GT-shadows for the gentle version of the Grothendieck-Teichmueller group

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The absolute Galois group $G_{\mathbb{Q}}$ of rationals and $\widehat{\mathrm{GT}}$

$G_{\mathbb{Q}}$ is the group of (field) automorphisms of the algebraic closure $\overline{\mathbb{Q}}$ of the field $\mathbb{Q}$ of rational numbers. This group is uncountable. For every finite Galois extension $E \supset \mathbb{Q}$, any element $g \in \mathrm{Gal}(E/\mathbb{Q})$ can be extended (in infinitely many ways) to an element of $G_{\mathbb{Q}}$. The group $G_{\mathbb{Q}}$ is one of the most mysterious objects in mathematics!

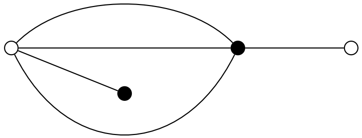
In 1990, Vladimir Drinfeld introduced yet another mysterious group $\widehat{\mathrm{GT}}$ (the Grothendieck-Teichmuellller group). $\widehat{\mathrm{GT}}$ consists pairs $(\hat{m}, \hat{f})$ in $\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$ satisfying some conditions and it receives a one-to-one homomorphism

$$G_{\mathbb{Q}} \hookrightarrow \widehat{\mathrm{GT}}.$$

Only two elements of $G_{\mathbb{Q}}$ **are known explicitly**: the identity element and the complex conjugation $a + bi \mapsto a - bi$. The corresponding images in $\widehat{\mathrm{GT}}$ are $(0, 1)$ and $(-1, 1)$.

A child's drawing of degree d is ...

An isom. class of a connected bipartite ribbon graph with d edges.



An equiv. class of a pair (g_1, g_2) of permutations in S_d for which the group $\langle g_1, g_2 \rangle$ acts transitively on $\{1, 2, \dots, d\}$.

A conjugacy class of an index d subgroup of $F_2 := \langle x, y \rangle$.

A conjugacy class of a group homomorphism $\psi : F_2 \to S_d$ (with the subgroup $\psi(F_2) \leq S_d$ being transitive.)

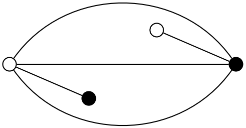
An isom. class of a (non-constant) holomorphic map $\varphi : \Sigma \to \mathbb{CP}^1$ from a compact connected Riemann surface (without boundary) that does not have branch points above every $w \in \mathbb{CP}^1 - \{0, 1, \infty\}$.

The action of $G_{\mathbb{Q}}$ on child's drawings

Given a child's drawing D, we can find a smooth projective curve X defined over $\overline{\mathbb{Q}}$ and an algebraic map $\varphi : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ that does not have branch points above every $w \in \mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}$. (X, φ) is called a **Belyi pair** representing D.

The coefficients defining the curve X and the map φ lie in some finite Galois extension E of $\mathbb{Q}$. Given $g \in \mathrm{Gal}(E/\mathbb{Q})$, the child's drawing $g(D)$ is represented by the new Belyi pair $(g(X), g(\varphi))$. We simply act by g on the coefficients defining X and φ!

The $G_{\mathbb{Q}}$-orbit of the above child's drawing has two elements. It's 'Galois conjugate' is



Basic invariants of child's drawings

- the **degree** d of a child's drawing $[\mathrm{F}_2 \xrightarrow{\psi} S_d]$;
- the conjugacy class of the subgroup $\psi(\mathrm{F}_2) \le S_d$ is call the **monodromy group** of $[\psi]$;
- for a child's drawing represented by $(g_1, g_2) \in S_d \times S_d$, its **passport** is the triple of partitions $\big(\mathrm{ct}(g_1), \mathrm{ct}(g_2), \mathrm{ct}(g_2^{-1} g_1^{-1})\big)$ of d, where $\mathrm{ct}(h)$ denotes the cycle type of a permutation $h \in S_d$;
- the **cartographic group** and more...

Let σ_1, σ_2 be the standard generators of Artin's braid group B_3. The formulas (here, $g_3 := g_2^{-1} g_1^{-1}$)

$$\sigma_1(g_1, g_2, g_3) := (g_2, g_2^{-1} g_1 g_2, g_3), \quad \sigma_2(g_1, g_2, g_3) := (g_1, g_3, g_3^{-1} g_2 g_3)$$

define an action of B_3 on child's drawings. Since the pure braid group PB_3 acts trivially, we actually get an action of S_3 on child's drawings. The action of $G_{\mathbb{Q}}$ commutes with this action of S_3.

A bit about (the gentle version of) $\widehat{GT}$

For $(\hat{m}, \hat{f}) \in \widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$, the formulas

$$E_{\hat{m},\hat{f}}(x) := x^{2\hat{m}+1}, \qquad E_{\hat{m},\hat{f}}(y) := \hat{f}^{-1} y^{2\hat{m}+1} \hat{f}$$

define a continuous endomorphism $E_{\hat{m},\hat{f}}$ of $\widehat{\mathrm{F}}_2$.

$\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$ is a monoid with the binary operation

$$(\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2) := \big(2\hat{m}_1 \hat{m}_2 + \hat{m}_1 + \hat{m}_2,\ \hat{f}_1 E_{\hat{m}_1,\hat{f}_1}(\hat{f}_2) \big)$$

and the identity element $(0, 1)$.

Let σ_1, σ_2 be the standard generators of Artin's braid group B_3,

$$c := (\sigma_1 \sigma_2 \sigma_1)^2 \quad \text{and} \quad x_{12} := \sigma_1^2, \qquad x_{23} := \sigma_2^2.$$

The gentle version $\widehat{\mathrm{GT}}_{gen}$ of $\widehat{\mathrm{GT}}$

Let $\widehat{\mathrm{GT}}_{mon}$ be the submonoid of $\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$ that consists of pairs $(\hat{m}, \hat{f})$ satisfying the **hexagon relations**:

$$\sigma_1^{2\hat{m}+1} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} = \hat{f}^{-1} \sigma_1 \sigma_2 \, x_{12}^{-\hat{m}} c^{\hat{m}} \,,$$

$$\hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \, \sigma_1^{2\hat{m}+1} = \sigma_2 \sigma_1 x_{23}^{-\hat{m}} c^{\hat{m}} \, \hat{f}$$

and $\hat{f} \in [\widehat{\mathrm{F}}_2, \widehat{\mathrm{F}}_2]^{top.\, cl.}$.

$\widehat{\mathrm{GT}}_{gen}$ is the group of invertible elements of the monoid $\widehat{\mathrm{GT}}_{mon}$.

The formula

$$\chi_{vir}(\hat{m}, \hat{f}) := 2\hat{m} + 1$$

defines a (continuous) group homomorphism $\widehat{\mathrm{GT}}_{gen} \to \widehat{\mathbb{Z}}^{\times}$.
χ_{vir} is called the **virtual cyclotomic character.**

In the remaining slides, $\widehat{\mathrm{GT}}$ denotes $\widehat{\mathrm{GT}}_{gen} = \widehat{\mathrm{GT}}_0$. *D. Harbater, L. Schneps, (2000).*

The action of $\widehat{\mathrm{GT}}$ on child's drawings

Let $(\hat{m}, \hat{f})$ be an element of $\widehat{\mathrm{GT}}$ and D be a child's drawing. It is convenient to represent D by a group homomorphism

$$\varphi : \mathrm{F}_2 \to S_d\,,$$

where $\varphi(\mathrm{F}_2)$ is transitive. (D corresponds to the conjugacy class of the stabilizer of 1.)

φ extends, by continuity, to a (continuous) group homomorphism $\hat{\varphi} : \widehat{\mathrm{F}}_2 \to S_d$. The child's drawing $D^{(\hat{m},\hat{f})}$ corresponds to the group homomorphism

$$\hat{\varphi} \circ \hat{T}\big|_{\mathrm{F}_2} : \mathrm{F}_2 \to S_d\,,$$

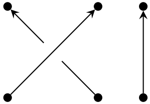
where

$$\hat{T}(x) := x^{2\hat{m}+1} \quad \text{and} \quad \hat{T}(y) := \hat{f}^{-1}\, y^{2\hat{m}+1} \hat{f}.$$

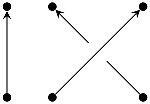
See Y. Ihara's paper "On the embedding of Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$ into $\widehat{\mathrm{GT}}$".

The Artin braid group B_3 and PB_3

B_3 (resp. PB_3) denotes the Artin braid group (resp. the pure braid group) on 3 strands. σ_1, σ_2 are the standard generators of B_3



σ_1



σ_2

We set $\Delta := \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.
PB_3 is generated by

$$x_{12} := \sigma_1^2, \qquad x_{23} := \sigma_2^2, \qquad c := \Delta^2 .$$

It is known that $\mathcal{Z}(B_3) = \mathcal{Z}(PB_3) = \langle\, c \,\rangle \cong \mathbb{Z}$, the subgroup $\langle\, x_{12}, x_{23} \,\rangle$ is isomorphic to F_2. In fact, $PB_3 \cong F_2 \times \langle\, c \,\rangle$.

A bit more about F_2, PB_3 and B_3

We tacitly identify F_2 with the subgroup $\langle x_{12}, x_{23} \rangle \le PB_3$. We also set

$$x := x_{12} \qquad \text{and} \qquad y := x_{23}.$$

We denote by θ and τ the following automorphisms of F_2:

$$\theta(x) := y, \quad \theta(y) := x, \qquad \tau(x) := y, \quad \tau(y) := y^{-1}x^{-1}.$$

We set

$$NFI_{PB_3}(B_3) := \{ N \trianglelefteq B_3 \mid N \le PB_3, \ \ |B_3 : N| < \infty \}$$

and abbreviate $NFI := NFI_{PB_3}(B_3)$.

Preparation

For $\mathsf{N} \in \mathsf{NFI}$, we set

$$\mathsf{N}_{\mathsf{F}_2} := \mathsf{F}_2 \cap \mathsf{N}, \quad N_{\mathrm{ord}} := \mathrm{lcm}\big(\mathrm{ord}(x_{12}\mathsf{N}), \mathrm{ord}(x_{23}\mathsf{N}), \mathrm{ord}(c\mathsf{N})\big).$$

We say that $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$ satisfies the **hexagon relations** modulo N if

$$\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f \,\mathsf{N} = f^{-1} \sigma_1 \sigma_2 x_{12}^{-m} c^m \,\mathsf{N}$$
$$f^{-1} \sigma_2^{2m+1} f \, \sigma_1^{2m+1} \,\mathsf{N} = \sigma_2 \sigma_1 x_{23}^{-m} c^m f \,\mathsf{N}.$$

Proposition

If $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$ satisfies the hexagon relations modulo N*, then the formulas*

$$T_{m,f}(\sigma_1) := \sigma_1^{2m+1} \mathsf{N}, \qquad T_{m,f}(\sigma_2) := f^{-1} \sigma_2^{2m+1} f \,\mathsf{N}$$

define a group homomorphism $T_{m,f} : \mathsf{B}_3 \to \mathsf{B}_3/\mathsf{N}$.

Let us restrict $T_{m,f}$ to PB_3 and to F_2

Restricting the above homomorphism $T_{m,f} : \mathrm{B}_3 \to \mathrm{B}_3/\mathrm{N}$ to PB_3 and to F_2, we get

$$T^{\mathrm{PB}_3}_{m,f} : \mathrm{PB}_3 \to \mathrm{PB}_3/\mathrm{N}, \qquad T^{\mathrm{F}_2}_{m,f} : \mathrm{F}_2 \to \mathrm{F}_2/\mathrm{N}_{\mathrm{F}_2},$$

$$T^{\mathrm{PB}_3}_{m,f}(x_{12}) = x_{12}^{2m+1}\,\mathrm{N}, \qquad T^{\mathrm{PB}_3}_{m,f}(x_{23}) = f^{-1} x_{23}^{2m+1} f\,\mathrm{N},$$

$$T^{\mathrm{PB}_3}_{m,f}(c) = c^{2m+1}\,\mathrm{N},$$

$$T^{\mathrm{F}_2}_{m,f}(x) = x^{2m+1}\,\mathrm{N}_{\mathrm{F}_2}, \qquad T^{\mathrm{F}_2}_{m,f}(y) = f^{-1} y^{2m+1} f\,\mathrm{N}_{\mathrm{F}_2},$$

$T_{m,f}$ is onto $\iff$ $T^{\mathrm{PB}_3}_{m,f}$ is onto $\iff$ $T^{\mathrm{F}_2}_{m,f}$ is onto.

$\ker(T_{m,f}) = \ker(T^{\mathrm{PB}_3}_{m,f})$. Hence $\ker(T_{m,f}) \in \mathrm{NFI}$.

A GT-shadow is . . .

Definition

Let $\mathsf{N} \in \mathsf{NFI}$. *A* GT-***shadow with the target*** N *is a pair*

$$[m, f] := (m + N_{\mathrm{ord}}\mathbb{Z}, f\mathsf{N}_{\mathsf{F}_2}) \in \mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}$$

satisfying the hexagon relations (modulo N*) and such that*

- $2m+1$ *represents a unit in the ring* $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z}$*,*
- $f\mathsf{N}_{\mathsf{F}_2} \in [\mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}, \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}]$*, or equivalently* $\exists\ w \in [\mathsf{F}_2, \mathsf{F}_2]$ *such that* $f\mathsf{N}_{\mathsf{F}_2} = w\mathsf{N}_{\mathsf{F}_2}$*, and*
- *the homomorphism* $T_{m,f} : \mathsf{B}_3 \to \mathsf{B}_3/\mathsf{N}$ *is onto (* $\iff$ $T^{\mathsf{PB}_3}_{m,f}$ *is onto* $\iff$ $T^{\mathsf{F}_2}_{m,f}$ *is onto).*

GT(N) is the set of GT-shadows with the target N.

The groupoid GTSh

Guess what?!.... GT-shadows form a groupoid GTSh.

$$Ob(\mathsf{GTSh}) := \mathsf{NFI}_{\mathsf{PB}_3}(\mathsf{B}_3); \quad \text{for} \quad \mathsf{K}, \mathsf{N} \in \mathsf{NFI},$$

$$\mathsf{GTSh}(\mathsf{K}, \mathsf{N}) := \Big\{ [m, f] \in \mathsf{GT}(\mathsf{N}) \mid \ker(T_{m,f}) = \mathsf{K} \Big\}.$$

Let $\mathsf{N}^{(1)}, \mathsf{N}^{(2)}, \mathsf{N}^{(3)} \in \mathsf{NFI}$ and

$$\mathsf{N}^{(3)} \xrightarrow{[m_2, f_2]} \mathsf{N}^{(2)} \xrightarrow{[m_1, f_1]} \mathsf{N}^{(1)}.$$

The composition of morphisms is defined by the formula:

$$[m_1, f_1] \circ [m_2, f_2] := \big[2m_1 m_2 + m_1 + m_2 \,,\; f_1 E_{m_1, f_1}(f_2) \big]$$

$\forall\, \mathsf{N} \in \mathsf{NFI}$, $[0, 1_{\mathsf{F}_2}]$ is the identity morphism in $\mathsf{GTSh}(\mathsf{N}, \mathsf{N})$.

A comment

For $(m, f) \in \mathbb{Z} \times \mathrm{F}_2$, the formulas

$$E_{m,f}(x) := x^{2m+1}, \qquad E_{m,f}(y) := f^{-1} y^{2m+1} f$$

define an endomorphism of F_2.

Moreover, for all $(m_1, f_1), (m_2, f_2) \in \mathbb{Z} \times \mathrm{F}_2$,

$$E_{m_1,f_1} \circ E_{m_2,f_2} = E_{m,f},$$

where $m := 2m_1 m_2 + m_1 + m_2$ and $f := f_1 E_{m_1,f_1}(f_2)$.

On can show that the set $\mathbb{Z} \times \mathrm{F}_2$ is a monoid with respect to the binary operation

$$(m_1, f_1) \bullet (m_2, f_2) := \big(2m_1 m_2 + m_1 + m_2\,,\ f_1 E_{m_1,f_1}(f_2)\big)$$

with $(0, 1_{\mathrm{F}_2})$ being the identity element.

Basic facts about GTSh

- GTSh has infinitely many objects. ($\mathsf{NFI}_{\mathsf{PB}_3}(\mathsf{B}_3)$ is infinite because PB_3 is residually finite.)
- GTSh is highly disconnected. However, for every $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_3}(\mathsf{B}_3)$, the connected component $\mathsf{GTSh}_{\mathsf{conn}}(\mathsf{N})$ of N is a finite groupoid.
- If $\mathsf{GTSh}_{\mathsf{conn}}(\mathsf{N})$ has only one object (i.e. GT(N) is group), then we say that N is **isolated**.
- For every $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_3}(\mathsf{B}_3)$, the object

$$\mathsf{N}^{\diamond} := \bigcap_{\mathsf{K} \in Ob(\mathsf{GTSh}_{\mathsf{conn}}(\mathsf{N}))} \mathsf{K}$$

  is isolated. In particular, the subposet $\mathsf{NFI}^{isol.} \subset \mathsf{NFI}$ of isolated objects is cofinal.

“Reduction modulo” H

Let $\mathsf{N}, \mathsf{H} \in \mathsf{NFI}$ with $\mathsf{N} \le \mathsf{H}$. We have $\mathsf{N}_{\mathsf{F}_2} \le \mathsf{H}_{\mathsf{F}_2}$ and $H_{\mathrm{ord}} | N_{\mathrm{ord}}$.

If a pair $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$ represents a GT-shadow with the target N, then **the same pair** also represents a GT-shadow with the target H.

Hence we have a natural map

$$\mathcal{R}_{\mathsf{N},\mathsf{H}} : \mathsf{GT}(\mathsf{N}) \to \mathsf{GT}(\mathsf{H})$$

If N, H are isolated (i.e. $\mathsf{GT}(\mathsf{N}), \mathsf{GT}(\mathsf{H})$ are groups) then $\mathcal{R}_{\mathsf{N},\mathsf{H}}$ is a group homomorphism.

$\widehat{\mathsf{GT}}$ versus GTSh

For every $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}$ and $\mathsf{N} \in \mathsf{NFI}$ the pair

$$\mathsf{PR}_{\mathsf{N}}(\hat{m}, \hat{f}) := \big(\mathcal{P}_{N_{\mathrm{ord}}}(\,\hat{m}\,)\,,\, \mathcal{P}_{\mathsf{N}_{\mathsf{F}_2}}(\,\hat{f}\,) \big) \;\in\; \mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}$$

is a GT-shadow with the target N. (For $\mathsf{K} \in \mathsf{NFI}(G)$, $\mathcal{P}_{\mathsf{K}}$ denotes the standard continuous homomorphism $\widehat{G} \to G/\mathsf{K}$.) $\mathsf{PR}_{\mathsf{N}}(\hat{m}, \hat{f})$ is an **approximation** of the element $(\hat{m}, \hat{f})$.

A GT-shadow $[m, f] \in \mathsf{GT}(\mathsf{N})$ is called **genuine** if $\exists\, (\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}$ such that $\mathsf{PR}_{\mathsf{N}}(\hat{m}, \hat{f}) = [m, f]$. Otherwise, it is called **fake.**

A GT-shadow $[m, f] \in \mathsf{GT}(\mathsf{N})$ **survives into** $\mathsf{K} \in \mathsf{NFI}$ (with $\mathsf{K} \le \mathsf{N}$) if $[m, f] \in \mathcal{R}_{\mathsf{K},\mathsf{N}}\big(\mathsf{GT}(\mathsf{K})\big)$.

Proposition. A GT-shadow $[m, f] \in \mathsf{GT}(\mathsf{N})$ is genuine $\iff$ $[m, f]$ survives into K for every $\mathsf{K} \in \mathsf{NFI}_{\mathsf{PB}_3}(\mathsf{B}_3)$ such that $\mathsf{K} \le \mathsf{N}$.

The Main Line functor

Let $\mathsf{K}, \mathsf{N} \in \mathsf{NFI}$ be isolated objects of the groupoid GTSh and $\mathsf{K} \le \mathsf{N}$.

Since $\mathcal{R}_{\mathsf{K},\mathsf{N}}$ is a group homomorphism

$$\mathsf{GT}(\mathsf{K}) \to \mathsf{GT}(\mathsf{N}),$$

the assignments

$$\mathsf{ML}(\mathsf{N}) := \mathsf{GT}(\mathsf{N}), \qquad \mathsf{ML}(\,\mathsf{K} \le \mathsf{N}\,) := \mathcal{R}_{\mathsf{K},\mathsf{N}}$$

define a functor from the poset $\mathsf{NFI}^{isol.}$ to the category of finite groups.

Theorem. The limit of ML is isomorphic to (the gentle version of) $\widehat{\mathsf{GT}}$.

Can GT-shadows act on child's drawings? Sure!

Recall that a child's drawing of degree d can be represented by a group homomorphism $\psi : \mathsf{F}_2 \to S_d$ (with the subgroup $\psi(\mathsf{F}_2)$ being transitive). $[\psi]$ denotes the child's drawing represented by a homomorphism $\psi : \mathsf{F}_2 \to S_d$.

We say that a child's drawing $[\psi]$ is **subordinate** to $\mathsf{N} \in \mathsf{NFI}$ (or N **dominates** $[\psi]$), if

$$\mathsf{N}_{\mathsf{F}_2} \subset \ker(\psi).$$

Dessin(N) denotes the set of child's drawings subordinate to N.
We denote by Dessin the category whose objects are elements of NFI.
For $\mathsf{K}, \mathsf{N} \in \mathsf{NFI}$, morphisms from K to N are all functions from Dessin(K) to Dessin(N).

The action of GTSh on child's drawings

Proposition

Let $\mathsf{K}, \mathsf{N} \in \mathsf{NFI}$ *and* $[m, f] \in \mathsf{GTSh}(\mathsf{K}, \mathsf{N})$*. Let* $\psi : \mathrm{F}_2 \to S_d$ *be a homomorphism that represents* $[\psi] \in \mathsf{Dessin}(\mathsf{N})$*. Then*

- *the homomorphism* $\tilde{\psi} : \mathrm{F}_2 \to S_d$

$$\tilde{\psi}(x) := \psi(x^{2m+1}), \qquad \tilde{\psi}(y) := \psi(f^{-1} y^{2m+1} f)$$

  represents a child's drawing subordinate to K *and*

- *the assignments* $\mathscr{A}^\sharp(\mathsf{N}) := \mathsf{N}$*,* $[\psi]^{[m,f]} := [\psi \circ E_{m,f}]$ *define a cofunctor* $\mathscr{A}^\sharp : \mathsf{GTSh} \to \mathsf{Dessin}$*.*

What can be proved about the action of GTSh?

- The action of GTSh on child's drawings is compatible with the action of S_3. Hence the passport of a child's drawing is invariant with respect to the GTSh-action.
- The GTSh-action is compatible with the partial order on the set of child's drawings. (We say that $[\tilde{H}] \le [H]$ if $\exists\ w \in \mathrm{F}_2$ such that $\tilde{H} \le w\, H\, w^{-1}$, i.e the child's drawing $[\tilde{H}]$ "covers" $[H]$.)
- If a child's drawing $D \in \mathrm{Dessin(N)}$ is Galois, then so is $D^{[m,f]}$ for every $[m, f] \in \mathrm{GT(N)}$.
- The GTSh-action commutes with the operation of taking the Galois (normal) closure of a child's drawing.
- If a child's drawing $[\psi] \in \mathrm{Dessin(N)}$ is abelian (i.e. the monodromy group $\psi(\mathrm{F}_2)$ is abelian), then the orbit $\mathrm{GT(N)}([\psi])$ is a singleton.

GT-shadows for the dihedral subposet

Let $n \in \mathbb{Z}_{\geq 3}$ and $D_n := \langle\, r, s \mid r^n, s^2, rsrs \,\rangle$ be the dihedral group of order $2n$. Let φ be the following homomorphism $\mathrm{PB}_3 \to D_n^3$

$$\varphi(x_{12}) := (r^{-1}, s, s), \qquad \varphi(x_{23}) := (rs, r, rs), \qquad \varphi(c) := \mathrm{id}$$

and

$$\mathsf{K}^{(n)} := \ker(\mathrm{PB}_3 \stackrel{\varphi}{\longrightarrow} D_n^3).$$

One can show that $\mathsf{K}^{(n)} \trianglelefteq \mathrm{B}_3$, i.e. $\mathsf{K}^{(n)} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$.

We call

$$\big\{\mathsf{K}^{(n)} \;:\; n \in \mathbb{Z}_{\geq 3}\big\} \;\subset\; \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$$

the **dihedral subposet** of $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$.

Basic facts about $\mathsf{GTSh}_{\mathrm{conn}}(\mathsf{K}^{(n)})$

- For every $n \geq 3$, $\mathsf{K}^{(n)}$ is isolated. Hence $\mathsf{GT}(\mathsf{K}^{(n)})$ is a (finite) group and the Galois child's drawing represented by $\mathsf{K}^{(n)}_{\mathsf{F}_2} \leq \mathsf{F}_2$ admits a Belyi pair defined over $\mathbb{Q}$.

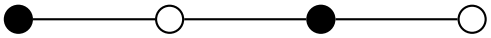
- If $q, n \geq 3$ are odd and $q | n$ then the group homomorphism

$$\mathcal{R}_{\mathsf{K}^{(n)},\mathsf{K}^{(q)}} : \mathsf{GT}(\mathsf{K}^{(n)}) \to \mathsf{GT}(\mathsf{K}^{(q)})$$

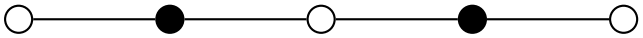
  is **onto**. *I do not think that one can find fake* GT*-shadows using the dihedral subposet.*

- For **odd** $n \geq 3$, the group $\mathsf{GT}(\mathsf{K}^{(n)})$ can be described explicitly and the limit of the corresponding functor $\mathsf{K}^{(n)} \mapsto \mathsf{GT}(\mathsf{K}^{(n)})$ can be also computed explicitly.

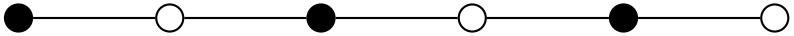
Greetings to a distant mathematical ancestor! :-)



$d = 3$



$d = 4$



$d = 5$

$\ldots$

$$T_0(z) := 1, \qquad T_1(z) := z, \qquad T_2(z) := 2z^2 - 1,$$

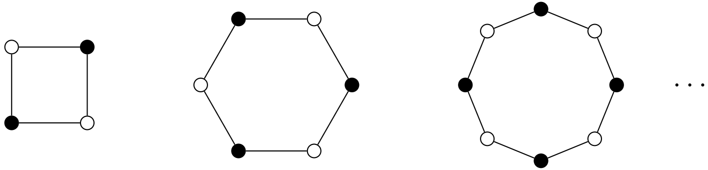
$$T_3(z) := 4z^3 - 3z, \qquad T_4(z) := 8z^4 - 8z^2 + 1,$$

$$T_5(z) := 16z^5 - 20z^3 + 5z, \qquad \ldots$$

$$T_{d+1}(z) := 2zT_d(z) - T_{d-1}(z), \qquad \varphi_d(z) := \frac{1}{2}(T_d(z) + 1).$$

More child's drawings? Sure!

GTSh-orbits of the following child's drawings are singletons:



Weeeeell... I would not try to draw child's drawings from the following family (here n is odd):

$$G := \langle\, g_{12}, g_{23} \,\rangle \le D_n^3, \quad g_{12} := (r^{-1}, s, s), \quad g_{23} := (rs, r, rs),$$

Let C_n be the child's drawing corresponding to the action of g_{12} and g_{23} on the set G/H of left cosets, where

$$H := \langle\, (r, 1, 1)\,, (1, s, s) \,\rangle \;\le\; G$$

A bit more about the family C_n, $n \geq 3$, odd

- The degree of C_n is $2n^2$.
- C_n is not Galois.
- C_n is subordinate to $\mathsf{K}^{(n)}$ and $\mathsf{GT}(\mathsf{K}^{(n)})(C_n) = \{C_n\}$.
- C_3 is represented by this permutation pair

$$(3,5)(4,6)(7,18)(8,17)(9,15)(10,16)(11,14)(12,13),$$

$$(1,7,15,3,9,13)(2,8,16,4,10,14)(5,11,18,6,12,17).$$

Its passport is $(2^8\,1^2,\ 6^3,\ 6^3)$. Its genus is 2.

Dear Collaborators! Thank you!

These people worked (are working) with me on GT-shadows for the gentle version of the Grothendieck-Teichmueller group:

- Jacob Guynee (currently, a PhD student at Georgia Tech)
- Jessica Radford (currently, a PhD student at the University of Oklahoma)
- Jingfeng Xia (currently, a PhD student at Temple University)
- Hm... your name can be here! :-)
- Are you an “adult mathematician”? Your name can be here too! :-)