (-1)^{|\gamma_1|(|\gamma_2|-1)} \gamma_2 \wedge [\gamma_1, \gamma_3]_{SN}\,, \quad \gamma_i \in T^\bullet_{poly}(M) \tag{3.20}$$

with respect to the $\wedge$-product to an arbitrary pair of polyvector fields.

Furthermore, the DGLA $D^\bullet_{poly}(M)$-module structure on $C^{poly}_\bullet(M)$ induces a $T^\bullet_{poly}(M)$-module structure on the vector space $\mathcal{A}^\bullet(M)$ which coincides with the one defined by

[1]See also [15], in which this statement was proven for any compact smooth manifold.

the action of a polyvector field on exterior forms via the Lie derivative

$$L_\gamma = d\, i_\gamma + (-)^k i_\gamma\, d\,, \qquad \gamma \in T^k_{poly}(M)\,, \tag{3.21}$$

where as above $d$ stands for the De Rham differential and $i_\gamma$ denotes the contraction of the polyvector field $\gamma$ with an exterior form.

**Remark.** In what follows I will restrict myself to the third version (3.15) of $C^{poly}_\bullet(M)$ and since all $D^\bullet_{poly}(M)$-modules (3.13), (3.14), (3.15) are naturally quasi-isomorphic the further results will hold for versions (3.13), (3.14) as well.

For my purposes it will be very convenient to represent the chains (3.15) as $C^\infty(M)$-linear homomorphisms from $D^\bullet_{poly}(M)$ to $C^\infty(M)$. Namely, one can equivalently define ($k \ge 0$)

$$C^{poly}_k(M) = Hom_{C^\infty(M)}(D^{k-1}_{poly}(M), C^\infty(M))\,. \tag{3.22}$$

To avoid the shift in the above formula let me introduce the auxiliary graded bundle of polyjets placed in non-negative degrees

$$J_\bullet = \bigoplus_{k\ge 0} J_k\,, \qquad J_k = Hom_{\mathcal{O}_M}(D^k_{poly}, \mathcal{O}_M)\,, \tag{3.23}$$

where $\mathcal{O}_M$ denotes the structure sheaf of (smooth) functions on $M$ and $D^\bullet_{poly}$ is the sheaf of polydifferential operators.

Note that although

$$J_k(M) = C^{poly}_{k+1}(M), \qquad k \ge 0$$

I would like to reserve special notation for the bundle (3.23) and distinguish $J_\bullet(M)$ and $C^{poly}_\bullet(M)$. Let me, from now on, refer to elements of $C^{poly}_\bullet(M)$ as *Hochschild chains* and to elements of $J_\bullet(M)$ as *polyjets*.

The bundle $J_\bullet$ is endowed with a canonical flat connection $\nabla^G$ which is called the *Grothendieck connection* and defined by the formula

$$\nabla^G_u(j)(P) := u(j(P)) - j(u\bullet P)\,, \tag{3.24}$$

where $j \in J_k(M)$, $P \in D^k_{poly}(M)$, and $u$ is a vector field which is viewed, in the right hand side, as a differential operator. The operation $\bullet$ is defined in (3.3).

For this connection we have the following remarkable proposition:

**Proposition 7** *Let $\chi$ be a linear map ($k \geq 0$)*

$$\chi : J_k(M) \to C^{poly}_k(M)$$

*defined by the formula*

$$\chi(a)(P) = a(1 \otimes P)\,, \quad P \in D^{k-1}_{poly}(M)\,, \quad a \in J_k(M)\,. \tag{3.25}$$

*The restriction of the map $\chi$ to the $\nabla^G$-flat polyjets gives the ($\mathbb{R}$-linear) isomorphism ($k \geq 0$)*

$$\chi : \ker \nabla^G \cap J_k(M) \stackrel{\sim}{\to} C^{poly}_k(M)\,. \tag{3.26}$$

**Proof.** To see that the map (3.26) is surjective one has to notice that for any Hochschild chain $b \in C^{poly}_k(M)$ the equations

$$a(1 \otimes P) = b(P)\,, \qquad P \in D^{k-1}_{poly}(M)$$

and

$$a(u \cdot Q \otimes P) = u\, a(Q \otimes P) - a(Q \otimes (u\bullet P))\,, \tag{3.27}$$

$$Q \in D^0_{poly}(M)\,, \qquad u \in \Gamma(M, TM)$$

define a $\nabla^G$-flat polyjet $a$ of degree $k$.

On the other hand, if $a$ is a $\nabla^G$-flat polyjet of degree $k$ equation (3.27) is automatically satisfied. Thus $a$ is uniquely determined by its image $\chi(a)$. $\square$

Let $t$ be the cyclic permutation acting on the sheaf $J_\bullet$ of polyjets

$$t(a)(P_0 \otimes \cdots \otimes P_l) := a(P_1 \otimes \cdots \otimes P_l \otimes P_0)\,, \tag{3.28}$$

$$a \in J_l(M)\,, \qquad P_i \in D^0_{poly}(M)\,.$$

Using this operation and the bilinear product (3.3) I define the map

$$\hat{R} : D^k_{poly} \otimes J_l \to J_{l-k}\,, \qquad P \otimes a \mapsto \hat{R}_P(a)\,,$$

$$\hat{R}_P(a)(Q_0 \otimes Q) = a((Q_0 \otimes Q) \bullet P) + \sum_{j=1}^{k} (-1)^{lj} t^j(a)\big((Q_0 \bullet P) \otimes Q\big)\,, \tag{3.29}$$

$$P \in D^k_{poly}(M)\,, \qquad a \in J_l(M)\,, \qquad Q \in D^{l-k-1}_{poly}(M)\,, \qquad Q_0 \in D^0_{poly}(M)\,.$$

Following the lines of the proof of lemma 2.3 in [21] one can show that

$$\hat{R}_{[P_1,P_2]_G} = \hat{R}_{P_1}\hat{R}_{P_2} - (-)^{|P_1||P_2|}\hat{R}_{P_2}\hat{R}_{P_1}\,, \tag{3.30}$$

and hence, (3.23) is a sheaf of graded modules over the sheaf of graded Lie algebras $D^\bullet_{poly}$. Furthermore, using the multiplication $\mu_0 \in D^1_{poly}(M)$ in $C^\infty(M)$ one can turn the $D^\bullet_{poly}$-module (3.23) into a DGLA $D^\bullet_{poly}$-module by introducing the following differential

$$\hat{\mathfrak{b}} = \hat{R}_{\mu_0} : J_k \mapsto J_{k-1}\,. \tag{3.31}$$

It follows from the construction that both the action (3.29) and the differential (3.31) commute with the Grothendieck connection (3.24). Thus the $\nabla^G$-flat polyjets $\ker \nabla^G \cap J_\bullet(M)$ form a DG module over the DGLA $D^\bullet_{poly}(M)$.

A direct but slightly tedious computation shows that

**Proposition 8** *The DG module structure on $C^{poly}_\bullet(M)$ over the DGLA $D^\bullet_{poly}(M)$ induced from (3.29) and (3.31) via the isomorphism (3.26) coincides with standard one given by (3.5) and (3.9).* $\square$

This proposition allows me to identify $\nabla^G$-flat polyjets $\ker \nabla^G \cap J_\bullet(M)$ and Hochschild chains $C^{poly}_\bullet(M)$ as DGLA modules. This identification will be very handy for the construction of the Fedosov resolution for $C^{poly}_\bullet(M)$.

Unfortunately, the maps (3.17) and (3.19) are not compatible with the Lie brackets on $T^\bullet_{poly}(M)$ and $D^\bullet_{poly}(M)$ and with the respective actions (3.5) and (3.21). In particular, the equation

$$\mathfrak{C} \circ R_{\mathcal{V}(\gamma)} \stackrel{?}{=} L_\gamma \mathfrak{C} \tag{3.32}$$

does not hold in general. In [53] B. Tsygan suggested that this defect could be cured by the following statement:

**Conjecture 1 (B. Tsygan, [53])** *For any smooth manifold $M$ the DGLA modules $(T^\bullet_{poly}(M), \mathcal{A}^\bullet(M))$ and $(D^\bullet_{poly}(M), C^{poly}_\bullet(M))$ are quasi-isomorphic.*

The following theorem gives a positive answer to the question of B. Tsygan.

**Theorem 1** *For any smooth manifold $M$ there exists a commutative diagram of DGLAs and DGLA modules*

$$\begin{array}{ccccccc}
T^\bullet_{poly}(M) & \succ\!\!\longrightarrow & \mathcal{L}_1 & \succ\!\!\longrightarrow & \mathcal{L}_2 & \longleftarrow\!\!\prec & D^\bullet_{poly}(M) \\
\downarrow_{mod} & & \downarrow_{mod} & & \downarrow_{mod} & & \downarrow_{mod} \\
\mathcal{A}^\bullet(M) & \succ\!\!\succ\!\!\longrightarrow & \mathcal{M}_1 & \longleftarrow\!\!\prec\!\!\prec & \mathcal{M}_2 & \longleftarrow\!\!\prec\!\!\prec & C^{poly}_\bullet(M)\,,
\end{array} \tag{3.33}$$

*in which the horizontal arrows in the upper row are quasi-isomorphisms of DGLAs and the horizontal arrows in the lower row are quasi-isomorphisms of $L_\infty$-modules. The terms ($\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{M}_1$, $\mathcal{M}_2$) and the quasi-isomorphisms of diagram (3.33) are functorial for diffeomorphisms of pairs "manifold $M$ + a torsion free connection on $TM$".*

The construction of the quasi-isomorphisms in diagram (3.33) is explicit and in chapter 5 I show how this result allows us to prove Tsygan's conjecture (see the first part of corollary 4.0.3 in [53]) about Hochschild homology of the quantum algebra of functions on an arbitrary Poisson manifold, and in particular, to describe the space of traces on this algebra.

The main part of the proof of theorem 1 concerns the construction of Fedosov resolutions of the DGLA modules $(T^{\bullet}_{poly}(M), \mathcal{A}^{\bullet}(M))$ and $(D^{\bullet}_{poly}(M), C^{poly}_{\bullet}(M))$. After completing this stage it will only remain to use Kontsevich's [38] and Shoikhet's [46] formality theorems for $\mathbb{R}^d_{formal}$ and apply the twisting procedures developed in the previous chapter.

Let us now recall these formality theorems.

**Theorem 2 (M. Kontsevich, [38])** *There exists a quasi-isomorphism* $\mathcal{K}$

$$\mathcal{K} \,:\, T^{\bullet}_{poly}(\mathbb{R}^d) \succ\!\!\!\rightarrow D^{\bullet}_{poly}(\mathbb{R}^d) \tag{3.34}$$

*from the DGLA* $T^{\bullet}_{poly}(\mathbb{R}^d)$ *of polyvector fields to the DGLA* $D^{\bullet}_{poly}(\mathbb{R}^d)$ *of polydifferential operators on the space* $\mathbb{R}^d$ *such that*

1. *One can replace* $\mathbb{R}^d$ *in (3.34) by its formal completion* $\mathbb{R}^d_{formal}$ *at the origin.*

2. *The quasi-isomorphism* $\mathcal{K}$ *is equivariant with respect to linear transformations of the coordinates on* $\mathbb{R}^d_{formal}$.

3. *If* $n > 1$ *then*
$$\mathcal{K}_n(u_1, u_2, \dots, u_n) = 0 \tag{3.35}$$
*for any set of vector fields* $u_1, u_2, \dots, u_n \in T^0_{poly}(\mathbb{R}^d_{formal})$.

4. *If* $n \geq 2$ *and* $u \in T^0_{poly}(\mathbb{R}^d_{formal})$ *is linear in the coordinates on* $\mathbb{R}^d_{formal}$ *then for any set of polyvector fields* $\gamma_2, \dots, \gamma_n \in T^{\bullet}_{poly}(\mathbb{R}^d_{formal})$
$$\mathcal{K}_n(u, \gamma_2, \dots, \gamma_n) = 0\,. \tag{3.36}$$

Composing the quasi-isomorphism $\mathcal{K}$ with the action (3.5) of $D^{\bullet}_{poly}(\mathbb{R}^d)$ on $C^{poly}_{\bullet}(\mathbb{R}^d)$ I get an $L_{\infty}$-module structure on $C^{poly}_{\bullet}(\mathbb{R}^d)$ over the DGLA $T^{\bullet}_{poly}(\mathbb{R}^d)$. For this module structure we have the following results:

**Theorem 3 (B. Shoikhet, [46])** *There exists a quasi-isomorphism* $\mathcal{S}$

$$\mathcal{S} \,:\, C^{poly}_{\bullet}(\mathbb{R}^d) \succ\!\!\succ\!\!\longrightarrow \mathcal{A}^{\bullet}(\mathbb{R}^d) \tag{3.37}$$

*of* $L_\infty$*-modules over* $T_{poly}(\mathbb{R}^d)$*, the zeroth structure map* $\mathcal{S}_0$ *of which is the map (3.19) of Connes and such that*

1. *One can replace* $\mathbb{R}^d$ *in (3.37) by its formal completion* $\mathbb{R}^d_{formal}$ *at the origin.*

2. *The quasi-isomorphism* $\mathcal{S}$ *is equivariant with respect to linear transformations of the coordinates on* $\mathbb{R}^d_{formal}$*.*

**Proposition 9** *If* $\mathcal{S}$ *be the quasi-isomorphism (3.37) of B. Shoikhet,* $n \geq 1$*, and* $u \in T^0_{poly}(\mathbb{R}^d_{formal})$ *is linear in the coordinates on* $\mathbb{R}^d_{formal}$ *then for any set of polyvector fields* $\gamma_2, \dots, \gamma_n \in T^{\bullet}_{poly}(\mathbb{R}^d_{formal})$ *and any Hochschild chain* $a \in C^{poly}_{\bullet}(\mathbb{R}^d_{formal})$

$$\mathcal{S}_n(u, \gamma_2, \dots, \gamma_n; a) = 0\,. \tag{3.38}$$

**Proof.** The proof of (3.38) reduces to calculation of integrals entering the construction of the structure maps $\mathcal{S}_n$ (see section 2.2 of [46]). To do this calculation I first transform the unit disk $\{|\zeta| \leq 1\}$ used in section 2.2 of [46] into the upper half plane $\mathcal{H}^+ = \{z,\, Im\,(z) \geq 0\}$ via the standard fractional linear transformation

$$z = -i\frac{\zeta+1}{\zeta-1}\,. \tag{3.39}$$

The origin of the unit disk goes to $z = i$ and the point $\zeta = 1$ goes to $z = \infty$. The angle function corresponding to an edge of the first type [46] (see figure A-1) connecting $p \neq i$ and $q \neq i$ looks as follows

$$\alpha^{Sh}(p,q) = Arg(p-q) - Arg(\bar{p}-q) - Arg(p-i) + Arg(\bar{p}-i)\,. \tag{3.40}$$

If I fix the rotation symmetry by placing the first function of the Hochschild chain at the point $z = \infty$ then the angle function corresponding to an edge of the second type (see figure A-2) connecting $p = i$ and $q$ takes the form

$$\beta^{Sh}(q) = Arg(i - q) - Arg(-i - q)\,. \tag{3.41}$$

Let us suppose that $u$ is a vector linear in coordinates on $\mathbb{R}^d_{formal}$. Then there are three types of the diagrams corresponding to $\mathcal{S}_n(u, \dots)$ $n \geq 2$. In the diagram of the first type (see figure A-3) there are no edges ending at the vertex $z$ corresponding to the vector $u$. In the diagrams of the second type (see figure A-4) there is exactly one edge ending at the vertex $z$ and this wedge does not start at the vertex $i$. In the diagrams of the third type (see figure A-5) there is exactly one edge ending at the vertex $z$ and this wedge starts at the vertex $i$.

The coefficient corresponding to a diagram of the first type vanishes because the angle functions entering the integrand form turn out to be dependent. The coefficients corresponding to diagrams of the second and the third type vanish since so do the following integrals

$$\int_{z \in \mathcal{H}^+ \setminus \{w,v,i\}} d\alpha^{Sh}(w,z) d\alpha^{Sh}(z,v) = 0\,, \quad \int_{z \in \mathcal{H}^+ \setminus \{v,i\}} d\beta^{Sh}(z) d\alpha^{Sh}(z,v) = 0\,. \tag{3.42}$$

Equations (3.42) follow immediately from lemmas 7.3, 7.4, and 7.5 in [38]. □

**Remark.** Hopefully, alternative proofs of theorems 2, 3, and proposition 9 may be obtained along the lines of Tamarkin and Tsygan [48, 49, 50].

# Chapter 4

# Fedosov resolutions of the DGLA modules $(T^{\bullet}_{poly}(M),\ \mathcal{A}^{\bullet}(M))$ and $(D^{\bullet}_{poly}(M),\ C^{poly}_{\bullet}(M))$

In paper [22] B. Fedosov proposed a simple geometric construction for star-products on an arbitrary symplectic manifold. The key idea of Fedosov's construction has various incarnations and it is referred to as the Gelfand-Fuchs trick [27] or formal geometry [28] in the sense of I.M. Gelfand and D.A. Kazhdan , or mixed resolutions [57]. This idea can be roughly formulated as the following slogan: "In order to linearize a problem one has to formulate it in terms of jets".

If $M$ is smooth manifold the bundle of jets $J_0$ (3.23) is non-canonically isomorphic to the bundle $\mathcal{S}M$ of the formally completed symmetric algebra of the cotangent bundle $T^*M$. For this reason I start with the definition of this bundle.

**Definition 10** *The bundle $\mathcal{S}M$ of the formally completed symmetric algebra of the cotangent bundle $T^*M$ is defined as a bundle over the manifold $M$ whose sections are infinite collections of symmetric covariant tensors $a_{i_1 \dots i_p}(x)$ , where $x^i$ are local coordinates, $p$ runs from $0$ to $\infty$ , and the indices $i_1, \dots, i_p$ run from $1$ to $d$ .*

It is convenient to introduce auxiliary variables $y^i$ , which transform as contravariant vectors. These variables allow us to rewrite any section $a \in \Gamma(M, \mathcal{S}M)$ in the form of the formal power series

$$a = a(x,y) = \sum_{p=0}^{\infty} a_{i_1 \dots i_p}(x) y^{i_1} \dots y^{i_p} \,. \tag{4.1}$$

It is easy to see that the vector space $\Gamma(M, \mathcal{S}M)$ is naturally endowed with the commutative product which is induced by a fiberwise multiplication of formal power series in $y^i$. This product makes $\Gamma(M, \mathcal{S}M)$ into a commutative algebra with a unit.

Now I recall from [19] definitions of formal fiberwise polyvector fields and formal fiberwise polydifferential operators on $\mathcal{S}M$.

**Definition 11** *A bundle $\mathcal{T}^k_{poly}$ of formal fiberwise polyvector fields of degree $k$ is a bundle over $M$ whose sections are $C^\infty(M)$-linear operators $\mathfrak{v} : \wedge^{k+1}\Gamma(M, \mathcal{S}M) \mapsto \Gamma(M, \mathcal{S}M)$ of the form*

$$\mathfrak{v} = \sum_{p=0}^{\infty} \mathfrak{v}^{j_0 \dots j_k}_{i_1 \dots i_p}(x) y^{i_1} \dots y^{i_p} \frac{\partial}{\partial y^{j_0}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_k}} \,, \tag{4.2}$$

*where I assume that the infinite sum in $y$'s is formal and $\mathfrak{v}^{j_0 \dots j_k}_{i_1 \dots i_p}(x)$ are tensors symmetric in indices $i_1, \dots, i_p$ and antisymmetric in indices $j_0, \dots, j_k$.*

Extending the definition of the formal fiberwise polyvector field by allowing the fields to be inhomogeneous I define the total bundle $\mathcal{T}_{poly}$ of formal fiberwise polyvector fields

$$\mathcal{T}_{poly} = \bigoplus_{k=-1}^{\infty} \mathcal{T}^k_{poly} \,, \qquad \mathcal{T}^{-1}_{poly} = \mathcal{S}M \,. \tag{4.3}$$

The fibers of the bundle $\mathcal{T}_{poly}$ are endowed with the DGLA structure $T_{poly}(\mathbb{R}^d_{formal})$ of polyvector fields on the formal completion $\mathbb{R}^d_{formal}$ of $\mathbb{R}^d$ at the origin. This turns $\mathcal{T}_{poly}$ into a sheaf of DGLAs (with the vanishing differential).

**Definition 12** *A bundle $\mathcal{D}^k_{poly}$ of formal fiberwise polydifferential operator of degree $k$ is a bundle over $M$ whose sections are $C^\infty(M)$-polylinear maps $\mathfrak{P} : \bigotimes^{k+1} \Gamma(M, \mathcal{S}M) \mapsto$*

*$\Gamma(M, \mathcal{S}M)$ of the form*

$$\mathfrak{P} = \sum_{\alpha_0 \dots \alpha_k} \sum_{p=0}^{\infty} \mathfrak{P}^{\alpha_0 \dots \alpha_k}_{i_1 \dots i_p}(x) y^{i_1} \dots y^{i_p} \frac{\partial}{\partial y^{\alpha_0}} \otimes \dots \otimes \frac{\partial}{\partial y^{\alpha_k}} \,, \tag{4.4}$$

*where $\alpha$'s are multi-indices $\alpha = j_1 \dots j_l$ and*

$$\frac{\partial}{\partial y^{\alpha}} = \frac{\partial}{\partial y^{j_1}} \dots \frac{\partial}{\partial y^{j_l}} \,,$$

*the infinite sum in $y$'s is formal, and the sum in the orders of derivatives $\partial/\partial y$ is finite.*

Notice that the tensors $\mathfrak{P}^{\alpha_0 \dots \alpha_k}_{i_1 \dots i_p}(x)$ are symmetric in covariant indices $i_1, \dots, i_p$.

As well as for polyvector fields I define the total bundle $\mathcal{D}_{poly}$ of formal fiberwise polydifferential operators as the direct sum

$$\mathcal{D}_{poly} = \bigoplus_{k=-1}^{\infty} \mathcal{D}^k_{poly} \,, \qquad \mathcal{D}^{-1}_{poly} = \mathcal{S}M \,. \tag{4.5}$$

The fibers of the bundle $\mathcal{D}_{poly}$ are endowed with the DGLA structure (and DGA structure) $D_{poly}(\mathbb{R}^d_{formal})$ of polydifferential operators on $\mathbb{R}^d_{formal}$. This turns $\mathcal{D}_{poly}$ into a sheaf of DGLAs and a sheaf of DGAs.

**Definition 13** *A bundle $\mathcal{C}^{poly}_k$ of formal fiberwise Hochschild chains of degree $k$ ($k \geq 0$) is a bundle over $M$ whose sections are formal power series in $k+1$ collections of fiber coordinates $y^i_0, \dots, y^i_k$ of the tangent bundle*

$$a(x, y_0, \dots, y_k) = \sum_{\alpha_0 \dots \alpha_k} a_{\alpha_0 \dots \alpha_k}(x) y_0^{\alpha_0} \dots y_k^{\alpha_k} \,, \tag{4.6}$$

*where $\alpha$'s are multi-indices $\alpha = j_1 \dots j_l$ and*

$$y^{\alpha} = y^{j_1} y^{j_2} \dots y^{j_l} \,.$$

The total bundle $\mathcal{C}^{poly}$ of formal fiberwise Hochschild chains is the direct sum

$$\mathcal{C}^{poly} = \bigoplus_{k=0}^{\infty} \mathcal{C}_k^{poly}, \qquad \mathcal{C}_0^{poly} = \mathcal{S}M\,. \tag{4.7}$$

The operations $R$ (3.5) and $\mathfrak{b}$ (3.9) turn each fiber of $\mathcal{C}^{poly}$ into a DGLA $D_{poly}(\mathbb{R}^d_{formal})$-module. Thus $\mathcal{C}^{poly}$ is a sheaf of DG modules over the sheaf of DGLAs $\mathcal{D}_{poly}$.

As above, I denote by $\mathcal{A}^\bullet(M)$ the space of exterior forms

$$\mathcal{A}^\bullet(M) = \bigoplus_{k=0}^{\infty} \mathcal{A}^k(M)\,, \qquad \mathcal{A}^k(M) = \{a = a_{i_1\dots i_k}(x) dx^{i_1}\dots dx^{i_k}\}\,. \tag{4.8}$$

Furthermore,

**Definition 14** *The bundle $\mathcal{E}$ of fiberwise exterior forms is a bundle over $M$ whose sections are exterior forms with values in $\mathcal{S}M$. These sections are given by the following formal power series*

$$a(x,y,dx) = \sum_{p,q\ge 0} a_{i_1\dots i_p\,;\,j_1\dots j_q}(x) y^{i_1}\dots y^{i_p} dx^{j_1}\dots dx^{j_q}\,, \tag{4.9}$$

*where $a_{i_1\dots i_p\,;\,j_1\dots j_q}(x)$ are components of covariant tensors symmetric in indices $i_1,\dots,i_p$ and antisymmetric in indices $j_1,\dots,j_q$.*

The fiberwise analogue of the Lie derivative (3.21) allows me to speak of $\mathcal{E}$ as of a sheaf of modules over the sheaf of DGLAs $\mathcal{T}_{poly}$.

For my purposes I will also need "exterior forms with values in exterior forms". This forces me to introduce an additional copy $\{dy^i\}$ of the local basis $\{dx^i\}$ of exterior forms on $M$. Having these two copies I reserve the notation $\Omega^\bullet(M,\mathcal{B})$ for the graded vector space of $dy$-exterior forms with values in the bundle $\mathcal{B}$. In particular, I would like to distinguish the graded vector spaces $\Omega^\bullet(M,\mathcal{S}M)$ and $\Gamma(M,\mathcal{E})$. $\Omega^\bullet(M,\mathcal{S}M)$ consists of *$dy$-forms* and $\Gamma(M,\mathcal{E})$ consists of *$dx$-forms*.

For the relations between $dx^i$ and $dy^j$ I accept the following convention

$$dx^i dy^j = -dy^j dx^i\,.$$

Homogeneous elements of the graded vector spaces $\Omega^\bullet(M, \mathcal{T}_{poly})$ and $\Omega^\bullet(M, \mathcal{D}_{poly})$ are the following formal series in $y$'s

$$\mathfrak{v} = \sum_{p\ge 0} dy^{l_1} \dots dy^{l_q} \mathfrak{v}^{j_0 \dots j_k}_{l_1 \dots l_q\,; i_1 \dots i_p}(x) y^{i_1} \dots y^{i_p} \frac{\partial}{\partial y^{j_0}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_k}}\,, \tag{4.10}$$

and

$$\mathfrak{P} = \sum_{\alpha_0 \dots \alpha_k} \sum_{p \ge 0} dy^{l_1} \dots dy^{l_q} \mathfrak{P}^{\alpha_0 \dots \alpha_k}_{l_1 \dots l_q\,; i_1 \dots i_p}(x) y^{i_1} \dots y^{i_p} \frac{\partial}{\partial y^{\alpha_0}} \otimes \dots \otimes \frac{\partial}{\partial y^{\alpha_k}}\,, \tag{4.11}$$

where as above $\alpha$'s are multi-indices $\alpha = j_1 \dots j_l$ and

$$\frac{\partial}{\partial y^{\alpha}} = \frac{\partial}{\partial y^{j_1}} \dots \frac{\partial}{\partial y^{j_l}}\,.$$

Similarly, homogeneous elements of $\Omega^\bullet(M, \mathcal{E})$ and $\Omega^\bullet(M, \mathcal{C}^{poly})$ are the formal series

$$a(x, dy, y, dx) = \sum_{p \ge 0} dy^{l_1} \dots dy^{l_q} a_{l_1 \dots l_q\,; i_1 \dots i_p j_1 \dots j_k}(x) y^{i_1} \dots y^{i_p} dx^{j_1} \dots dx^{j_k}\,, \tag{4.12}$$

and

$$b(x, dy, y_0, \dots, y_k) = \sum_{\alpha_0 \dots \alpha_k} dy^{l_1} \dots dy^{l_q} b_{l_1 \dots l_q\,; \alpha_0 \dots \alpha_k}(x) y_0^{\alpha_0} \dots y_k^{\alpha_k}\,, \tag{4.13}$$

where as above $\alpha$'s are multi-indices $\alpha = j_1 \dots j_l$ and

$$y^{\alpha} = y^{j_1} y^{j_2} \dots y^{j_l}\,.$$

The symmetries of tensor indices in formulas (4.10), (4.11), (4.12), and (4.13) are obvious.

The space $\Omega^\bullet(M, \mathcal{S}M)$ is naturally endowed with the structure of a $\mathbb{Z}$-graded commutative algebra and it is also filtered with respect to the powers in $y$'s. The graded vector spaces $\Omega^\bullet(M, \mathcal{T}_{poly})$ and $\Omega^\bullet(M, \mathcal{D}_{poly})$ are, in turn, endowed with fiberwise DGLA structures induced by those on $T_{poly}(\mathbb{R}^d_{formal})$ and $D_{poly}(\mathbb{R}^d_{formal})$. Similarly,

$\Omega^\bullet(M,\mathcal{E})$ and $\Omega^\bullet(M,\mathcal{C}^{poly})$ become fiberwise DGLA modules over $\Omega^\bullet(M,\mathcal{T}_{poly})$ and $\Omega^\bullet(M,\mathcal{D}_{poly})$, respectively[1]. I denote the Lie bracket in $\Omega^\bullet(M,\mathcal{D}_{poly})$ by $[,]_G$ and the Lie bracket in $\Omega^\bullet(M,\mathcal{T}_{poly})$ by $[,]_{SN}$ . For fiberwise Lie derivative on $\Omega^\bullet(M,\mathcal{E})$ and for the fiberwise action of $\Omega^\bullet(M,\mathcal{D}_{poly})$ on $\Omega^\bullet(M,\mathcal{C}^{poly})$ I also use the same notation $L$ and $R$, respectively. It is not hard to see that the formulas for the fiberwise differentials on $\Omega^\bullet(M,\mathcal{D}_{poly})$ and $\Omega^\bullet(M,\mathcal{C}^{poly})$ can be written similarly to (3.8) and (3.9)

$$\partial = [\mu, \cdot\,]\,, \qquad \mathfrak{b} = R_\mu\,,$$

where $\mu \in \Gamma(M,\mathcal{D}^1_{poly})$ is the (commutative) multiplication in $\Gamma(M,\mathcal{S}M)$ . Notice that $\Omega^\bullet(M,\mathcal{D}_{poly})$ is also endowed with a fiberwise DGA structure induced by that on $D_{poly}(\mathbb{R}^d_{formal})$ .

The parity of elements in the algebras $\Omega^\bullet(M,\mathcal{T}_{poly})$, $\Omega^\bullet(M,\mathcal{D}_{poly})$ and the modules $\Omega^\bullet(M,\mathcal{E})$ and $\Omega^\bullet(M,\mathcal{C}^{poly})$ is defined by the sum of the exterior degree and the degree in the respective fiberwise algebra or the respective fiberwise module.

The following proposition shows that I have a distinguished sheaf of graded Lie algebras which acts on the sheaves $\mathcal{S}M$, $\mathcal{T}_{poly}$, $\mathcal{E}$, $\mathcal{D}_{poly}$, and $\mathcal{C}^{poly}$ .

**Proposition 10** *$\mathcal{T}^0_{poly}$ is a sheaf of graded Lie algebras. $\mathcal{S}M$, $\mathcal{E}$, $\mathcal{T}_{poly}$, $\mathcal{D}_{poly}$, and $\mathcal{C}^{poly}$ are sheaves of modules over $\mathcal{T}^0_{poly}$ and the action of $\mathcal{T}^0_{poly}$ is compatible with the (DG) algebraic structures on $\mathcal{S}M$, $\mathcal{E}$, $\mathcal{T}_{poly}$, $\mathcal{D}_{poly}$, and $\mathcal{C}^{poly}$ .*

**Proof.** Since the Schouten-Nijenhuis bracket (3.20) has degree zero $\mathcal{T}^0_{poly} \subset \mathcal{T}_{poly} \subset \mathcal{D}_{poly}$ is a subsheaf of graded Lie algebras. While the action of $\mathcal{T}^0_{poly}$ on the sections of $\mathcal{S}M$ is obvious, the action on $\mathcal{E}$ is given by the Lie derivative, the action on $\mathcal{T}_{poly}$ is the adjoint action corresponding to the Schouten-Nijenhuis bracket, the action on $\mathcal{D}_{poly}$ is given by the Gerstenhaber bracket and the action on $\mathcal{C}^{poly}$ is induced by the action of Hochschild cochains on Hochschild chains (3.5). The compatibility of the action with the cup product (3.10) in $\mathcal{D}_{poly}$ essentially follows from the fact that $\mathcal{T}^0_{poly}$ acts by derivations on the sheaf of algebras $\mathcal{S}M$. The compatibility with the remaining DG algebraic structures follows from the definitions. $\square$

[1] I regard $\Omega^\bullet(M,\mathcal{T}_{poly})$ and $\Omega^\bullet(M,\mathcal{E})$ as a DGLA and a DGLA module with vanishing differentials.

This proposition implies that the canonical vector field $dy^i \frac{\partial}{\partial y^i} \in \Omega^1(M, \mathcal{T}^0_{poly})$ defines the differential

$$\delta = dy^i \frac{\partial}{\partial y^i} \cdot \, : \, \Omega^\bullet(M, \mathcal{B}) \mapsto \Omega^{\bullet+1}(M, \mathcal{B}) \, , \qquad \delta^2 = 0 \, , \tag{4.14}$$

where $\mathcal{B}$ is either of the bundles $\mathcal{S}M$, $\mathcal{T}_{poly}$, $\mathcal{D}_{poly}$, $\mathcal{E}$, or $\mathcal{C}^{poly}$ and $\cdot$ denotes the corresponding action of $\mathcal{T}^0_{poly}$. Due to the above proposition the differential $\delta$ is compatible with the corresponding DG algebraic structures.

The subspaces $ker \, \delta \cap \Gamma(M, \mathcal{T}_{poly})$ and $ker \, \delta \cap \Gamma(M, \mathcal{D}_{poly})$ will subsequently play an important role in our construction. They can be described in the following way. Elements of $ker \, \delta \cap \Gamma(M, \mathcal{T}_{poly})$ are fiberwise polyvector fields (4.2)

$$\mathfrak{v} = \sum_k \mathfrak{v}^{j_0 \dots j_k}(x) \frac{\partial}{\partial y^{j_0}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_k}}$$

whose components do not depend on $y$'s. Similarly, elements of $ker \, \delta \cap \Gamma(M, \mathcal{D}_{poly})$ are fiberwise polydifferential operators (4.4)

$$\mathfrak{P} = \sum_k \sum_{\alpha_0 \dots \alpha_k} \mathfrak{P}^{\alpha_0 \dots \alpha_k}(x) \frac{\partial}{\partial y^{\alpha_0}} \otimes \dots \otimes \frac{\partial}{\partial y^{\alpha_k}}$$

whose coefficients do not depend on $y$'s.

In the following proposition I describe cohomology of the differential $\delta$ in $\Omega^\bullet(M, \mathcal{S}M)$, $\Omega^\bullet(M, \mathcal{T}_{poly})$, $\Omega^\bullet(M, \mathcal{D}_{poly})$, and $\Omega^\bullet(M, \mathcal{E})$

**Proposition 11** *For $\mathcal{B}$ be either of the bundles $\mathcal{S}M$, $\mathcal{T}_{poly}$, $\mathcal{D}_{poly}$, or $\mathcal{E}$*

$$H^{>0}(\Omega^\bullet(M, \mathcal{B}), \delta) = 0 \, .$$

*Furthermore,*

$$H^0(\Omega^\bullet(M, \mathcal{T}_{poly}), \delta) = ker \, \delta \cap \Gamma(M, \mathcal{T}_{poly}) \, ,$$

$$H^0(\Omega^\bullet(M, \mathcal{D}_{poly}), \delta) = ker \, \delta \cap \Gamma(M, \mathcal{D}_{poly}) \, ,$$

$$H^0(\Omega^\bullet(M, \mathcal{S}M), \delta) = C^\infty(M) \, ,$$

$$H^0(\Omega^\bullet(M,\mathcal{E}),\delta) = \mathcal{A}^\bullet(M)\,.$$

**Proof.** The proposition will follow immediately if I construct an operator

$$\delta^{-1} \,:\, \Omega^\bullet(M,\mathcal{B}) \mapsto \Omega^{\bullet-1}(M,\mathcal{B})$$

such that for any $a \in \Omega^\bullet(M,\mathcal{B})$

$$a = \sigma(a) + \delta\delta^{-1}a + \delta^{-1}\delta a\,, \tag{4.15}$$

where

$$\sigma a = a\Big|_{y^i=dy^i=0}\,. \tag{4.16}$$

First, I define this operator on $\Omega^\bullet(M,\mathcal{S}M)$

$$\delta^{-1}(a) = \begin{cases} y^k \dfrac{\vec{\partial}}{\partial(dy^k)} \displaystyle\int_0^1 a(x,ty,tdy)\frac{dt}{t}, & \text{if } a \in \Omega^{>0}(M,\mathcal{S}M)\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{4.17}$$

where the arrow over $\partial$ denotes the left derivative with respect to the anti-commuting variable $dy^k$.

Next, I extend $\delta^{-1}$ to the vector spaces $\Omega^\bullet(M,\mathcal{E})$, $\Omega^\bullet(M,\mathcal{T}_{poly})$, $\Omega^\bullet(M,\mathcal{D}_{poly})$ in the componentwise manner. A direct computation shows that equation (4.15) holds and the proposition follows. $\square$

It is worth noting that the operator $\delta^{-1}$ is 2-nilpotent for either of complexes

$$(\delta^{-1})^2 = 0\,. \tag{4.18}$$

For our purposes I fix an affine torsion free connection $\nabla$ on $M$. Since the bundles $\mathcal{S}M$, $\mathcal{T}_{poly}$, $\mathcal{D}_{poly}$, $\mathcal{E}$, or $\mathcal{C}^{poly}$ are obtained from the tangent bundle the connection $\nabla$ extends to them in the natural way. I use the same notation for all these connections

$$\nabla \,:\, \Omega^\bullet(M,\mathcal{B}) \mapsto \Omega^{\bullet+1}(M,\mathcal{B})\,, \tag{4.19}$$

where $\mathcal{B}$ is either $\mathcal{S}M$, $\mathcal{T}_{poly}$, $\mathcal{D}_{poly}$, $\mathcal{E}$, or $\mathcal{C}^{poly}$.

The following statement is an easy exercise of differential geometry.

**Proposition 12** *Let $\mathcal{B}$ be either $\mathcal{S}M$, $\mathcal{T}_{poly}$, $\mathcal{D}_{poly}$, $\mathcal{E}$, or $\mathcal{C}^{poly}$ and let $\cdot$ denote the action of $\mathcal{T}^0_{poly}$. Then the connection $\nabla$ is given by the following operator*

$$\nabla = dy^i \frac{\partial}{\partial x^i} + \Gamma \cdot \, : \, \Omega^\bullet(M, \mathcal{B}) \mapsto \Omega^{\bullet+1}(M, \mathcal{B}) \,, \qquad (4.20)$$

*where*

$$\Gamma = -dy^i \Gamma^k_{ij}(x) y^j \frac{\partial}{\partial y^k} \,, \qquad (4.21)$$

*and $\Gamma^k_{ij}(x)$ are the corresponding Christoffel symbols. Furthermore,*

$$\nabla^2 a = \mathcal{R} \cdot a \, : \, \Omega^\bullet(M, \mathcal{B}) \mapsto \Omega^{\bullet+2}(M, \mathcal{B}) \,, \qquad (4.22)$$

*where*

$$\mathcal{R} = -\frac{1}{2} dy^i dy^j (R_{ij})^k_l(x) y^l \frac{\partial}{\partial y^k} \,,$$

*and $(R_{ij})^k_l(x)$ is the standard Riemann curvature tensor of the connection $\nabla$.* $\Box$

Notice that, due proposition 10 the operator (4.20) is compatible with the (DG) algebraic structures on $\Omega^\bullet(M, \mathcal{S}M)$, $\Omega^\bullet(M, \mathcal{T}_{poly})$, $\Omega^\bullet(M, \mathcal{E})$, $\Omega^\bullet(M, \mathcal{D}_{poly})$, and $\Omega^\bullet(M, \mathcal{C}^{poly})$. Moreover, since the connection $\nabla$ is torsion free the derivations (4.14) and (4.20) (anti)commute

$$\delta \nabla + \nabla \delta = 0 \,. \qquad (4.23)$$

I would like to combine the operators (4.14) and (4.20) into a 2-nilpotent derivation

$$D = \nabla - \delta + A \cdot \, : \, \Omega^\bullet(M, \mathcal{B}) \mapsto \Omega^{\bullet+1}(M, \mathcal{B}) \,, \qquad (4.24)$$

where $\mathcal{B}$ and $\cdot$ are as in proposition 12 and

$$A = \sum_{p=2}^{\infty} dy^k A^j_{k i_1 \dots i_p}(x) y^{i_1} \dots y^{i_p} \frac{\partial}{\partial y^j} \in \Omega^1(M, \mathcal{T}^0_{poly}) \,.$$

is a $dy$- 1-form with values in the fiberwise vector fields $\mathcal{T}^0_{poly}$.

Due to the following theorem it is always possible to find the 1-form $A$ such that the derivation (4.24) is 2-nilpotent.

**Theorem 4** *Iterating the equation*

$$A = \delta^{-1}\mathcal{R} + \delta^{-1}(\nabla A + \frac{1}{2}[A, A]_{SN}) \tag{4.25}$$

*in degrees in $y$ one constructs $A \in \Omega^1(M, \mathcal{T}^0_{poly})$ such that $\delta^{-1}A = 0$ and the derivation $D$ (4.24) is 2-nilpotent*

$$D^2 = 0\,.$$

In what follows I refer to the differential $D$ (4.24) as *the Fedosov differential.*

**Proof.** First, I observe that the recurrent procedure in (4.25) converges to an element $A \in \Omega^1(M, \mathcal{T}^0_{poly})$ since the operator $\delta^{-1}$ raises the degree in $y$. Moreover, due to equation (4.18)

$$\delta^{-1}A = 0\,. \tag{4.26}$$

Second, the equation $D^2 = 0$ is equivalent to

$$\mathcal{R} - \delta A + \nabla A + \frac{1}{2}[A, A]_{SN} = 0\,. \tag{4.27}$$

Denoting by $C \in \Omega^2(M, \mathcal{T}^0_{poly})$ the left hand side of (4.27)

$$C = -\delta A + \mathcal{R} + \nabla A + \frac{1}{2}[A, A]_{SN}$$

using (4.15), (4.25), and (4.26) one gets that

$$\delta^{-1}C = 0\,. \tag{4.28}$$

On the other hand $[D, D^2] = 0$ and hence

$$\nabla C - \delta C + [A, C]_{SN} = 0\,. \tag{4.29}$$

Thus, applying (4.15) to $C$ and using (4.28) one gets the equation

$$C = \delta^{-1}(\nabla C + [A, C]_{SN})\,.$$

This equation has the unique vanishing solution since the operator $\delta^{-1}$ raises the degree in $y$. The theorem is proved. $\Box$

In the next theorem I compute cohomology of the Fedosov differential (4.24) for $\Omega^{\bullet}(M, \mathcal{S}M)$, $\Omega^{\bullet}(M, \mathcal{T}_{poly})$, $\Omega^{\bullet}(M, \mathcal{E})$, and $\Omega^{\bullet}(M, \mathcal{D}_{poly})$.

**Theorem 5** *If $\mathcal{B}$ is either $\mathcal{S}M$, $\mathcal{E}$, $\mathcal{T}_{poly}$, or $\mathcal{D}_{poly}$ then*

$$H^{>0}(\Omega^{\bullet}(M, \mathcal{B}), D) = 0\,. \tag{4.30}$$

*Furthermore,*

$$\begin{aligned} H^0(\Omega(M, \mathcal{S}M), D) &\cong C^{\infty}(M)\,, \\ H^0(\Omega(M, \mathcal{E}), D) &\cong \mathcal{A}^{\bullet}(M) \end{aligned} \tag{4.31}$$

*as graded commutative algebras,*

$$H^0(\Omega(M, \mathcal{T}_{poly}), D) \cong ker\,\delta \cap \Gamma(M, \mathcal{T}_{poly}) \tag{4.32}$$

*as graded vector spaces, and*

$$H^0(\Omega(M, \mathcal{D}_{poly}), D) \cong ker\,\delta \cap \Gamma(M, \mathcal{D}_{poly}) \tag{4.33}$$

*as graded associative algebras.*

**Proof.** Although the first statement follows easily from the spectral sequence argument I need a more explicit proof.

To prove (4.30) I construct an $\mathbb{R}$-linear map

$$\Phi : \Omega^{\bullet}(M, \mathcal{B}) \mapsto \Omega^{\bullet - 1}(M, \mathcal{B}) \tag{4.34}$$

such that for any $a \in \Omega^{>0}(M, \mathcal{B})$

$$D\Phi(a) + \Phi D(a) = a\,. \tag{4.35}$$

I define the map $\Phi$ with the help of the following recurrent procedure

$$\Phi(a) = -\delta^{-1} a + \delta^{-1}(\nabla\Phi(a) + A \cdot \Phi(a))\,, \tag{4.36}$$

where $\cdot$ denotes the action of $\mathcal{T}^0_{poly}$ (see proposition 10) and the procedure (4.36) converges since $\delta^{-1}$ (4.17) raises the degree in the fiber coordinates $y^i$.

Due to equation (4.18) $\delta^{-1}\Phi(a) = 0$ and therefore

$$\Phi^2 = 0\,. \tag{4.37}$$

Let me prove that for any element $a \in \Omega^{>0}(M, \mathcal{B}) \cap \ker D$

$$a = D\Phi(a)\,. \tag{4.38}$$

For this I denote by $h$ the element

$$h = a - D\Phi(a) \in \Omega^{>0}(M, \mathcal{B})$$

and mention that $Dh = 0$ or equivalently

$$\delta h = \nabla h + A \cdot h\,. \tag{4.39}$$

Since $\delta^{-1}\Phi(a) = 0$ and $\sigma(\Phi(a)) = 0$ equation (4.15) for $\Phi(a)$ boils down to

$$\Phi(a) = \delta^{-1}\delta\Phi(a)\,.$$

Thus, using (4.36), I conclude that

$$\delta^{-1} h = 0 \, .$$

Furthermore, since $h \in \Omega^{>0}(M, \mathcal{B})$

$$\sigma h = 0 \, .$$

Hence applying (4.15) to $h$ and using (4.39) I get

$$h = \delta^{-1}(\nabla h + A \cdot h) \, .$$

The latter equation has the unique vanishing solution since $\delta^{-1}$ raises the degree in the fiber coordinates $y^i$ . Thus (4.38) is proved.

Using (4.38) I conclude that

$$D \circ \Phi \circ D = D \, . \tag{4.40}$$

Let me now turn to our combination

$$b = a - D\Phi(a) - \Phi D(a) \, ,$$

where $a \in \Omega^{>0}(M, \mathcal{B})$ .

Thanks to (4.40) and $D^2 = 0$

$$Db = 0 \, .$$

Hence, applying (4.38) to $b$ I get

$$b = D\Phi(b) \, .$$

Using (4.37) and (4.40) once again I get that $b = 0$, and therefore, (4.35) holds.

Thus the first statement (4.30) is proved.

Let $\mathcal{H}$ denote either $C^\infty(M)$, $\mathcal{A}^\bullet(M)$, $ker\,\delta \cap \Gamma(M, \mathcal{T}_{poly})$, or $ker\,\delta \cap \Gamma(M, \mathcal{D}_{poly})$ and $\cdot$ denote the action of $\mathcal{T}^0_{poly}$ (see proposition 10) on $\mathcal{S}M$, $\mathcal{E}$, $\mathcal{T}_{poly}$, and $\mathcal{D}_{poly}$. I claim that iterating the equation

$$\tau(a) = a + \delta^{-1}(\nabla\tau(a) + A\cdot\tau(a))\,, \qquad a \in \mathcal{H} \tag{4.41}$$

in degrees in $y$ one gets an isomorphism

$$\tau : \mathcal{H} \mapsto \ker D \cap \Gamma(M, \mathcal{B})\,. \tag{4.42}$$

Indeed, let $a \in \mathcal{H}$. Then, due to formula (4.15) $\tau(a)$ satisfies the following equation

$$\delta^{-1}(D(\tau(a))) = 0\,. \tag{4.43}$$

Let us denote $D\tau(a)$ by $Y$

$$Y = D\tau(a)\,.$$

The equation $D^2 = 0$ implies that $DY = 0$, or in other words

$$\delta Y = \nabla Y + A \cdot Y\,. \tag{4.44}$$

Applying (4.15) to $Y$ and using equations (4.43), (4.44) I get

$$Y = \delta^{-1}(\nabla Y + A\cdot Y)\,.$$

The latter equation has the unique vanishing solution since the operator $\delta^{-1}$ (4.17) raises the degree in the fiber coordinates $y^i$.

The map (4.42) is injective since $\sigma$ (4.16) is a section of (4.42)

$$\sigma \circ \tau = Id\,. \tag{4.45}$$

To prove surjectivity of (4.42) it suffices to show that if $b \in \Gamma(M, \mathcal{B}) \cap \ker D$ and

$$\sigma b = 0 \tag{4.46}$$

then $b$ vanishes.

The condition $b \in \ker D$ is equivalent to the equation

$$\delta b = \nabla b + A \cdot b \,.$$

Hence, applying (4.15) to $a$ and using (4.46) I get

$$b = \delta^{-1}(\nabla b + A \cdot b)\,.$$

The latter equation has the unique vanishing solution since the operator $\delta^{-1}$ (4.17) raises the degree in the fiber coordinates $y^i$. Thus, the map (4.42) is bijective and the map $\sigma$ (4.16) provides me with the inverse of (4.42)

$$\tau \circ \sigma \Big|_{\ker D \,\cap\, \Gamma(M,\mathcal{B})} = Id\,. \tag{4.47}$$

Since $\sigma$ respects the multiplications in $\Omega^\bullet(M, \mathcal{S}M)$, $\Omega^\bullet(M, \mathcal{E})$, $\mathcal{D}^\bullet_{poly}(M)$, $C^\infty(M)$, $\mathcal{A}^\bullet(M)$, and $ker\, \delta \cap \Gamma(M, \mathcal{D}_{poly})$ so does the map $\tau$ and the theorem follows. $\square$

Notice that since the Fedosov differential (4.24) is compatible with the DGLA structure on $\Omega^\bullet(M, \mathcal{T}_{poly})$ and $\Omega^\bullet(M, \mathcal{D}_{poly})$, the cohomology groups $H^\bullet(\Omega^\bullet(M, \mathcal{T}_{poly}), D)$ and $H^\bullet(\Omega^\bullet(M, \mathcal{D}_{poly}), D)$ acquire structures of a DGLA, and $H^\bullet(\Omega^\bullet(M, \mathcal{D}_{poly}), D)$ also becomes a DGA. To analyze these structures let me, first, observe that for any function $a \in C^\infty(M)$ and for any integer $p \geq 0$

$$\frac{\partial}{\partial y^{i_1}} \dots \frac{\partial}{\partial y^{i_p}} \tau(a) \Big|_{y=0} = \partial_{x^{i_1}} \dots \partial_{x^{i_p}} a(x) + \text{lower order derivatives of } a\,. \tag{4.48}$$

Due to this observation the following map

$$\nu \,:\, ker\, \delta \cap \Gamma(M, \mathcal{D}_{poly}) \mapsto D^\bullet_{poly}(M)\,,$$

$$\nu(\mathfrak{P})(a_0, \dots, a_k) = \Big(\mathfrak{P}(\tau(a_0), \dots, \tau(a_k))\Big)\Big|_{y=0}, \tag{4.49}$$

$$\mathfrak{P} \in \ker \delta \cap \Gamma(M, \mathcal{D}^k_{poly}), \qquad a_i \in C^\infty(M)$$

is an isomorphism of graded associative algebras $ker\, \delta \cap \Gamma(M, \mathcal{D}_{poly})$ and $D^\bullet_{poly}(M)$.

I claim that

**Proposition 13** *The composition*

$$\lambda_D = \tau \circ \nu^{-1} : D^\bullet_{poly}(M) \mapsto \ker D \cap \Gamma(M, \mathcal{D}^\bullet_{poly}) \tag{4.50}$$

*induces an isomorphism from the DGLA (and DGA) $D^\bullet_{poly}(M)$ to the DGLA (and DGA) $H^\bullet(\Omega^\bullet(M, \mathcal{D}_{poly}), D)$.*

**Proof.** Since both the map $\tau$ (4.42) and the map $\nu$ (4.49) respect the cup-product (3.10) it suffices to prove the compatibility with the DGLA structures. I will prove that inverse map

$$\lambda_D^{-1} = \nu \circ \sigma : \ker D \cap \Gamma(M, \mathcal{D}^\bullet_{poly}) \mapsto D^\bullet_{poly}(M),$$

$$\lambda_D^{-1}(\mathfrak{P})(a_0, \dots, a_k) = \Big(\mathfrak{P}(\tau(a_0), \dots, \tau(a_k))\Big)\Big|_{y=0}, \tag{4.51}$$

$$\mathfrak{P} \in \ker D \cap \Gamma(M, \mathcal{D}^k_{poly}), \qquad a_i \in C^\infty(M)$$

respects the Gerstenhaber bracket (3.2) and the Hochschild differential (3.8).

To prove the compatibility with the bracket I observe that applying $\tau$ (4.42) to both sides of (4.51) and using (4.47) one gets

$$\tau\Big(\lambda_D^{-1}(\mathfrak{P})(a_0, \dots, a_k)\Big) = \mathfrak{P}(\tau(a_0), \dots, \tau(a_k)), \tag{4.52}$$

$$\forall \qquad \mathfrak{P} \in \ker D \cap \Gamma(M, \mathcal{D}^k_{poly}), \qquad a_i \in C^\infty(M).$$

Using this equation I conclude that for any $\mathfrak{P}_1 \in \ker D \cap \Gamma(M, \mathcal{D}^{k_1}_{poly})$ and $\mathfrak{P}_2 \in$

$\ker D \cap \Gamma(M, \mathcal{D}^{k_2}_{poly})$

$$\lambda_D^{-1}(\mathfrak{P}_1)(a_0, \dots, \lambda_D^{-1}(\mathfrak{P}_2)(a_i, \dots, a_{i+k_2}), \dots, a_{k_1+k_2}) =$$

$$\mathfrak{P}_1(\tau(a_0), \dots, \mathfrak{P}_2(\tau(a_i), \dots, \tau(a_{i+k_2})), \dots, \tau(a_{k_1+k_2}))\,.$$

Therefore, for any $\mathfrak{P}_1 \in \ker D \cap \Gamma(M, \mathcal{D}^{k_1}_{poly})$ and $\mathfrak{P}_2 \in \ker D \cap \Gamma(M, \mathcal{D}^{k_2}_{poly})$

$$\lambda_D^{-1}(\mathfrak{P}_1) \bullet \lambda_D^{-1}(\mathfrak{P}_2) = \lambda_D^{-1}(\mathfrak{P}_1 \bullet \mathfrak{P}_2)\,, \tag{4.53}$$

where the operation $\bullet$ is defined in (3.3).

Thus $\lambda_D^{-1}$ is compatible with the Gerstenhaber bracket (3.2).

To prove that $\lambda_D^{-1}$ respects the differentials (3.8) in $D^{\bullet}_{poly}(M)$ and $H^{\bullet}(\Omega^{\bullet}(M, \mathcal{D}_{poly}), D)$ it suffices to show that the multiplication $\mu \in \Gamma(M, \mathcal{D}^1_{poly})$ in $\Gamma(M, \mathcal{S}M)$ is sent to the multiplication $\mu_0 \in D^1_{poly}(M)$ in $C^{\infty}(M)$. This is immediate from the definition of $\lambda_D^{-1}$ (4.51). Thus the proposition is proved. $\Box$

It is obvious that the restriction of the map $\nu$ (4.49) to $\Gamma(M, \mathcal{T}_{poly})$ gives a map

$$\nu : ker\, \delta \cap \Gamma(M, \mathcal{T}_{poly}) \mapsto T^{\bullet}_{poly}(M)\,. \tag{4.54}$$

By the abuse of notation I denote this map by the same letter.

It is easy to see that due to equation (4.48) the map (4.54) is also an isomorphism of graded vector spaces. Furthermore,

**Proposition 14** *The composition*

$$\lambda_T = \tau \circ \nu^{-1} : T^{\bullet}_{poly}(M) \mapsto \ker D \cap \Gamma(M, \mathcal{T}^{\bullet}_{poly}) \tag{4.55}$$

*induces an isomorphism from the graded Lie algebra $T^{\bullet}_{poly}(M)$ to the graded Lie algebra $H^{\bullet}(\Omega^{\bullet}(M, \mathcal{T}_{poly}), D)$.*

**Proof.** To show that $\lambda_T$ is compatible with Lie brackets I observe that the following diagram

$$
\begin{array}{ccc}
T^\bullet_{poly}(M) & \stackrel{\lambda_T}{\to} & \Gamma(M, \mathcal{T}^\bullet_{poly}) \\
 & & \\
\downarrow^{\mathcal{V}} & & \downarrow^{\mathcal{V}^{fib}} \\
 & & \\
D^\bullet_{poly}(M) & \stackrel{\lambda_D}{\to} & \Gamma(M, \mathcal{D}^\bullet_{poly}),
\end{array}
\tag{4.56}
$$

commutes. Here $\mathcal{V}$ is the map of Vey (3.17) and $\mathcal{V}^{fib}$ denotes its fiberwise analogue.

Thus for any pair $\gamma_1, \gamma_2 \in T^\bullet_{poly}(M)$ I have

$$
\mathcal{V}^{fib}(\lambda_T([\gamma_1, \gamma_2]_{SN}) - [\lambda_T(\gamma_1), \lambda_T(\gamma_2)]_{SN}) = \lambda_D \mathcal{V}([\gamma_1, \gamma_2]_{SN}) - [\mathcal{V}^{fib} \lambda_T(\gamma_1), \mathcal{V}^{fib} \lambda_T(\gamma_2)]_{SN}
$$

modulo $\partial$-exact terms in $\Gamma(M, \mathcal{D}^\bullet_{poly})$. Continuing this line of equations and using proposition 13 I conclude that

$$
\mathcal{V}^{fib}(\lambda_T([\gamma_1, \gamma_2]_{SN}) - [\lambda_T(\gamma_1), \lambda_T(\gamma_2)]_{SN}) \in \partial(\Gamma(M, \mathcal{D}^\bullet_{poly})) .
$$

Therefore, since $\mathcal{V}^{fib}$ is a quasi-isomorphism of complexes $(\Gamma(M, \mathcal{T}^\bullet_{poly}), 0)$ and $(\Gamma(M, \mathcal{D}^\bullet_{poly}), \partial)$

$$
\lambda_T([\gamma_1, \gamma_2]_{SN}) - [\lambda_T(\gamma_1), \lambda_T(\gamma_2)]_{SN} = 0
$$

and the proposition follows. $\square$

Since the Fedosov differential (4.24) is compatible with the DGLA module structures on $\Omega^\bullet(M, \mathcal{E})$ and $\Omega^\bullet(M, \mathcal{C}^{poly})$ the cohomology groups $H^\bullet(\Omega^\bullet(M, \mathcal{E}), D)$ and $H^\bullet(\Omega^\bullet(M, \mathcal{C}^{poly}), D)$ acquire the DGLA module structures over $H^\bullet(\Omega^\bullet(M, \mathcal{T}_{poly}), D)$ and $H^\bullet(\Omega^\bullet(M, \mathcal{D}_{poly}), D)$, respectively. Due to theorem 5 and propositions 13, 14 $H^\bullet(\Omega^\bullet(M, \mathcal{T}_{poly}), D) \cong T^\bullet_{poly}(M)$ and $H^\bullet(\Omega^\bullet(M, \mathcal{D}_{poly}), D) \cong D^\bullet_{poly}(M)$ as DGLAs. My next task is to show that $H(\Omega^\bullet(M, \mathcal{E}), D) \cong \mathcal{A}^\bullet(M)$ and $H(\Omega^\bullet(M, \mathcal{C}^{poly}), D) \cong C^{poly}_\bullet(M)$ as modules over the corresponding DGLAs.

The desired statement about chains follows from proposition 8 and

**Proposition 15** *For any $q > 0$*

$$
H^q(\Omega^\bullet(M, \mathcal{C}^{poly}), D) = 0 . \tag{4.57}
$$

*The map*

$$\varrho : \Gamma(M, \mathcal{C}^{poly}_{\bullet}) \to J_{\bullet}(M)\,, \qquad \varrho(a)(P) = (\lambda_D(P))(a)\Big|_{y^i=0}\,, \tag{4.58}$$

$$a \in \Gamma(M, \mathcal{C}^{poly}_k)\,, \quad P \in D^k_{poly}(M)$$

*is an isomorphism of the DG modules over the DGLA* $D^{\bullet}_{poly}(M) \cong \ker D \cap \Gamma(M, \mathcal{D}^{\bullet}_{poly})$*. Moreover, this isomorphism sends the Fedosov connection (4.24) on* $\mathcal{C}^{poly}$ *to the Grothendieck connection (3.24) on* $J_{\bullet}$ *.*

**Proof.** The first statement (4.57) follows easily from the spectral sequence argument. Indeed, using the zeroth collection of the fiber coordinates $y^i_0$ (4.13) I introduce the decreasing filtration on the sheaf $\Omega^{\bullet}(M, \mathcal{C}^{poly})$

$$\dots \subset F^p(\Omega^{\bullet}(M, \mathcal{C}^{poly})) \subset F^{p-1}(\Omega^{\bullet}(M, \mathcal{C}^{poly})) \subset \dots \subset F^0(\Omega^{\bullet}(M, \mathcal{C}^{poly})) = \Omega^{\bullet}(M, \mathcal{C}^{poly})\,,$$

where the components of the forms (4.13) in $F^p(\Omega^{\bullet}(M, \mathcal{C}^{poly}))$ have degree in $y^i_0 \geq p$.

Since $D(F^p(\Omega^{\bullet}(M, \mathcal{C}^{poly}))) \subset F^{p-1}(\Omega^{\bullet}(M, \mathcal{C}^{poly}))$ the corresponding spectral sequence starts with

$$E^{p,q}_{-1} = F^p(\Omega^{p+q}(M, \mathcal{C}^{poly}))\,.$$

Next, I observe that

$$d_{-1} = dy^i \frac{\partial}{\partial y^i_0}\,,$$

and hence, due to the Poincaré lemma for the formal disk I have

$$E^{p,q}_0 = E^{p,q}_1 = \dots = E^{p,q}_{\infty} = 0$$

whenever $p + q > 0$. Thus, the first statement (4.57) of the proposition follows.

Thanks to observation (4.48) the map (4.58) is indeed an isomorphism of graded vector spaces.

The compatibility with the action (3.5) of the sheaf of DGLAs $\mathcal{D}^{\bullet}_{poly}$ on the sheaf

$\mathcal{C}_\bullet^{poly}$ and with the action (3.29) of the sheaf of DGLAs $D_{poly}^\bullet$ on the sheaf $J_\bullet$

$$\hat{R}_P(\varrho(a)) = \varrho(R_{\lambda_D(P)}(a))$$

follows from the compatibility of $\lambda_D$ with the operation $\bullet$ (3.3) (see (4.53)), with the cyclic permutations, and with the cup products (3.10) in $D_{poly}^\bullet(M)$ and $\mathcal{D}_{poly}^\bullet(M)$ (see proposition 13).

It remains to prove that the map (4.58) sends the Fedosov connection (4.24) to the Grothendieck connection (3.24). This statement is proved by the following line of equations:

$$\begin{aligned}
\varrho(D_u a)(P) &= (\lambda_D(P))(D_u a)\Big|_{y^i=0} = (D_u[\lambda_D(P)(a)])\Big|_{y^i=0} \\
&= u[\lambda_D(P)(a)]\Big|_{y^i=0} - (i_u\delta \bullet [\lambda(P)(a)])\Big|_{y^i=0} \\
&= u[\lambda_D(P)(a)]\Big|_{y^i=0} - (\lambda_D(u)\bullet\lambda_D(P))(a)\Big|_{y^i=0} \\
&= u[\lambda_D(P)(a)]\Big|_{y^i=0} - \lambda_D(u\bullet P)(a)\Big|_{y^i=0} \\
&= u(\varrho(a))(P) - (\varrho(a))(u\bullet P) = (\nabla_u^G \varrho(a))(P)\,,
\end{aligned}$$

where $u \in \Gamma(M, TM)$, $a \in \Gamma(M, \mathcal{C}_k^{poly})$, $P \in D_{poly}^k(M)$ , $i$ denotes the contraction of a vector field with differential forms, $\bullet$ is as in (3.3), and $u$ is viewed both as a vector field and a differential operator of the first order. $\square$

Let me conclude this chapter with

**Proposition 16** *The map (4.42)*

$$\tau \,:\, \mathcal{A}^\bullet(M) \mapsto \Omega^\bullet(M, \mathcal{E}) \tag{4.59}$$

*induces an isomorphism of DG modules $\mathcal{A}^\bullet(M)$ and $H^\bullet(\Omega^\bullet(M,\mathcal{E}), D)$ over the DGLA $H^\bullet(\Omega^\bullet(M, \mathcal{T}_{poly}), D) \cong T_{poly}^\bullet(M)$ .*

**Proof.** I have to prove that for any exterior form $a = a_{i_1\dots i_q}(x)dx^{i_1}\dots dx^{i_q}$ and any polyvector field $\gamma = \gamma^{i_0 \dots i_k}(x)\partial_{x^{i_0}} \wedge \dots \wedge \partial_{x^{i_k}}$

$$\tau(L_\gamma(a)) = L_{\tau\circ\nu^{-1}(\gamma)}(\tau(a))\,. \tag{4.60}$$

Since Fedosov differential $D$ is compatible with the fiberwise Lie derivative $L$ the form $L_{\tau\circ\nu^{-1}(\gamma)}(\tau(a))$ is $D$-closed. Therefore by (4.47) it suffices to the show that

$$L_{\tau\circ\nu^{-1}(\gamma)}(\tau(a))\Big|_{y=0} = L_\gamma(a)\,. \tag{4.61}$$

To prove (4.61) I need the expressions for $\tau(\nu^{-1}(\gamma))$ and $\tau(a)$ only up to the second order terms in $y$. They are

$$\tau(\nu^{-1}(\gamma)) = \nu^{-1}(\gamma) + y^i \frac{\partial \nu^{-1}(\gamma)}{\partial x^i} - y^i[\,\Gamma_i(x), \nu^{-1}(\gamma)\,]_{SN} \quad mod \quad (y)^2\,, \tag{4.62}$$

$$\tau(a) = a + y^i \frac{\partial a}{\partial x^i} - y^i L_{\Gamma_i(x)}(a) \quad mod \quad (y)^2\,, \tag{4.63}$$

$$\Gamma_i = \Gamma^k_{ij}(x) y^j \partial_{y^k}\,,$$

where $\Gamma^k_{ij}(x)$ are Christoffel symbols and

$$\nu^{-1}(\gamma) = \gamma^{i_0 \dots i_k}(x)\partial_{y^{i_0}} \wedge \dots \wedge \partial_{y^{i_k}}\,.$$

Using symmetry of indices for the Christoffel symbols $\Gamma^k_{ij} = \Gamma^k_{ji}$ I can rewrite (4.62) and (4.63) in the form

$$\tau(\nu^{-1}(\gamma)) = \nu^{-1}(\gamma) + y^i \frac{\partial \nu^{-1}(\gamma)}{\partial x^i} - [\,\widetilde{\Gamma}(x), \nu^{-1}(\gamma)\,]_{SN} \quad mod \quad (y)^2\,, \tag{4.64}$$

$$\tau(a) = a + y^i \frac{\partial a}{\partial x^i} - L_{\widetilde{\Gamma}(x)}(a) \quad mod \quad (y)^2\,, \tag{4.65}$$

where $\widetilde{\Gamma} = \dfrac{1}{2}\Gamma^k_{ij} y^i y^j \dfrac{\partial}{\partial y^k}$. Using these formulas it is not hard to show that equation (4.61) is equivalent to

$$L_{\nu^{-1}(\gamma)} L_{\widetilde{\Gamma}}(a) + L_{[\widetilde{\Gamma}, \nu^{-1}(\gamma)]_{SN}}(a) = 0\,,$$

which obviously holds because $L_{\nu^{-1}(\gamma)}(a) = 0$. $\square$

# Chapter 5

# Formality theorems for $(D^{\bullet}_{poly}(M)$, $C^{poly}_{\bullet}(M))$ and their applications

## 5.1 Proof of the formality theorem for $C^{poly}_{\bullet}(M)$

The results of the previous chapter can be represented in the form of the following commutative diagrams of DGLAs, their modules, and quasi-isomorphisms given by honest (not $L_\infty$) morphisms

$$
\begin{array}{ccc}
T^{\bullet}_{poly}(M) & \stackrel{\lambda_T}{\longrightarrow} & (\Omega^{\bullet}(M, \mathcal{T}_{poly}), D, [,]_{SN}) \\
\downarrow^{L}_{mod} & & \downarrow^{L}_{mod} \\
\mathcal{A}^{\bullet}(M) & \stackrel{\lambda_{\mathcal{A}}}{\longrightarrow} & (\Omega^{\bullet}(M, \mathcal{E}), D), \\
 & & \\
(\Omega^{\bullet}(M, \mathcal{D}_{poly}), D + \partial, [,]_G) & \stackrel{\lambda_D}{\longleftarrow} & D^{\bullet}_{poly}(M) \\
\downarrow^{R}_{mod} & & \downarrow^{R}_{mod} \\
(\Omega^{\bullet}(M, \mathcal{C}^{poly}), D + \mathfrak{b}) & \stackrel{\lambda_C}{\longleftarrow} & C^{poly}_{\bullet}(M),
\end{array}
\tag{5.1}
$$

where $\lambda_T = \tau \circ \nu^{-1}\Big|_{T^{\bullet}_{poly}(M)}$, $\lambda_{\mathcal{A}} = \tau\Big|_{\mathcal{A}^{\bullet}(M)}$, $\lambda_D = \tau \circ \nu^{-1}$, $\lambda_C = \varrho^{-1} \circ \chi^{-1}$, the map $\chi$ is defined in (3.25), the map $\tau$ is defined in (4.41), and the map $\varrho$ is defined in (4.58).

Next, due to properties 1 and 2 in theorem 2 I have a fiberwise quasi-isomorphism (which I denote by the same letter $\mathcal{K}$)

$$\mathcal{K} : (\Omega^\bullet(M, \mathcal{T}_{poly}), 0, [,]_{SN}) \succ\!\!\!\longrightarrow (\Omega^\bullet(M, \mathcal{D}_{poly}), \partial, [,]_G) \,. \tag{5.2}$$

from the DGLA $(\Omega^\bullet(M, \mathcal{T}_{poly}), 0, [,]_{SN})$ to the DGLA $(\Omega^\bullet(M, \mathcal{D}_{poly}), \partial, [,]_G)$ .

Due to properties 1 and 2 in theorem 3 I have a fiberwise quasi-isomorphism (which I denote by the same letter $\mathcal{S}$)

$$\mathcal{S} : (\Omega^\bullet(M, \mathcal{C}^{poly}), \mathfrak{b}) \succ\!\succ\!\!\!\longrightarrow (\Omega^\bullet(M, \mathcal{E}), 0, L) \tag{5.3}$$

from the $L_\infty$-module $\Omega^\bullet(M, \mathcal{C}^{poly})$ to the DGLA module $\Omega^\bullet(M, \mathcal{E})$ over $\Omega^\bullet(M, \mathcal{T}_{poly})$ .

Thus I get the following commutative diagram

$$\begin{array}{ccc} (\Omega^\bullet(M, \mathcal{T}_{poly}), 0, [,]_{SN}) & \overset{\mathcal{K}}{\succ\!\!\longrightarrow} & (\Omega^\bullet(M, \mathcal{D}_{poly}), \partial, [,]_G) \\ \downarrow^{L}_{mod} & & \downarrow^{R}_{mod} \\ (\Omega^\bullet(M, \mathcal{E}), 0) & \overset{\mathcal{S}}{\longleftarrow\!\!\prec\!\prec} & (\Omega^\bullet(M, \mathcal{C}^{poly}), \mathfrak{b}), \end{array} \tag{5.4}$$

where by commutativity I mean that $\mathcal{S}$ is a morphism of the $L_\infty$-modules $(\Omega^\bullet(M, \mathcal{C}^{poly}), \mathfrak{b})$ and $(\Omega^\bullet(M, \mathcal{E}), 0)$ over the DGLA $(\Omega^\bullet(M, \mathcal{T}_{poly}), 0, [,]_{SN})$, and the $L_\infty$-module structure on $(\Omega^\bullet(M, \mathcal{C}^{poly}), \mathfrak{b})$ over $(\Omega^\bullet(M, \mathcal{T}_{poly}), 0, [,]_{SN})$ is obtained by composing the quasi-isomorphism $\mathcal{K}$ with the action $R$ of $(\Omega^\bullet(M, \mathcal{D}_{poly}), \partial, [,]_G)$ on $(\Omega^\bullet(M, \mathcal{C}^{poly}), \mathfrak{b})$ .

Having the complete decreasing filtration on $\Omega^\bullet(M, \mathcal{T}_{poly})$, $\Omega^\bullet(M, \mathcal{D}_{poly})$, $\Omega^\bullet(M, \mathcal{E})$, and $\Omega^\bullet(M, \mathcal{C}^{poly})$ induced by the exterior degree I can now apply the technique developed in section 2.4. To do this I first restrict myself to an open coordinate subset

$$V \subset M \,.$$

On $V$ it makes sense to speak about the ordinary De Rham differential[1] in the DGLA

[1]Let me recall that $\Omega(\cdot)$ stands for $dy$-forms.

modules $(\Omega^\bullet(V,\mathcal{T}_{poly}),\Omega^\bullet(V,\mathcal{E}))$, and $(\Omega^\bullet(V,\mathcal{D}_{poly}),\Omega^\bullet(V,\mathcal{C}^{poly}))$

$$d = dy^i\partial_{x^i} : \Omega^\bullet(V,\mathcal{B}) \mapsto \Omega^{\bullet+1}(V,\mathcal{B})\,, \tag{5.5}$$

where $\mathcal{B}$ is either $\mathcal{T}_{poly}$ or $\mathcal{D}_{poly}$, $\mathcal{E}$, or $\mathcal{C}^{poly}$.

Since the quasi-isomorphisms (5.2) and (5.3) are fiberwise I can add to all the differentials in diagram (5.4) the De Rham differential (5.5), and thus, get the new commutative diagram

$$\begin{array}{ccc} (\Omega^\bullet(V,\mathcal{T}_{poly}), d, [,]_{SN}) & \stackrel{\mathcal{K}}{\succ\!\!\!\longrightarrow} & (\Omega^\bullet(V,\mathcal{D}_{poly}), d+\partial, [,]_G) \\ \downarrow^L_{mod} & & \downarrow^R_{mod} \\ (\Omega^\bullet(V,\mathcal{A}), d) & \stackrel{\mathcal{S}}{\longleftarrow\!\!\!\prec\!\!\prec} & (\Omega^\bullet(V,\mathcal{C}^{poly}), d+\mathfrak{b})\,. \end{array} \tag{5.6}$$

I claim that

**Proposition 17** *The $L_\infty$-morphism $\mathcal{K}$ and the morphism of $L_\infty$-modules $\mathcal{S}$ in (5.6) are quasi-isomorphisms.*

**Proof.** This statement follows easily from the standard argument of the spectral sequence. Indeed, the $L_\infty$-morphism $\mathcal{K}$ (resp. the morphism of $L_\infty$-modules $\mathcal{S}$) is compatible the descending filtration induced by the exterior degree

$$\mathcal{F}^p(\Omega^\bullet(V,\mathcal{B})) = \bigoplus_{k\ge p}\Omega^k(V,\mathcal{B})\,, \tag{5.7}$$

where $\mathcal{B}$ is either $\mathcal{T}_{poly}$ or $\mathcal{D}_{poly}$ (resp. $\mathcal{E}$ or $\mathcal{C}^{poly}$).

The corresponding versions of Vey's [54] and Hochschild-Kostant-Rosenberg-Connes-Teleman [15], [36], [52] theorems for $\mathbb{R}^d_{formal}$ imply that the first structure map $\mathcal{K}_1$ (resp. the zeroth structure map $\mathcal{S}_0$) induces a quasi-isomorphism on the level of $E_0$. Therefore, $\mathcal{K}_1$ (resp. $\mathcal{S}_0$) induces a quasi-isomorphism on the level of $E_\infty$. The standard snake lemma argument of homological algebra implies that $\mathcal{K}_1$ (resp. $\mathcal{S}_0$) is a quasi-isomorphism. Hence, so is $\mathcal{K}$ (resp. $\mathcal{S}$). $\square$

On the subset $V$ I can represent the Fedosov differential (4.24) in the following form

$$D = d + B \cdot \,,$$

where

$$B = \sum_{p=0}^{\infty} dy^i B^k_{i\,;\,j_1 \dots j_p}(x) y^{j_1} \dots y^{j_p} \frac{\partial}{\partial y^k} \in \Omega^1(V, \mathcal{T}^0_{poly})\,, \tag{5.8}$$

and $\cdot$ denotes the action of the sheaf $\mathcal{T}^0_{poly}$. (See proposition 10.)

The nilpotency condition $D^2 = 0$ says that $B$ is a Maurer-Cartan element of the DGLA $(\Omega^\bullet(V, \mathcal{T}_{poly}), d, [,]_{SN})$ with the filtration (5.7). Thus using the terminology of section 2.4 one can say that the DGLA $(\Omega^\bullet(V, \mathcal{T}_{poly}), D, [,]_{SN})$ is obtained from $(\Omega^\bullet(V, \mathcal{T}_{poly}), d, [,]_{SN})$ by twisting via $B$.

Due to property 3 in theorem 2 the Maurer-Cartan element in $(\Omega(V, \mathcal{D}_{poly}), d + \partial, [,]_G)$

$$B_D = \sum_{m=1}^{\infty} \frac{1}{m!} \mathcal{K}_m(B, \dots, B)$$

corresponding to the Maurer-Cartan element $B$ in $(\Omega(V, \mathcal{T}_{poly}), d, [,]_{SN})$ coincides with $B$ viewed as an element of $\Omega^1(V, \mathcal{D}_{poly})$. Thus twisting of the quasi-isomorphism $\mathcal{K}$ via the Maurer-Cartan element $B$ I get the $L_\infty$-morphism

$$\mathcal{K}^{tw} \,:\, (\Omega^\bullet(V, \mathcal{T}_{poly}), D, [,]_{SN}) \succ\!\!\rightarrow (\Omega^\bullet(V, \mathcal{D}_{poly}), D + \partial, [,]_G)\,, \tag{5.9}$$

which is a quasi-isomorphism due to claim 5 of proposition 1.

Next, using (2.59) it is not hard to see that the graded module structures on $\Omega^\bullet(V, \mathcal{E})$ and $\Omega^\bullet(V, \mathcal{C}^{poly})$ over $(\Omega^\bullet(V, \mathcal{T}_{poly}), D, [,]_{SN})$ and $(\Omega^\bullet(V, \mathcal{D}_{poly}), D + \partial, [,]_G)$, respectively, remain unchanged under the twisting procedures, while the differentials get shifted. Namely, $d$ on $\Omega^\bullet(V, \mathcal{E})$ gets replaced by $D$ and $d + \mathfrak{b}$ on $\Omega^\bullet(V, \mathcal{C}^{poly})$ gets replaced by $D + \mathfrak{b}$.

Hence, by virtue of propositions 2 and 3 twisting procedure turns diagram (5.6) into the commutative diagram

$$
\begin{array}{ccc}
(\Omega^\bullet(V, \mathcal{T}_{poly}), D, [,]_{SN}) & \stackrel{\mathcal{K}^{tw}}{\succ\!\!\longrightarrow} & (\Omega^\bullet(V, \mathcal{D}_{poly}), D + \partial, [,]_G) \\
\downarrow^L_{mod} & & \downarrow^R_{mod} \\
(\Omega^\bullet(V, \mathcal{E}), D) & \stackrel{\mathcal{S}^{tw}}{\longleftarrow\!\!\prec\!\!\prec} & (\Omega^\bullet(V, \mathcal{C}^{poly}), D + \mathfrak{b}),
\end{array}
\tag{5.10}
$$

where $\mathcal{S}^{tw}$ is morphism of $L_\infty$-modules obtained from $\mathcal{S}$ by twisting via the Maurer-Cartan element $B \in \Omega^1(V, \mathcal{T}_{poly})$. Due to claim 5 of proposition 2 $\mathcal{S}^{tw}$ is a quasi-isomorphism.

Surprisingly, due to property 4 in theorem 2 and proposition 9 the "morphisms" $\mathcal{K}^{tw}$ and $\mathcal{S}^{tw}$ are defined globally. Indeed, using (2.49) and (2.60) I get the structure maps of $\mathcal{K}^{tw}$ and $\mathcal{S}^{tw}$

$$
\mathcal{K}^{tw}_n(\gamma_1, \dots, \gamma_n) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathcal{K}_{n+k}(B, \dots, B, \gamma_1, \dots, \gamma_n), \tag{5.11}
$$

$$
\mathcal{S}^{tw}_n(\gamma_1, \dots, \gamma_n, a) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathcal{S}_{n+k}(B, \dots, B, \gamma_1, \dots, \gamma_n, a), \tag{5.12}
$$

$$
\gamma_i \in \Omega(V, \mathcal{T}_{poly}), \qquad a \in \Omega(V, \mathcal{C}^{poly})
$$

in terms of the structure maps of $\mathcal{K}$ and $\mathcal{S}$. But the only term in $B$ that does not transform as a tensor is

$$
\Gamma = -dy^i \Gamma^k_{ij} y^j \frac{\partial}{\partial y^k}. \tag{5.13}
$$

This term contributes neither to $\mathcal{K}^{tw}_n$ nor to $\mathcal{S}^{tw}_n$ since it is linear in $y$'s.

Thus the quasi-isomorphisms $\mathcal{K}^{tw}$ and $\mathcal{S}^{tw}$ are defined globally and I arrive at the following commutative diagram

$$
\begin{array}{ccc}
(\Omega^\bullet(M, \mathcal{T}_{poly}), D, [,]_{SN}) & \stackrel{\mathcal{K}^{tw}}{\succ\!\!\longrightarrow} & (\Omega^\bullet(M, \mathcal{D}_{poly}), D + \partial, [,]_G) \\
\downarrow^L_{mod} & & \downarrow^R_{mod} \\
(\Omega^\bullet(M, \mathcal{E}), D) & \stackrel{\mathcal{S}^{tw}}{\longleftarrow\!\!\prec\!\!\prec} & (\Omega^\bullet(M, \mathcal{C}^{poly}), D + \mathfrak{b}).
\end{array}
\tag{5.14}
$$

Assembling (5.14) with (5.1) I get the desired commutative diagram

$$\begin{array}{ccccccc}
T^\bullet_{poly}(M) & \stackrel{\lambda_T}{\longrightarrow} & \Omega^\bullet(M,\mathcal{T}_{poly}) & \stackrel{\mathcal{K}^{tw}}{\succ\!\!\longrightarrow} & \Omega^\bullet(M,\mathcal{D}_{poly}) & \stackrel{\lambda_D}{\longleftarrow} & D^\bullet_{poly}(M) \\
\downarrow^L_{mod} & & \downarrow^L_{mod} & & \downarrow^R_{mod} & & \downarrow^R_{mod} \\
\mathcal{A}^\bullet(M) & \stackrel{\lambda_{\mathcal{A}}}{\longrightarrow} & \Omega^\bullet(M,\mathcal{E}) & \stackrel{\mathcal{S}^{tw}}{\longleftarrow\!\!\prec\!\!\prec} & \Omega^\bullet(M,\mathcal{C}^{poly}) & \stackrel{\lambda_C}{\longleftarrow} & C^{poly}_\bullet(M)\,,
\end{array} \qquad (5.15)$$

where the DGLAs $\Omega^\bullet(M,\mathcal{T}_{poly})$ and $\Omega^\bullet(M,\mathcal{D}_{poly})$ are taken with the differentials $D$ and $D+\partial$, respectively, where as the DGLA modules $\Omega^\bullet(M,\mathcal{E})$ and $\Omega^\bullet(M,\mathcal{C}^{poly})$ are taken with the differentials $D$ and $D+\mathfrak{b}$, respectively.

Let $f$ be a diffeomorphism of the pairs $(M,\nabla)$, $(\widetilde{M},\widetilde{\nabla})$

$$f:(M,\nabla)\mapsto(\widetilde{M},\widetilde{\nabla})\,,$$

where $M$ and $\widetilde{M}$ are $d$-dimensional manifolds and $\nabla$, $\widetilde{\nabla}$ are torsion free connections on $M$ and $\widetilde{M}$, respectively.

It is obvious that the corresponding isomorphisms between the DGLA modules

$$f_*:(T^\bullet_{poly}(M),\mathcal{A}^\bullet(M))\mapsto(T^\bullet_{poly}(\widetilde{M}),\mathcal{A}^\bullet(\widetilde{M}))\,,$$

$$f_*:(D^\bullet_{poly}(M),C^{poly}_\bullet(M))\mapsto(D^\bullet_{poly}(\widetilde{M}),C^{poly}_\bullet(\widetilde{M}))\,,$$

$$f_*:(\Omega^\bullet(M,\mathcal{T}_{poly}),\Omega^\bullet(M,\mathcal{E}))\mapsto(\Omega^\bullet(\widetilde{M},\mathcal{T}_{poly}),\Omega^\bullet(\widetilde{M},\mathcal{E}))\,,$$

and

$$f_*:(\Omega^\bullet(M,\mathcal{D}_{poly}),\Omega^\bullet(M,\mathcal{C}^{poly}))\mapsto(\Omega^\bullet(\widetilde{M},\mathcal{D}_{poly}),\Omega^\bullet(\widetilde{M},\mathcal{C}^{poly}))$$

commute with the maps in the diagrams (5.1) for $M$ and $\widetilde{M}$.

Furthermore, since the term (5.13) of the Fedosov connection form (5.8) does not enter the definition of the $L_\infty$-morphism $\mathcal{K}^{tw}$ (5.11) (resp. the morphism of $L_\infty$-modules $\mathcal{S}^{tw}$ (5.12)) the isomorphism $f_*$ commutes with $\mathcal{K}^{tw}$, and $\mathcal{S}^{tw}$ as well. Thus the terms and the morphisms of diagram (5.15) are functorial for diffeomorphisms of pairs $(M,\nabla)$, where $\nabla$ is a torsion free connection on $M$.

Theorem 1 is proved. $\square$

## 5.2 Kontsevich's formality theorem revisited

In this section I prove the existence of a quasi-isomorphism from $T^\bullet_{poly}(M)$ to $D^\bullet_{poly}(M)$ which is functorial for diffeomorphisms of pairs $(M, \nabla)$, where $\nabla$ is a torsion free connection on $M$. Although a proof of this statement is outlined in Appendix 3 of [37] some people [8] think that my proof is more thorough and refer to my paper [19] instead of [37].

First, I observe that composing the quasi-isomorphisms $\lambda_T$ and $\mathcal{K}^{tw}$ one can shorten the upper row in diagram (5.15) to

$$T^\bullet_{poly}(M) \quad \stackrel{\mathcal{U}}{\succ\!\!\longrightarrow} \quad (\Omega^\bullet(M, \mathcal{D}_{poly}), D + \partial, [,]_G) \quad \stackrel{\lambda_D}{\longleftarrow} \quad D^\bullet_{poly}(M)\,, \tag{5.16}$$

in which $\mathcal{U}$ is a quasi-isomorphism of DGLAs.

On the other hand due to proposition 13 the DGLA $D^\bullet_{poly}(M)$ is isomorphic to the DG Lie subalgebra

$$\ker D \cap \Gamma(M, \mathcal{D}_{poly}) \subset \Omega^\bullet(M, \mathcal{D}_{poly})\,. \tag{5.17}$$

This observation raises the question of whether one can contract the quasi-isomorphism $\mathcal{U}$ in (5.16) to a quasi-isomorphism

$$\mathcal{U}^c : T^\bullet_{poly}(M) \succ\!\!\longrightarrow \ker D \cap \Gamma(M, \mathcal{D}^\bullet_{poly}) \tag{5.18}$$

in a functorial way with respect to diffeomorphisms of the pair $(M, \nabla)$. The following theorem gives a positive answer to this question:

**Theorem 6 (M. Konstevich, [37], construction 4)** *For smooth real manifolds $M$ there exists a construction of DGLA quasi-isomorphisms*

$$\mathcal{U}^K : T^\bullet_{poly}(M) \succ\!\!\longrightarrow D^\bullet_{poly}(M) \tag{5.19}$$

*which is functorial for diffeomorphisms of pairs $(M, \nabla)$, where $\nabla$ is a (torsion free) connection on $M$ .*

**Proof.** First, I construct a collection of quasi-isomorphisms ($n \geq 0$)

$$\mathcal{U}^{(n)} \, : \, T^{\bullet}_{poly}(M) \succ\!\!\!\rightarrow (\Omega^{\bullet}(M, \mathcal{D}_{poly}), D + \partial, [,]_G) \tag{5.20}$$

satisfying the following properties:

$$\mathcal{U}^{(n)}_m(\wedge^m T^{\bullet}_{poly}(M)) \subset \ker D \cap \Gamma(M, \mathcal{D}^{\bullet}_{poly}) \, , \qquad \forall \; m \leq n \, , \tag{5.21}$$

$$\mathcal{U}^{(n-1)}_m = \mathcal{U}^{(n)}_m \, , \qquad \forall \; m < n \, , \tag{5.22}$$

where $\mathcal{U}^{(n)}_m$ denote the structure maps of $\mathcal{U}^{(n)}$ .

I start with $\mathcal{U}^{(0)} = \mathcal{U}$ and observe that due to (2.17)

$$(D + \partial) \, \mathcal{U}^{(0)}_1(\gamma) = 0 \, , \qquad \forall \; \gamma \in T^{\bullet}_{poly}(M) \, . \tag{5.23}$$

Since the map $\Phi$ (4.34), (4.36) satisfies equation (4.35) I conclude that for any $\gamma \in T^{\bullet}_{poly}(M)$ the combination

$$\mathcal{U}^{(0)}_1(\gamma) - (D + \partial) \, \Phi(\mathcal{U}^{(0)}_1(\gamma))$$

does not have the top exterior degree component. Thus, applying lemma 1 from section 2.3 for $n = 1$ I get a quasi-isomorphism

$$\widetilde{\mathcal{U}} \, : \, T^{\bullet}_{poly}(M) \succ\!\!\!\rightarrow (\Omega^{\bullet}(M, \mathcal{D}_{poly}), D + \partial, [,]_G) \, ,$$

such that for any $\gamma \in T^{\bullet}_{poly}(M)$

$$\widetilde{\mathcal{U}}_1(\gamma) \in \bigoplus_{k=1}^{d-1} \Omega^k(M, \mathcal{D}^{\bullet}_{poly}) \, ,$$

where $d = \dim M$ .

Proceeding in this way I construct a quasi-isomorphism of DGLAs

$$\mathcal{U}^{(1)} \,:\, T^{\bullet}_{poly}(M) \succ\!\!\!\longrightarrow (\Omega^{\bullet}(M, \mathcal{D}_{poly}), D + \partial, [,]_G)\,,$$

such that for any $\gamma \in T^{\bullet}_{poly}(M)$

$$\mathcal{U}^{(1)}_1(\gamma) \in \Gamma(M, \mathcal{D}^{\bullet}_{poly})\,.$$

On the other hand due to equation (2.17)

$$(D + \partial)\,\mathcal{U}^{(1)}_1(\gamma) = 0$$

and hence, $\mathcal{U}^{(1)}_1(\gamma)$ belongs to the kernel of the Fedosov differential $D$. Thus $\mathcal{U}^{(1)}$ satisfies (5.21).

Suppose that I have already constructed $\mathcal{U}^{(k)}$ up to $k = n-1$ satisfying (5.21) and (5.22). Due to (2.17)

$$(D + \partial)\,\mathcal{U}^{(n-1)}_n(\gamma_1, \gamma_2, \dots, \gamma_n) =$$

$$= \frac{1}{2}\sum_{p,q\geq 1}^{p+q=n} \sum_{\varepsilon \in Sh(p,q)} \pm[\mathcal{U}^{(n-1)}_p(\gamma_{\varepsilon_1}, \dots, \gamma_{\varepsilon_p}), \mathcal{U}^{(n-1)}_q(\gamma_{\varepsilon_{p+1}}, \dots, \gamma_{\varepsilon_n})] - \tag{5.24}$$

$$- \sum_{i \neq j} \pm \mathcal{U}^{(n-1)}_{n-1}([\gamma_i, \gamma_j]_{SN}, \gamma_1, \dots, \hat{\gamma}_i, \dots, \hat{\gamma}_j, \dots \gamma_n), \qquad \gamma_i \in T^{\bullet}_{poly}(M)\,.$$

By the assumption (5.21) of induction the right hand side of equation (5.24) is of exterior degree zero. Hence, using the map $\Phi$ (4.34), (4.35), (4.36) once again I conclude that for any collection $\gamma_i \in T^{\bullet}_{poly}(M)$ the combination

$$\mathcal{U}^{(n-1)}_n(\gamma_1, \dots, \gamma_n) - (D + \partial)\,\Phi(\mathcal{U}^{(n-1)}_n(\gamma_1, \dots, \gamma_n))$$

does not have the top exterior degree component.

Thus applying lemma 1 enough times I get a quasi-isomorphism of DGLAs

$$\mathcal{U}^{(n)} : T^{\bullet}_{poly}(M) \succ\!\!\rightarrow (\Omega^{\bullet}(M, \mathcal{D}_{poly}), D + \partial, [,]_G) ,$$

such that for any $\gamma \in T^{\bullet}_{poly}(M)$

$$\mathcal{U}^{(n)}_n(\gamma) \in \Gamma(M, \mathcal{D}^{\bullet}_{poly}) ,$$

and for any $m < n$

$$\mathcal{U}^{(n)}_m = \mathcal{U}^{(n-1)}_m .$$

Due to the corresponding version of (5.24) $\mathcal{U}^{(n)}_n(\gamma)$ is also annihilated by the Fedosov differential $D$.

Thus, I have constructed the desired collection (5.20). The projective limit of this inverse system gives me a quasi-isomorphism $\mathcal{U}^c$ (5.18). Composing it with $\lambda_D^{-1}$ I get the desired quasi-isomorphism $\mathcal{U}^K$ (5.19).

The construction of $\mathcal{K}^{tw}$ (5.9) is functorial for diffeomorphisms of pairs $(M, \nabla)$ since the term (5.13) of the Fedosov connection form (5.8) does not contribute to (5.11). Thus, the construction of (5.19) is functorial for diffeomorphisms of pairs $(M, \nabla)$ since so are the constructions of $\tau$ (4.42), $\Phi$ (4.34), $\lambda_D$ (4.50), and $\lambda_T$ (4.55). $\square$

## 5.3 Applications

The first obvious applications of the formality theorem for $C^{poly}_{\bullet}(M)$ are related to computation of Hochschild homology for the quantum algebra of functions on a Poisson manifold and to description of traces on this algebra. These applications were suggested in Tsygan's paper [53] (see the first part of corollary 4.0.3 and corollary 4.0.5) as immediate corollaries of the conjectural formality theorem (conjecture 3.3.1 in [53]).

Let $M$ be, as above, a smooth manifold. Recall that

**Definition 15 ([3, 4])** *A deformation quantization of $M$ is a Maurer-Cartan element (2.40) $\Pi \in \hbar D^1_{poly}(M)[[\hbar]]$ of the DGLA $D^{\bullet}_{poly}(M)[[\hbar]]$, where $\hbar$ is an auxiliary variable which plays the role of the deformation parameter. Furthermore, two deformation quantizations $\Pi$ and $\widetilde{\Pi}$ are called equivalent if they are connected by the action (2.41) of an element $U$ in the prounipotent group*

$$\mathfrak{G}_D = \{ I + \hbar D^0_{poly}(M)[[\hbar]] \}$$

*corresponding to the Lie algebra $\hbar D^0_{poly}(M)[[\hbar]]$*

Notice that, since the DGLA $D^{\bullet}_{poly}(M)[[\hbar]]$ is endowed with the complete filtration given by degrees in $\hbar$ the above definition makes sense.

In plain English, the Maurer-Cartan element $\Pi$ in the above definition gives rise to an associative product $*$ (the so-called star-product) in the algebra $C^{\infty}(M)[[\hbar]]$

$$a * b = a \cdot b + \Pi(a, b)\,, \qquad a, b \in C^{\infty}(M)[[\hbar]]\,, \tag{5.25}$$

which deforms the ordinary commutative multiplication in $C^{\infty}(M)[[\hbar]]$. Moreover, two deformation quantizations $\Pi$ and $\widetilde{\Pi}$ corresponding to star-products $*$ and $\widetilde{*}$ are equivalent if there exists a formal series of differential operators (the element $U$ in $\mathfrak{G}_D$)

$$U = I + \hbar U_1 + \hbar^2 U_2 + \cdots \in \{ I + \hbar D^0_{poly}(M)[[\hbar]] \}\,,$$

which establishes an isomorphism between the algebras $(C^{\infty}(M)[[\hbar]], *)$ and $(C^{\infty}(M)[[\hbar]], \widetilde{*})$

$$U(a * b) = U(a) \,\widetilde{*}\, U(b)\,, \qquad a, b \in C^{\infty}(M)[[\hbar]]\,. \tag{5.26}$$

**Remark.** Sometimes it is required that the Maurer-Cartan element $\Pi$ belongs to the subalgebra of normalized polydifferential operator. This requirement corresponds to the compatibility condition with the unit function:

$$a * 1 = 1 * a = 0\,.$$

However, since the subcomplex of normalized Hochschild chains is quasi-isomorphic to the total complex it is very easy to switch from one definition to another using the action of the group $\{I + \hbar D^0_{poly}(M)[[\hbar]]\}$.

Thanks to quasi-isomorphisms of the upper row in the diagram (5.15) and proposition 4 I have a bijective correspondence between the moduli space of Maurer-Cartan elements of the DGLA $T^{\bullet}_{poly}(M)[[\hbar]]$ of polyvector fields and the moduli space of Maurer-Cartan elements of the DGLA $D^{\bullet}_{poly}(M)[[\hbar]]$ of polydifferential operators (tensored with $\mathbb{R}[[\hbar]]$). In other words, if we consider the cone

$$\begin{gathered} \alpha = \hbar\alpha_1 + \hbar^2\alpha_2 + \hbar^3\alpha_3 + \dots \in \hbar\, T^1_{poly}(M)[[\hbar]]\,, \\ [\alpha, \alpha]_{SN} = 0 \end{gathered} \tag{5.27}$$

acted upon by the Lie algebra $\hbar\,\Gamma(M, TM)[[\hbar]]$

$$\alpha \to [u, \alpha]_{SN}\,, \qquad u \in \hbar\,\Gamma(M, TM)[[\hbar]] \tag{5.28}$$

then

**Corollary 1 (M. Kontsevich, [38], theorem 1.1)** *The deformation quantizations (5.25) of $M$ modulo the equivalence relation (5.26) are in a bijective correspondence with the points of the cone (5.27) modulo the action (5.28) of the prounipotent group corresponding to the Lie algebra $\hbar\,\Gamma(M, TM)[[\hbar]]$.* $\Box$

An orbit $[\alpha]$ on the cone (5.27) corresponding to a deformation $\Pi$ is called *Kontsevich's class of the deformation* $\Pi$ and any point $\alpha$ of this orbit is called *a representative* of this class.

Theorem 1 allows me to describe Hochschild homology of the algebra $(C^\infty(M)[[\hbar]], *)$ for any deformation quantization $\Pi$ (5.25) of $M$. Namely[2]

**Corollary 2** *If $\Pi$ is a deformation quantization and $\alpha \in \hbar\,T^1_{poly}(M)[[\hbar]]$ represents*

[2]see the first part of corollary 4.0.3 in [53]

*Kontsevich's class $[\alpha]$ of $\Pi$ then the complex of Hochschild homology*

$$(C_\bullet^{poly}(M)[[\hbar]], \mathfrak{b} + R_\Pi) \tag{5.29}$$

*is quasi-isomorphic to the complex of exterior forms*

$$(\mathcal{A}^\bullet(M)[[\hbar]], L_\alpha) \tag{5.30}$$

*with the differential $L_\alpha$.*

Here, as above, $R$ denotes the action (3.5) of Hochschild cochains on Hochschild chains and $L$ stands for the Lie derivative (3.21).

**Remark.** In the symplectic case the above corollary reduces to the well-known theorem of R. Nest and B. Tsygan (theorem A2.1 in [42]) which is proved for the quantum algebra of compactly supported functions of a smooth symplectic manifold. An equivariant version of this result in the symplectic case is discussed in paper [20] (see proposition 4) and paper [41] (see theorem 5.2).

**Proof.** Since $\alpha$ represents Kontsevich's class of the deformation quantization $\Pi$ the Maurer-Cartan elements $\lambda_D(\Pi)$ and

$$\mathfrak{P} = \sum_{m=1}^{\infty} \frac{1}{m!} \mathcal{K}_m^{tw}(\lambda_T(\alpha), \dots, \lambda_T(\alpha)) \tag{5.31}$$

are connected by the action (2.41) of an element $\mathfrak{U}$ of the prounipotent group $\mathfrak{H}$ corresponding to the Lie algebra

$$\mathfrak{h} = (\Omega^0(M, \mathcal{D}^0_{poly}) \oplus \Omega^1(M, \mathcal{D}^{-1}_{poly})) \otimes \hbar\mathbb{R}[[\hbar]]\,.$$

Therefore $\mathfrak{U}$ provides me with a quasi-isomorphism (actually isomorphism)

$$R_{\mathfrak{U}} : (\Omega^\bullet(M, \mathcal{C}^{poly})[[\hbar]], D + \mathfrak{b} + R_{\lambda_D(\Pi)}) \mapsto (\Omega^\bullet(M, \mathcal{C}^{poly})[[\hbar]], D + \mathfrak{b} + R_{\mathfrak{P}}) \tag{5.32}$$

from the complex $(\Omega^\bullet(M,\mathcal{C}^{poly})[[\hbar]], D+\mathfrak{b}+R_{\lambda_D(\Pi)})$ to the complex $(\Omega^\bullet(M,\mathcal{C}^{poly})[[\hbar]], D+\mathfrak{b}+R_{\mathfrak{P}})$.

Twisting the terms in the second diagram in (5.1) by the Maurer-Cartan element $\Pi$ I get the new commutative diagram

$$\begin{array}{ccc} (\Omega^\bullet(M,\mathcal{D}_{poly})[[\hbar]], D+\partial+[\lambda_D(\Pi),\cdot\,]_G) & \overset{\lambda_D}{\longleftarrow} & (D^\bullet_{poly}(M)[[\hbar]], \partial+[\Pi,\cdot\,]_G) \\ \downarrow^R_{mod} & & \downarrow^R_{mod} \\ (\Omega^\bullet(M,\mathcal{C}^{poly})[[\hbar]], D+\mathfrak{b}+R_{\lambda_D(\Pi)}) & \overset{\lambda_C}{\longleftarrow} & (C^{poly}_\bullet(M)[[\hbar]], \mathfrak{b}+R_\Pi)\,, \end{array} \tag{5.33}$$

in which the DGLAs $\Omega^\bullet(M,\mathcal{D}_{poly})[[\hbar]]$ and $D^\bullet_{poly}(M)[[\hbar]]$ carry the initial Lie bracket $[,]_G$ (3.2).

Due to claim 5 of proposition 2 the map $\lambda_C$ in the above diagram is a quasi-isomorphism of complexes.

On the other hand twisting the terms in the left part of diagram (5.15) by the Maurer-Cartan element $\alpha \in T^\bullet_{poly}(M)[[\hbar]]$ I get the new commutative diagram

$$\begin{array}{ccc} (T^\bullet_{poly}(M)[[\hbar]], [\alpha,\cdot\,]_{SN}) & \overset{\lambda_T}{\longrightarrow} & (\Omega^\bullet(M,\mathcal{T}_{poly})[[\hbar]], D+[\lambda_T(\alpha),\cdot\,]_{SN}) \\ \downarrow^L_{mod} & & \downarrow^L_{mod} \\ (\mathcal{A}^\bullet(M)[[\hbar]], L_\alpha) & \overset{\lambda_{\mathcal{A}}}{\longrightarrow} & (\Omega^\bullet(M,\mathcal{E})[[\hbar]], L_{\lambda_T(\alpha)})\,, \end{array} \tag{5.34}$$

in which the DGLAs $\Omega^\bullet(M,\mathcal{T}_{poly})[[\hbar]]$ and $T^\bullet_{poly}(M)[[\hbar]]$ carry the initial Lie bracket $[,]_{SN}$ (3.20).

Due to claim 5 of proposition 2 the map $\lambda_{\mathcal{A}}$ in diagram (5.34) is a quasi-isomorphism of complexes.

Similarly, twisting the terms in the middle part of diagram (5.15) by the Maurer-

Cartan element $\lambda_T(\alpha) \in \Omega^\bullet(M, \mathcal{T}_{poly})[[\hbar]]$ I get

$$
\begin{array}{ccc}
(\Omega^\bullet(\mathcal{T}_{poly})[[\hbar]], D + [\lambda_T(\alpha), \cdot]_{SN}) & \stackrel{\mathcal{K}^\alpha}{\succ\!\!\longrightarrow} & (\Omega^\bullet(\mathcal{D}_{poly})[[\hbar]], D + \partial + [\mathfrak{P}, \cdot]_G) \\
\downarrow^{L}_{mod} & & \downarrow^{R}_{mod} \\
(\Omega^\bullet(\mathcal{E})[[\hbar]], L_{\lambda_T(\alpha)}) & \stackrel{\mathcal{S}^\alpha}{\longleftarrow\!\!\prec\!\!\prec} & (\Omega^\bullet(\mathcal{C}^{poly})[[\hbar]], D + \mathfrak{b} + R_{\mathfrak{P}}) \,,
\end{array}
\tag{5.35}
$$

where $\mathcal{K}^\alpha$ and $\mathcal{S}^\alpha$ are obtained from $\mathcal{K}^{tw}$ and $\mathcal{S}^{tw}$, respectively, by twisting via $\lambda_T(\alpha) \in \Omega^\bullet(M, \mathcal{T}_{poly})[[\hbar]]$ , $\mathfrak{P}$ is defined in (5.31), $\Omega^\bullet(M, \mathcal{T}_{poly})[[\hbar]]$ goes with the initial bracket $[,]_{SN}$, and $\Omega^\bullet(M, \mathcal{D}_{poly})[[\hbar]]$ goes with the initial bracket $[,]_G$ .

Again, due to claim 5 of proposition 2 the morphism of $L_\infty$-modules $\mathcal{S}^\alpha$ in diagram (5.35) is a quasi-isomorphism.

The desired statement is proved since the complexes (5.29) and (5.30) are connected by a chain of quasi-isomorphisms. $\square$

Another application of theorem 1 is related to description of traces on the algebra $(C^\infty_c(M)[[\hbar]], *)$, where by $C^\infty_c(M)$ I denote the vector space of smooth functions with a compact support.

By definition *a trace* is a continuous $\mathbb{R}[[\hbar]]$-linear $\mathbb{R}[[\hbar]]$-valued functional $tr$ on $C^\infty_c(M)[[\hbar]]$ vanishing on commutators

$$
tr(R_\Pi(a)) = 0 \,,
$$

where $a = a(x_0, x_1)$ is a function in $C^\infty(M \times M)$ with a compact support in its first argument, and $R$ is as in (3.5).

One can easily verify that my constructions still make sense if I replace the first version (3.13) of $C^{poly}_\bullet(M)$ by

$$
C^{poly-com}(M) = \bigoplus_{n \ge 0} C^\infty_{com}(M^{n+1}) \,,
$$

and the vector space of exterior forms $\mathcal{A}^\bullet(M)$ by the vector space $\mathcal{A}^\bullet_c(M)$ of exterior forms with a compact support. Here by $C^\infty_{com}(M^{n+1})$ I denote the vector space of smooth functions on $M^{n+1}$ with a compact support in the first argument.

Then the corresponding version of the above corollary implies that

**Corollary 3 ([53], Corollary 4.0.5)** *If* $\Pi$ *is deformation quantization (5.25) and* $\alpha$ *represents Kontsevich's class of* $\Pi$ *then the vector space of traces on the algebra* $(C_c^\infty(M)[[\hbar]], *)$ *is isomorphic to the vector space of continuous* $\mathbb{R}[[\hbar]]$*-linear* $\mathbb{R}[[\hbar]]$*-valued functionals on* $C_c^\infty(M)[[\hbar]]$ *vanishing on all Poisson brackets* $\alpha(a,b)$ *for* $a, b \in C_c^\infty(M)[[\hbar]]$ *.* $\Box$

For a symplectic manifold this statement has been proved in [16, 23, 42].

**Remark.** Corollary 3 still holds if one replaces real valued functions (resp. traces) by smooth complex valued functions (resp. complex valued traces), as well as the ring $\mathbb{R}[[\hbar]]$ by the field $\mathbb{C}[[\hbar, \hbar^{-1}]$.

I would like to mention that the functoriality of the chain of quasi-isomorphisms (3.33) in theorem 1 implies the following interesting results

**Corollary 4** *Let* $M$ *be a smooth manifold equipped with a smooth action of a group* $G$*. If one can construct a* $G$*-invariant connection* $\nabla$ *on* $M$ *then there exists a chain of* $G$*-equivariant quasi-isomorphisms between the DGLA modules* $(T^\bullet_{poly}(M), \mathcal{A}^\bullet(M))$ *and* $(D^\bullet_{poly}(M), C^{poly}_\bullet(M))$ *.* $\Box$

In particular,

**Corollary 5** *If* $M$ *is equipped with a smooth action of a finite or compact group* $G$ *then the DGLA modules* $\Big((T^\bullet_{poly}(M))^G,\, (\mathcal{A}^\bullet(M))^G\Big)$ *and* $\Big((D^\bullet_{poly}(M))^G,\, (C^{poly}_\bullet(M))^G\Big)$ *of* $G$*-invariants are quasi-isomorphic.* $\Box$

Due to the functoriality of the quasi-isomorphism (5.19) in theorem 6 I have the following result:

**Corollary 6** *If* $M$ *is equipped with a smooth action of a finite or compact group* $G$ *then there exists a quasi-isomorphism from the DGLA* $(T^\bullet_{poly}(M))^G$ *of* $G$*-invariant polyvector fields to the DGLA* $(D^\bullet_{poly}(M))^G$ $G$*-invariant polydifferential operators on* $M$*.* $\Box$

Using this corollary and proposition 4 I get a solution of a deformation quantization problem for an arbitrary Poisson orbifold. Namely,

**Corollary 7** *Given a smooth action of a finite group $G$ on a manifold $M$ and a $G$-invariant Poisson structure $\alpha \in (\wedge^2 TM)^G$ one can always construct a $G$-invariant star-product $*$, corresponding to $\alpha$. Furthermore, $G$-invariant star-products on $M$ corresponding to the Poisson bracket $\alpha$ are classified up to equivalence by non-trivial $G$-invariant deformations of $\alpha$.* $\square$

# Chapter 6

# Conclusion

I am glad that the results of my thesis have been already applied to two interesting problems. In his PhD thesis [51] X. Tang used theorem 1 to compute Hochschild homology of formal symplectic deformations of proper étale Lie groupoids[1] and in [8] D. Calaque used theorem 6 in order to quantize a class of formal classical dynamical $r$-matrices in the reductive case.

I would like to mention paper [7] in which D. Calaque generalized theorem 6 to the case when the tangent bundle of $M$ is replaced by an arbitrary smooth Lie algebroid. In our joint paper [9] we generalized the result of [7] to Hochschild chains and extended our constructions to the holomorphic setting. In this way we proved a version of Tsygan's formality conjecture for Hochschild chains for any complex manifold.

I would like to mention parallel results of the MIT alumnus A. Yekutieli. In his papers [56, 55, 57] he proved that for any smooth algebraic variety $X$ over a field $\mathbb{K}$ ($\mathbb{R} \subset \mathbb{K}$) the sheaf of polyvector fields and the sheaf of polydifferential operators are quasi-isomorphic as sheaves of DGLAs.

In [40] S.L. Lyakhovich and A.A. Sharapov suggested that a generalization of theorem 6 for super-manifolds can be applied to quantum reduction. In this paper they proposed the most general setting of a reduction which leads to a Poisson manifold

[1] See paper [41] in which cyclic and Hochschild (co)homology groups of formal symplectic deformations of proper étale Lie groupoids were computed without making use of formality theorems.

and showed that under certain cohomological conditions the “super”-version of Kontsevich's formality theorem would lead to deformation quantization of the reduced manifold. In [40] the authors also discussed a possible path integral approach [2, 11] to the “super”-version of Kontsevich's formality theorem.

Two relative versions of Kontsevich's formality theorem were suggested simultaneously in papers [5] and [12]. In both of these papers the authors considered a smooth submanifold $C$ of a smooth manifold $M$. In paper [5] it is conjectured that the DGLA (and more generally $G_\infty$-algebra) of polydifferential operators compatible with the ideal $I \subset C^\infty(M)$ of functions vanishing on $C$ is formal. In [5] the authors proved this conjecture for the case $M = \mathbb{R}^d$ and $C = \mathbb{R}^{d-k}$ if $k \geq 2$, and applied this result to the construction of representations of the star-product algebras. In paper [12] the authors proved the formality theorem for the DGLA of polydifferential operators acting on the exterior algebra of the conormal bundle of $C$ in $M$ and applied this result to the quantum reduction. I would like to mention that the question of globalization is not properly addressed in [12]. However, I do not think that this question is very difficult since the authors reduced their problem to a formal neighborhood of $C$ in $M$ .

There are still many interesting open questions in this subject. For example, it would be very interesting to develop the applications [12], [40] of the “super”-version of Kontsevich's formality theorem to the quantum reduction and find out how the characteristic classes of deformations fit into the reduction procedure [14]. It is also interesting to further investigate the relative versions [5, 12] of Kontsevich's formality theorem and apply them to the conjectural correspondence [10] between the category of Poisson manifolds with dual pairs as morphisms and the category of deformation quantization algebras with bimodules as morphisms. Finally, the cyclic formality conjecture [53] as well as the most general version of the algebraic index theorem [49] still remain open questions.

# Appendix A

# Figures

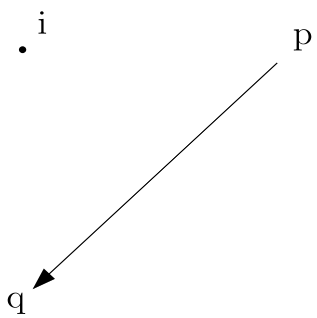


Figure A-1: Edge of the first type

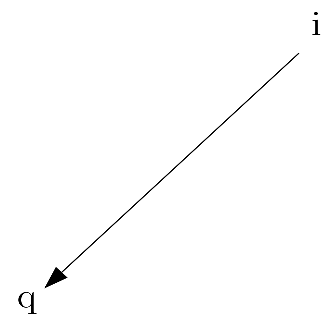


Figure A-2: Edge of the second type


Figure A-3: Diagrams of the first type

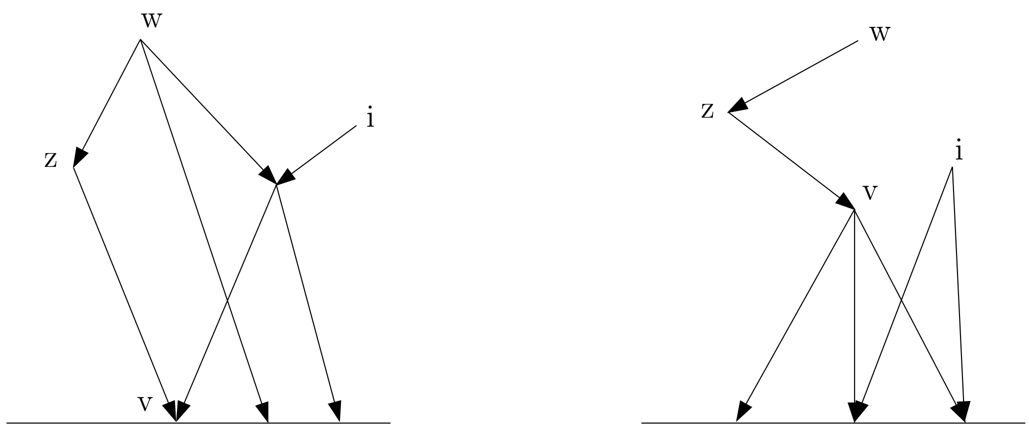


Figure A-4: Diagrams of the second type

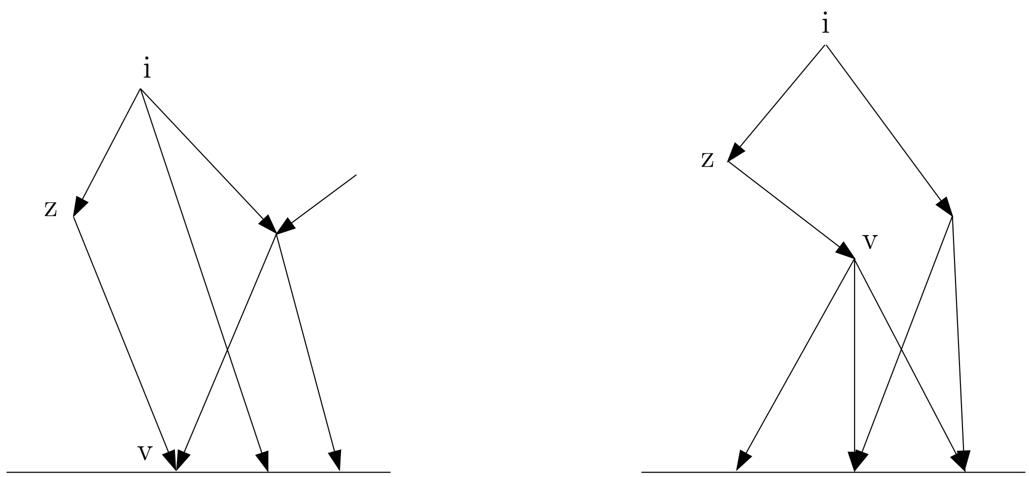


Figure A-5: Diagrams of the third type

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