Exploration of Grothendieck-Teichmueller(GT)-shadows and their action on child's drawings

Vasily Dolgushev

Temple University

The University of Minnesota, Topology Seminar, September 26, 2022

Loosely based on papers in preparation with Jacob Guynee, Jessica Radford and Jingfeng Xia

The absolute Galois group $G_{\mathbb{Q}}$ of rationals and $\widehat{\mathrm{GT}}$

$G_{\mathbb{Q}}$ is the group of (field) automorphisms of the algebraic closure $\overline{\mathbb{Q}}$ of the field $\mathbb{Q}$ of rational numbers. This group is uncountable. For every finite Galois extension $E \supset \mathbb{Q}$, any element $g \in \mathrm{Gal}(E/\mathbb{Q})$ can be extended (in infinitely many ways) to an element of $G_{\mathbb{Q}}$. The group $G_{\mathbb{Q}}$ is one of the most mysterious objects in mathematics!

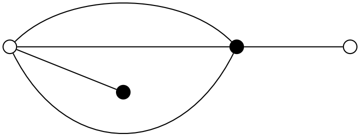
In 1990, Vladimir Drinfeld introduced yet another mysterious group $\widehat{\mathrm{GT}}$ (the Grothendieck-Teichmuellller group). $\widehat{\mathrm{GT}}$ consists of pairs $(\hat{m}, \hat{f})$ in $\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$ satisfying some conditions and it receives a one-to-one homomorphism

$$G_{\mathbb{Q}} \hookrightarrow \widehat{\mathrm{GT}}.$$

Only two elements of $G_{\mathbb{Q}}$ **are known explicitly**: the identity element and the complex conjugation $a + bi \mapsto a - bi$. The corresponding images in $\widehat{\mathrm{GT}}$ are $(0, 1)$ and $(-1, 1)$.

A child's drawing of degree d is ...

An isom. class of a connected bipartite ribbon graph with d edges.



An equiv. class of a pair (g_1, g_2) of permutations in S_d for which the group $\langle\, g_1, g_2 \,\rangle$ acts transitively on $\{1, 2, \dots, d\}$.

A conjugacy class of an index d subgroup of $\mathsf{F}_2 := \langle\, x, y \,\rangle$.

A conjugacy class of a group homomorphism $\psi : \mathsf{F}_2 \to S_d$ (with the subgroup $\psi(\mathsf{F}_2) \le S_d$ being transitive.)

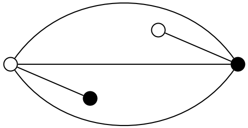
An isom. class of a (non-constant) holomorphic map $\varphi : \Sigma \to \mathbb{CP}^1$ from a compact connected Riemann surface (without boundary) that does not have branch points above every $w \in \mathbb{CP}^1 - \{0, 1, \infty\}$.

The action of $G_{\mathbb{Q}}$ on child's drawings

Given a child's drawing D, we can find a smooth projective curve X defined over $\overline{\mathbb{Q}}$ and an algebraic map $\varphi : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ that does not have branch points above every $w \in \mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}$. (X, φ) is called a **Belyi pair** representing D.

The coefficients defining the curve X and the map φ lie in some finite Galois extension E of $\mathbb{Q}$. Given $g \in \mathrm{Gal}(E/\mathbb{Q})$, the child's drawing $g(D)$ is represented by the (new) Belyi pair $(g(X), g(\varphi))$. We simply act by g on the coefficients defining X and φ!

The $G_{\mathbb{Q}}$-orbit of the above child's drawing has two elements. It's 'Galois conjugate' is



Basic invariants of child's drawings

- the **degree** d of a child's drawing $[\mathrm{F}_2 \stackrel{\psi}{\longrightarrow} S_d]$;
- the conjugacy class of the subgroup $\psi(\mathrm{F}_2) \le S_d$ is call the **monodromy group** of $[\psi]$;
- for a child's drawing represented by $(g_1, g_2) \in S_d \times S_d$, its **passport** is the triple of partitions $\big(\mathrm{ct}(g_1), \mathrm{ct}(g_2), \mathrm{ct}(g_2^{-1} g_1^{-1})\big)$ of d, where $\mathrm{ct}(h)$ denotes the cycle type of a permutation $h \in S_d$;
- the **cartographic group** and more...

Let σ_1, σ_2 be the standard generators of Artin's braid group B_3. The formulas (here, $g_3 := g_2^{-1} g_1^{-1}$)

$$\sigma_1(g_1, g_2, g_3) := (g_2, g_2^{-1} g_1 g_2, g_3), \quad \sigma_2(g_1, g_2, g_3) := (g_1, g_3, g_3^{-1} g_2 g_3)$$

define an action of B_3 on child's drawings. Since the pure braid group PB_3 acts trivially, we actually get an action of S_3 on child's drawings. The action of $G_{\mathbb{Q}}$ commutes with this action of S_3.

A bit about (the gentle version of) $\widehat{\mathrm{GT}}$

For $(\hat{m}, \hat{f}) \in \widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$, the formulas

$$E_{\hat{m},\hat{f}}(x) := x^{2\hat{m}+1}, \qquad E_{\hat{m},\hat{f}}(y) := \hat{f}^{-1} y^{2\hat{m}+1} \hat{f}$$

define a continuous endomorphism $E_{\hat{m},\hat{f}}$ of $\widehat{\mathrm{F}}_2$.

$\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$ is a monoid with the binary operation

$$(\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2) := \big(2\hat{m}_1 \hat{m}_2 + \hat{m}_1 + \hat{m}_2,\ \hat{f}_1 E_{\hat{m}_1, \hat{f}_1}(\hat{f}_2) \big)$$

and the identity element $(0, 1)$.

Let σ_1, σ_2 be the standard generators of Artin's braid group B_3,

$$c := (\sigma_1 \sigma_2 \sigma_1)^2 \quad \text{and} \quad x_{12} := \sigma_1^2, \qquad x_{23} := \sigma_2^2.$$

Let $\widehat{\mathsf{GT}}_{mon}$ be the submonoid of $\widehat{\mathbb{Z}} \times \widehat{\mathsf{F}}_2$ that consists of pairs $(\hat{m}, \hat{f})$ satisfying the **hexagon relations**:

$$\sigma_1^{2\hat{m}+1} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} = \hat{f}^{-1} \sigma_1 \sigma_2 \, x_{12}^{-\hat{m}} c^{\hat{m}} \,,$$

$$\hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \, \sigma_1^{2\hat{m}+1} = \sigma_2 \sigma_1 x_{23}^{-\hat{m}} c^{\hat{m}} \, \hat{f}$$

and $\hat{f} \in [\widehat{\mathsf{F}}_2, \widehat{\mathsf{F}}_2]^{top.\,cl.}$.

$\widehat{\mathsf{GT}}_{gen}$ is the group of invertible elements of the monoid $\widehat{\mathsf{GT}}_{mon}$.

The formula

$$\chi_{vir}(\hat{m}, \hat{f}) := 2\hat{m} + 1$$

defines a (continuous) group homomorphism $\widehat{\mathsf{GT}}_{gen} \to \widehat{\mathbb{Z}}^{\times}$.
χ_{vir} is called the **virtual cyclotomic character.**

In the remaining slides, $\widehat{\mathsf{GT}}$ denotes $\widehat{\mathsf{GT}}_{gen} = \widehat{\mathsf{GT}}_0$. *D. Harbater, L. Schneps, (2000).*

The action of $\widehat{\mathsf{GT}}$ on child's drawings

Let $(\hat{m}, \hat{f})$ be an element of $\widehat{\mathsf{GT}}$ and D be a child's drawing. It is convenient to represent D by a group homomorphism

$$\psi : \mathsf{F}_2 \to S_d \,,$$

where $\psi(\mathsf{F}_2)$ is transitive. (D corresponds to the conjugacy class of the stabilizer of 1 or 2 or 3 . . . .)

ψ extends, by continuity, to a (continuous) group homomorphism $\hat{\psi} : \widehat{\mathsf{F}}_2 \to S_d$. The child's drawing $D^{(\hat{m},\hat{f})}$ corresponds to the group homomorphism

$$\hat{\psi} \circ E_{\hat{m},\hat{f}}\big|_{\mathsf{F}_2} : \mathsf{F}_2 \to S_d \,.$$

$$E_{\hat{m},\hat{f}}(x) := x^{2\hat{m}+1} \quad \text{and} \quad E_{\hat{m},\hat{f}}(y) := \hat{f}^{-1}\, y^{2\hat{m}+1}\, \hat{f}.$$

See Y. Ihara's paper "On the embedding of Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$ into $\widehat{\mathsf{GT}}$" $+$ the appendix by M. Emsalem and P. Lochak.

The Artin braid group B_3 and PB_3

B_3 (resp. PB_3) denotes the Artin braid group (resp. the pure braid group) on 3 strands. σ_1, σ_2 are the standard generators of B_3

We set $\Delta := \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.
PB_3 is generated by

$$x_{12} := \sigma_1^2, \qquad x_{23} := \sigma_2^2, \qquad c := \Delta^2 .$$

It is known that $\mathcal{Z}(B_3) = \mathcal{Z}(PB_3) = \langle\, c \,\rangle \cong \mathbb{Z}$, the subgroup $\langle\, x_{12}, x_{23} \,\rangle$ is isomorphic to F_2. In fact, $PB_3 \cong F_2 \times \langle\, c \,\rangle$.

A bit more about F_2, PB_3 and B_3

We tacitly identify F_2 with the subgroup $\langle x_{12}, x_{23} \rangle \le PB_3$. We also set

$$x := x_{12} \qquad \text{and} \qquad y := x_{23}.$$

We denote by θ and τ the following automorphisms of F_2:

$$\theta(x) := y, \quad \theta(y) := x, \qquad \tau(x) := y, \quad \tau(y) := y^{-1}x^{-1}.$$

We set

$$NFI_{PB_3}(B_3) := \{ N \trianglelefteq B_3 \mid N \le PB_3, \ \ |B_3 : N| < \infty \}$$

and abbreviate $NFI := NFI_{PB_3}(B_3)$.

Preparation

For $\mathsf{N} \in \mathsf{NFI}$, we set

$$\mathsf{N}_{\mathsf{F}_2} := \mathsf{F}_2 \cap \mathsf{N}, \quad N_{\mathrm{ord}} := \mathrm{lcm}\left(\mathrm{ord}(x_{12}\mathsf{N}), \mathrm{ord}(x_{23}\mathsf{N}), \mathrm{ord}(c\mathsf{N})\right).$$

We say that $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$ satisfies the **hexagon relations** modulo N if

$$\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f \,\mathsf{N} = f^{-1} \sigma_1 \sigma_2 x_{12}^{-m} c^m \,\mathsf{N}$$
$$f^{-1} \sigma_2^{2m+1} f \, \sigma_1^{2m+1} \mathsf{N} = \sigma_2 \sigma_1 x_{23}^{-m} c^m f \,\mathsf{N}.$$

Proposition

If $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$ satisfies the hexagon relations modulo N*, then the formulas*

$$T_{m,f}(\sigma_1) := \sigma_1^{2m+1} \mathsf{N}, \qquad T_{m,f}(\sigma_2) := f^{-1} \sigma_2^{2m+1} f \,\mathsf{N}$$

define a group homomorphism $T_{m,f} : \mathsf{B}_3 \to \mathsf{B}_3/\mathsf{N}$.

Let us restrict $T_{m,f}$ to PB_3 and to F_2

Restricting the above homomorphism $T_{m,f} : \mathsf{B}_3 \to \mathsf{B}_3/\mathsf{N}$ to PB_3 and to F_2, we get

$$T^{\mathsf{PB}_3}_{m,f} : \mathsf{PB}_3 \to \mathsf{PB}_3/\mathsf{N}, \qquad T^{\mathsf{F}_2}_{m,f} : \mathsf{F}_2 \to \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2},$$

$$T^{\mathsf{PB}_3}_{m,f}(x_{12}) = x_{12}^{2m+1}\,\mathsf{N}, \qquad T^{\mathsf{PB}_3}_{m,f}(x_{23}) = f^{-1} x_{23}^{2m+1} f\,\mathsf{N},$$

$$T^{\mathsf{PB}_3}_{m,f}(c) = c^{2m+1}\,\mathsf{N},$$

$$T^{\mathsf{F}_2}_{m,f}(x) = x^{2m+1}\,\mathsf{N}_{\mathsf{F}_2}, \qquad T^{\mathsf{F}_2}_{m,f}(y) = f^{-1} y^{2m+1} f\,\mathsf{N}_{\mathsf{F}_2},$$

$T_{m,f}$ is onto $\iff$ $T^{\mathsf{PB}_3}_{m,f}$ is onto $\iff$ $T^{\mathsf{F}_2}_{m,f}$ is onto.

$\ker(T_{m,f}) = \ker(T^{\mathsf{PB}_3}_{m,f})$. Hence $\ker(T_{m,f}) \in \mathsf{NFI}$.

A GT-shadow is ...

Definition

Let $\mathsf{N} \in \mathsf{NFI}$. *A* GT-***shadow with the target*** N *is a pair*

$$[m, f] := (m + N_{\mathrm{ord}}\mathbb{Z}, f\mathsf{N}_{\mathsf{F}_2}) \in \mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}$$

satisfying the hexagon relations (modulo N*) and such that*

- $2m + 1$ *represents a unit in the ring* $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z}$,
- $f\mathsf{N}_{\mathsf{F}_2} \in [\mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}, \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}]$, *or equivalently* $\exists\ w \in [\mathsf{F}_2, \mathsf{F}_2]$ *such that* $f\mathsf{N}_{\mathsf{F}_2} = w\mathsf{N}_{\mathsf{F}_2}$, *and*
- *the homomorphism* $T_{m,f} : \mathsf{B}_3 \to \mathsf{B}_3/\mathsf{N}$ *is onto (* $\iff$ $T^{\mathsf{PB}_3}_{m,f}$ *is onto* $\iff$ $T^{\mathsf{F}_2}_{m,f}$ *is onto).*

GT(N) is the set of GT-shadows with the target N.

The groupoid GTSh

Guess what?!.... GT-shadows form a groupoid GTSh.

$$Ob(\mathsf{GTSh}) := \mathsf{NFI}_{\mathsf{PB}_3}(\mathsf{B}_3); \quad \text{for} \quad \mathsf{K}, \mathsf{N} \in \mathsf{NFI},$$

$$\mathsf{GTSh}(\mathsf{K}, \mathsf{N}) := \Big\{ [m, f] \in \mathsf{GT}(\mathsf{N}) \mid \ker(T_{m,f}) = \mathsf{K} \Big\}.$$

Let $\mathsf{N}^{(1)}, \mathsf{N}^{(2)}, \mathsf{N}^{(3)} \in \mathsf{NFI}$ and

$$\mathsf{N}^{(3)} \xrightarrow{[m_2, f_2]} \mathsf{N}^{(2)} \xrightarrow{[m_1, f_1]} \mathsf{N}^{(1)}.$$

The composition of morphisms is defined by the formula:

$$[m_1, f_1] \circ [m_2, f_2] := \big[2m_1 m_2 + m_1 + m_2 \, , \, f_1 E_{m_1, f_1}(f_2) \big]$$

$\forall \, \mathsf{N} \in \mathsf{NFI}$, $[0, 1_{\mathsf{F}_2}]$ is the identity morphism in $\mathsf{GTSh}(\mathsf{N}, \mathsf{N})$.

A comment

For $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$, the formulas

$$E_{m,f}(x) := x^{2m+1}, \qquad E_{m,f}(y) := f^{-1} y^{2m+1} f$$

define an endomorphism of F_2.

Moreover, for all $(m_1, f_1), (m_2, f_2) \in \mathbb{Z} \times \mathsf{F}_2$,

$$E_{m_1,f_1} \circ E_{m_2,f_2} = E_{m,f},$$

where $m := 2m_1 m_2 + m_1 + m_2$ and $f := f_1 E_{m_1,f_1}(f_2)$.

On can show that the set $\mathbb{Z} \times \mathsf{F}_2$ is a monoid with respect to the binary operation

$$(m_1, f_1) \bullet (m_2, f_2) := \big(2m_1 m_2 + m_1 + m_2\,,\ f_1 E_{m_1,f_1}(f_2)\big)$$

with $(0, 1_{\mathsf{F}_2})$ being the identity element.

Basic facts about GTSh

- GTSh has infinitely many objects. ($\mathsf{NFI}_{PB_3}(B_3)$ is infinite because PB_3 is residually finite.)
- GTSh is highly disconnected. However, for every $N \in \mathsf{NFI}_{PB_3}(B_3)$, the connected component $\mathsf{GTSh}_{conn}(N)$ of N is a finite groupoid.
- If $\mathsf{GTSh}_{conn}(N)$ has only one object (i.e. GT(N) is group), then we say that N is **isolated**.
- For every $N \in \mathsf{NFI}_{PB_3}(B_3)$, the object

$$N^{\diamond} := \bigcap_{K \in Ob(\mathsf{GTSh}_{conn}(N))} K$$

is isolated. In particular, the subposet $\mathsf{NFI}^{isol.} \subset \mathsf{NFI}$ of isolated objects is cofinal.

“Reduction modulo” H

Let $\mathsf{N}, \mathsf{H} \in \mathsf{NFI}$ with $\mathsf{N} \le \mathsf{H}$. We have $\mathsf{N}_{\mathsf{F}_2} \le \mathsf{H}_{\mathsf{F}_2}$ and $H_{\mathrm{ord}} | N_{\mathrm{ord}}$.

If a pair $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$ represents a GT-shadow with the target N, then **the same pair** also represents a GT-shadow with the target H.

Hence we have a natural map

$$\mathcal{R}_{\mathsf{N},\mathsf{H}} : \mathsf{GT}(\mathsf{N}) \to \mathsf{GT}(\mathsf{H})$$

If N, H are isolated (i.e. $\mathsf{GT}(\mathsf{N}), \mathsf{GT}(\mathsf{H})$ are groups) then $\mathcal{R}_{\mathsf{N},\mathsf{H}}$ is a group homomorphism.

$\widehat{\mathrm{GT}}$ versus GTSh

For every $(\hat{m}, \hat{f}) \in \widehat{\mathrm{GT}}$ and $\mathrm{N} \in \mathrm{NFI}$ the pair

$$\mathrm{PR}_{\mathrm{N}}(\hat{m}, \hat{f}) := \big(\mathcal{P}_{N_{\mathrm{ord}}}(\,\hat{m}\,)\,,\,\mathcal{P}_{\mathrm{N}_{\mathrm{F}_2}}(\,\hat{f}\,)\big) \;\in\; \mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathrm{F}_2/\mathrm{N}_{\mathrm{F}_2}$$

is a GT-shadow with the target N. (For $\mathrm{K} \in \mathrm{NFI}(G)$, $\mathcal{P}_{\mathrm{K}}$ denotes the standard continuous homomorphism $\widehat{G} \to G/\mathrm{K}$.) $\mathrm{PR}_{\mathrm{N}}(\hat{m}, \hat{f})$ is an **approximation** of the element $(\hat{m}, \hat{f})$.

A GT-shadow $[m, f] \in \mathrm{GT}(\mathrm{N})$ is called **genuine** if $\exists\,(\hat{m}, \hat{f}) \in \widehat{\mathrm{GT}}$ such that $\mathrm{PR}_{\mathrm{N}}(\hat{m}, \hat{f}) = [m, f]$. Otherwise, it is called **fake.**

A GT-shadow $[m, f] \in \mathrm{GT}(\mathrm{N})$ **survives into** $\mathrm{K} \in \mathrm{NFI}$ (with $\mathrm{K} \le \mathrm{N}$) if $[m, f] \in \mathcal{R}_{\mathrm{K},\mathrm{N}}\big(\mathrm{GT}(\mathrm{K})\big)$.

Proposition. A GT-shadow $[m, f] \in \mathrm{GT}(\mathrm{N})$ is genuine $\iff$ $[m, f]$ survives into K for every $\mathrm{K} \in \mathrm{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ such that $\mathrm{K} \le \mathrm{N}$.

The Main Line functor

Let $\mathsf{K}, \mathsf{N} \in \mathsf{NFI}$ be isolated objects of the groupoid GTSh and $\mathsf{K} \le \mathsf{N}$.

Since $\mathcal{R}_{\mathsf{K},\mathsf{N}}$ is a group homomorphism

$$\mathsf{GT}(\mathsf{K}) \to \mathsf{GT}(\mathsf{N}),$$

the assignments

$$\mathsf{ML}(\mathsf{N}) := \mathsf{GT}(\mathsf{N}), \qquad \mathsf{ML}(\,\mathsf{K} \le \mathsf{N}\,) := \mathcal{R}_{\mathsf{K},\mathsf{N}}$$

define a functor from the poset $\mathsf{NFI}^{isol.}$ to the category of finite groups.

Theorem. The limit of ML is isomorphic to (the gentle version of) $\widehat{\mathsf{GT}}$.

Can GT-shadows act on child's drawings? Sure!

Recall that a child's drawing of degree d can be represented by a group homomorphism $\psi : \mathsf{F}_2 \to S_d$ (with the subgroup $\psi(\mathsf{F}_2)$ being transitive). $[\psi]$ denotes the child's drawing represented by a homomorphism $\psi : \mathsf{F}_2 \to S_d$.

We say that a child's drawing $[\psi]$ is **subordinate** to $\mathsf{N} \in \mathsf{NFI}$ (or N **dominates** $[\psi]$), if

$$\mathsf{N}_{\mathsf{F}_2} \subset \ker(\psi).$$

Dessin(N) denotes the set of child's drawings subordinate to N.
We denote by Dessin the category whose objects are elements of NFI. For $\mathsf{K}, \mathsf{N} \in \mathsf{NFI}$, morphisms from K to N are all functions from Dessin(K) to Dessin(N).

The action of GTSh on child's drawings

Proposition

Let $\mathsf{K}, \mathsf{N} \in \mathsf{NFI}$ *and* $[m, f] \in \mathsf{GTSh}(\mathsf{K}, \mathsf{N})$. *Let* $\psi : \mathrm{F}_2 \to S_d$ *be a homomorphism that represents* $[\psi] \in \mathsf{Dessin}(\mathsf{N})$. *Then*

- *the homomorphism* $\tilde{\psi} : \mathrm{F}_2 \to S_d$

$$\tilde{\psi}(x) := \psi(x^{2m+1}), \qquad \tilde{\psi}(y) := \psi(f^{-1} y^{2m+1} f)$$

  represents a child's drawing subordinate to K *and*

- *the assignments* $\mathscr{A}^{sh}(\mathsf{N}) := \mathsf{N}$, $[\psi]^{[m,f]} := [\psi \circ E_{m,f}]$ *define a cofunctor* $\mathscr{A}^{sh} : \mathsf{GTSh} \to \mathsf{Dessin}$.

One can show that the action of GTSh on child's drawing is compatible with the action of $\widehat{\mathsf{GT}}$.

What can be proved about the action of GTSh?

- The action of GTSh on child's drawings is compatible with the action of S_3. Hence the passport of a child's drawing is invariant with respect to the GTSh-action.
- The GTSh-action is compatible with the partial order on the set of child's drawings. (We say that $[\tilde{H}] \le [H]$ if $\exists\, w \in \mathrm{F}_2$ such that $\tilde{H} \le w\, H\, w^{-1}$, i.e the child's drawing $[\tilde{H}]$ "covers" $[H]$.)
- If a child's drawing $D \in \mathrm{Dessin(N)}$ is Galois, then so is $D^{[m,f]}$ for every $[m, f] \in \mathrm{GT(N)}$.
- The GTSh-action commutes with the operation of taking the Galois (normal) closure of a child's drawing.
- If a child's drawing $[\psi] \in \mathrm{Dessin(N)}$ is abelian (i.e. the monodromy group $\psi(\mathrm{F}_2)$ is abelian), then the orbit $\mathrm{GT(N)}([\psi])$ is a singleton.

GT-shadows for the dihedral subposet

Let $n \in \mathbb{Z}_{\geq 3}$ and $D_n := \langle\, r, s \mid r^n, s^2, rsrs \,\rangle$ be the dihedral group of order $2n$. Let φ be the following homomorphism $\mathrm{PB}_3 \to D_n^3$

$$\varphi(x_{12}) := (r^{-1}, s, s), \qquad \varphi(x_{23}) := (rs, r, rs), \qquad \varphi(c) := \mathrm{id}$$

and

$$\mathrm{K}^{(n)} := \ker(\mathrm{PB}_3 \xrightarrow{\varphi} D_n^3).$$

One can show that $\mathrm{K}^{(n)} \trianglelefteq \mathrm{B}_3$, i.e. $\mathrm{K}^{(n)} \in \mathrm{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$.

We call

$$\{\mathrm{K}^{(n)} \;:\; n \in \mathbb{Z}_{\geq 3}\} \;\subset\; \mathrm{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$$

the **dihedral subposet** of $\mathrm{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$.

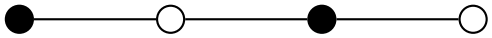
Basic facts about $\mathsf{GTSh}_{\mathrm{conn}}(\mathsf{K}^{(n)})$

- For every $n \geq 3$, $\mathsf{K}^{(n)}$ is isolated. Hence $\mathsf{GT}(\mathsf{K}^{(n)})$ is a (finite) group and the Galois child's drawing represented by $\mathsf{K}^{(n)}_{\mathsf{F}_2} \leq \mathsf{F}_2$ admits a Belyi pair defined over $\mathbb{Q}$.
- If $q, n \geq 3$ are odd and $q | n$ then the group homomorphism

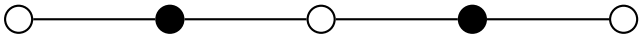
$$\mathcal{R}_{\mathsf{K}^{(n)},\mathsf{K}^{(q)}} : \mathsf{GT}(\mathsf{K}^{(n)}) \to \mathsf{GT}(\mathsf{K}^{(q)})$$

  is **onto**. *I do not think that one can find fake* GT*-shadows using the dihedral subposet.*
- For **odd** $n \geq 3$, the group $\mathsf{GT}(\mathsf{K}^{(n)})$ can be described explicitly and the limit of the corresponding functor $\mathsf{K}^{(n)} \mapsto \mathsf{GT}(\mathsf{K}^{(n)})$ can be also computed explicitly.

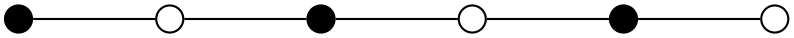
Greetings to a distant mathematical ancestor! :-)



$d = 3$



$d = 4$



$d = 5$

$\ldots$

$$T_0(z) := 1, \qquad T_1(z) := z, \qquad T_2(z) := 2z^2 - 1,$$

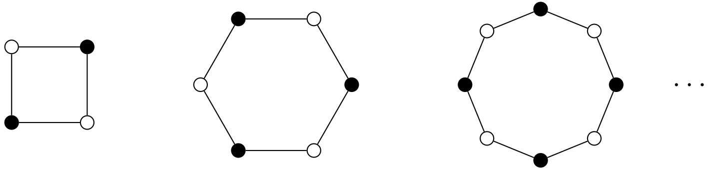
$$T_3(z) := 4z^3 - 3z, \qquad T_4(z) := 8z^4 - 8z^2 + 1,$$

$$T_5(z) := 16z^5 - 20z^3 + 5z, \qquad \ldots$$

$$T_{d+1}(z) := 2zT_d(z) - T_{d-1}(z), \qquad \varphi_d(z) := \frac{1}{2}(T_d(z) + 1).$$

More child's drawings? Sure!

GTSh-orbits of the following child's drawings are singletons:



Weeeeell... I would not try to draw child's drawings from the following family (here n is odd):

$$G := \langle\, g_{12}, g_{23} \,\rangle \leq D_n^3, \quad g_{12} := (r^{-1}, s, s), \quad g_{23} := (rs, r, rs),$$

Let C_n be the child's drawing corresponding to the action of g_{12} and g_{23} on the set G/H of left cosets, where

$$H := \langle\, (r, 1, 1)\,, (1, s, s) \,\rangle \;\leq\; G$$

A bit more about the family C_n, $n \geq 3$, odd

- The degree of C_n is $2n^2$.
- C_n is not Galois.
- C_n is subordinate to $\mathsf{K}^{(n)}$ and $\mathsf{GT}(\mathsf{K}^{(n)})(C_n) = \{C_n\}$.
- C_3 is represented by this permutation pair

$$(3,5)(4,6)(7,18)(8,17)(9,15)(10,16)(11,14)(12,13),$$

$$(1,7,15,3,9,13)(2,8,16,4,10,14)(5,11,18,6,12,17).$$

Its passport is $(2^8\,1^2,\ 6^3,\ 6^3)$. Its genus is 2.

Dear Collaborators! Thank you!

These people worked (are working) with me on GT-shadows for the gentle version of the Grothendieck-Teichmueller group:

- Jacob Guynee (currently, a PhD student at Georgia Tech)
- Jessica Radford (currently, a PhD student at the University of Oklahoma)
- Jingfeng Xia (currently, a PhD student at Temple University)
- Hm... your name can be here! :-)
- Are you an “adult mathematician”? Your name can be here too! :-)

Hierarchy of orbits

Consider a chain in the poset $\mathsf{NFI}_{\mathsf{PB}_3}(\mathsf{B}_3)$

$$\mathsf{N}^{(1)} \supset \mathsf{N}^{(2)} \supset \mathsf{N}^{(3)} \supset \dots$$

and a child's drawing $D \in \mathsf{Dessin}(\mathsf{N}^{(1)})$. It is clear that D is subordinate to $\mathsf{N}^{(i)}$ for every $\mathsf{N}^{(i)}$ in this chain.
Recall that $\mathsf{GT}(\mathsf{N})$ denotes the set of all GT-shadows with the target N. ($\mathsf{GT}(\mathsf{N})$ is finite!)

For every child's drawing D, we have the following *hierarchy of orbits*:

$$\mathsf{GT}(\mathsf{N}^{(1)})(D) \supset \mathsf{GT}(\mathsf{N}^{(2)})(D) \supset \mathsf{GT}(\mathsf{N}^{(3)})(D) \supset \dots \supset \widehat{\mathsf{GT}}(D) \supset G_{\mathbb{Q}}(D).$$

It is **very hard** to compute $G_{\mathbb{Q}}(D)$; **there are no tools in modern mathematics** to compute orbits $\widehat{\mathsf{GT}}(D)$; **it is relatively easy** to compute orbits $\mathsf{GT}(\mathsf{N})(D)$.