GT-shadows and their action on Grothendieck's child's drawings

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Seminar talk at the University of Angers, May 27, 2022

Loosely based on papers "What are GT*-shadows?" (joint with Khanh Q. Le and Aidan Lorenz) and "The action of* GT*-shadows on child's drawings."*

The absolute Galois group $G_{\mathbb{Q}}$ of rationals and $\widehat{\mathsf{GT}}$

$G_{\mathbb{Q}}$ is the group of (field) automorphisms of the algebraic closure $\overline{\mathbb{Q}}$ of the field $\mathbb{Q}$ of rational numbers. This group is uncountable. For every finite Galois extension $E \supset \mathbb{Q}$, any element $g \in \mathrm{Gal}(E/\mathbb{Q})$ can be extended (in infinitely many ways) to an element of $G_{\mathbb{Q}}$. The group $G_{\mathbb{Q}}$ is one of the most mysterious objects in mathematics!

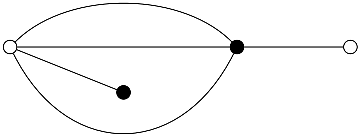
In 1990, Vladimir Drinfeld introduced yet another mysterious group $\widehat{\mathsf{GT}}$ (the Grothendieck-Teichmuellller group). $\widehat{\mathsf{GT}}$ consists pairs $(\hat{m}, \hat{f})$ in $\widehat{\mathbb{Z}} \times \widehat{\mathsf{F}}_2$ satisfying some conditions and it receives a one-to-one homomorphism

$$G_{\mathbb{Q}} \hookrightarrow \widehat{\mathsf{GT}}.$$

Only two elements of $G_{\mathbb{Q}}$ **are known explicitly**: the identity element and the complex conjugation $a + bi \mapsto a - bi$. The corresponding images in $\widehat{\mathsf{GT}}$ are $(0, 1)$ and $(-1, 1)$.

A child's drawing of degree d is ...

An isom. class of a connected bipartite ribbon graph with d edges.



An equiv. class of a pair (g_1, g_2) of permutations in S_d for which the group $\langle\, g_1, g_2 \,\rangle$ acts transitively on $\{1, 2, \dots, d\}$.

A conjugacy class of an index d subgroup of $\mathrm{F}_2 := \langle\, x, y \,\rangle$.

An isom. class of a degree d connected covering of $\mathbb{CP}^1 - \{0, 1, \infty\}$.

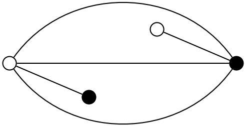
An isom. class of a (non-constant) holomorphic map $f : \Sigma \to \mathbb{CP}^1$ from a compact connected Riemann surface (without boundary) that does not have branch points above every $w \in \mathbb{CP}^1 - \{0, 1, \infty\}$.

The action of $G_{\mathbb{Q}}$ on child's drawings

Given a child's drawing D, we can find a smooth projective curve X defined over $\overline{\mathbb{Q}}$ and an algebraic map $f : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ that does not have branch points above every $w \in \mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}$. (X, f) is called a **Belyi pair** corresponding to D.

The coefficients defining the curve X and the map f lie in some finite Galois extension E of $\mathbb{Q}$. Given any $g \in \mathrm{Gal}(E/\mathbb{Q})$, the child's drawing $g(D)$ is the one corresponding to the new Belyi pair $(g(X), g(f))$. We simply act by g on the coefficients defining X and f!

The $G_{\mathbb{Q}}$-orbit of the above child's drawing has two elements. It's 'Galois conjugate' is



The action of $\widehat{\mathrm{GT}}$ on child's drawings

Let $(\hat{m}, \hat{f})$ be an element of $\widehat{\mathrm{GT}}$ and D be a child's drawing. It is convenient to represent D by a group homomorphism

$$\varphi : \mathrm{F}_2 \to S_d \,,$$

where $\varphi(\mathrm{F}_2)$ is transitive. (D corresponds to the conjugacy class of the stabilizer of 1.)

φ extends, by continuity, to a (continuous) group homomorphism $\hat{\varphi} : \widehat{\mathrm{F}}_2 \to S_d$. The child's drawing $D^{(\hat{m},\hat{f})}$ corresponds to the group homomorphism

$$\hat{\varphi} \circ \hat{T}\big|_{\mathrm{F}_2} : \mathrm{F}_2 \to S_d \,,$$

where

$$\hat{T}(x) := x^{2\hat{m}+1} \quad \text{and} \quad \hat{T}(y) := \hat{f}^{-1}\, y^{2\hat{m}+1} \hat{f}.$$

See Y. Ihara's paper "On the embedding of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into $\widehat{\mathrm{GT}}$".

The Artin braid group B_n and PB_n

The Artin braid group B_n is generated by element $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ subject to the following relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad |i - j| \geq 2,$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for} \quad 1 \leq i \leq n-2.$$

The formula $\rho(\sigma_i) := (i, i+1)$ defines an onto homomorphism ρ from B_n to the symmetric group S_n on n letters.

$$\mathrm{PB}_n := \ker \left(\mathrm{B}_n \stackrel{\rho}{\longrightarrow} S_n \right)$$

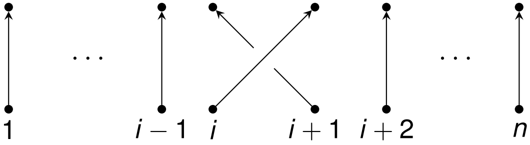
One can show that PB_n is generated by the elements

$$x_{ij} := \sigma_{j-1} \dots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \dots \sigma_{j-1}^{-1},$$

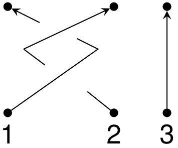
where $1 \leq i < j \leq n$.

Pictures for elements of B_n and PB_n

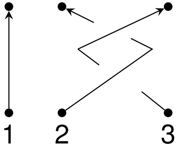
It is known that $PB_n \cong \pi_1\big(\operatorname{Conf}(n, \mathbb{C})\big)$ and $B_n \cong \pi_1\big(\operatorname{Conf}(n, \mathbb{C})/S_n\big)$.



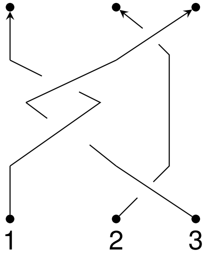
σ_i



x_{12}



x_{23}



x_{13}

$\mathrm{PB}_2 \cong \mathbb{Z}$, $\mathrm{PB}_3 \cong \mathrm{F}_2 \times \mathbb{Z}$

It is known that PB_2 is an infinite cyclic group generated by x_{12}, there are no relations on x_{12} and x_{23} in PB_3, the element

$$c := x_{23} x_{12} x_{13}$$

has infinite order and it generates the center of PB_3 (and the center of B_3).

We tacitly identify the free group F_2 on two generators with the subgroup $\langle\, x_{12}, x_{23} \,\rangle \le \mathrm{PB}_3$. We have

$$\mathrm{PB}_3 \cong \mathrm{F}_2 \times \langle\, c \,\rangle \cong \mathrm{F}_2 \times \mathbb{Z}.$$

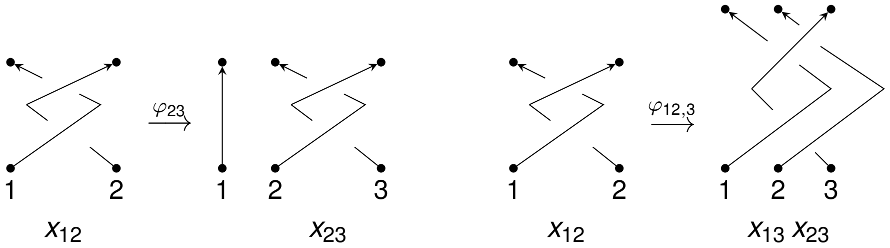
By adding a strand or doubling a stand, we get...

the collection of homomorphisms

$$\varphi_{23},\ \varphi_{12,3},\ \varphi_{1,23},\ \varphi_{12} : \mathrm{PB}_2 \to \mathrm{PB}_3,$$

$$\varphi_{234},\ \varphi_{12,3,4},\ \varphi_{1,23,4},\ \varphi_{1,2,34},\ \varphi_{123} : \mathrm{PB}_3 \to \mathrm{PB}_4.$$

For example,



The poset $\mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$ and the "cascade"

For a group G and a normal subgroup $H \trianglelefteq G$ of finite index,

$$\mathsf{NFI}_H(G)$$

denotes the poset of finite index normal subgroups $\mathsf{N} \trianglelefteq G$ such that $\mathsf{N} \le H$.

For every $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$, we form this cascade

$$\mathsf{N}_{\mathsf{PB}_3}, \quad \mathsf{N}_{\mathsf{F}_2}, \quad N_{\mathrm{ord}},$$

where $\mathsf{N}_{\mathsf{PB}_3} \in \mathsf{NFI}_{\mathsf{PB}_3}(\mathsf{B}_3)$, $\mathsf{N}_{\mathsf{F}_2} \in \mathsf{NFI}(\mathsf{F}_2)$ and N_{ord} is a positive integer.

Well... what are $\mathsf{N}_{\mathsf{PB}_3}$, $\mathsf{N}_{\mathsf{F}_2}$, N_{ord}?

For $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$, we set

$$\mathsf{N}_{\mathsf{PB}_3} := \varphi^{-1}_{123}(\mathsf{N}) \cap \varphi^{-1}_{12,3,4}(\mathsf{N}) \cap \varphi^{-1}_{1,23,4}(\mathsf{N}) \cap \varphi^{-1}_{1,2,34}(\mathsf{N}) \cap \varphi^{-1}_{234}(\mathsf{N}),$$

and

$$\mathsf{N}_{\mathsf{PB}_2} := \varphi^{-1}_{12}(\mathsf{N}_{\mathsf{PB}_3}) \cap \varphi^{-1}_{12,3}(\mathsf{N}_{\mathsf{PB}_3}) \cap \varphi^{-1}_{1,23}(\mathsf{N}_{\mathsf{PB}_3}) \cap \varphi^{-1}_{23}(\mathsf{N}_{\mathsf{PB}_3}).$$

It is not hard to show that $\mathsf{N}_{\mathsf{PB}_3} \in \mathsf{NFI}_{\mathsf{PB}_3}(\mathsf{B}_3)$ and $\mathsf{N}_{\mathsf{PB}_2} \in \mathsf{NFI}_{\mathsf{PB}_2}(\mathsf{B}_2)$.

Moreover, $\mathsf{N}_{\mathsf{PB}_2}$ is uniquely determined by its index $|\mathsf{PB}_2 : \mathsf{N}_{\mathsf{PB}_2}|$. So we set $N_{\mathrm{ord}} := |\mathsf{PB}_2 : \mathsf{N}_{\mathsf{PB}_2}|$.

N_{ord} is the *least common multiple* of the orders of $x_{12}\mathsf{N}_{\mathsf{PB}_3}$, $x_{23}\mathsf{N}_{\mathsf{PB}_3}$ and $c\mathsf{N}_{\mathsf{PB}_3}$ in $\mathsf{PB}_3/\mathsf{N}_{\mathsf{PB}_3}$.

Finally, we set

$$\mathsf{N}_{\mathsf{F}_2} := \mathsf{N}_{\mathsf{PB}_3} \cap \langle\, x_{12}, x_{23} \,\rangle.$$

So... what are GT-shadows???

For $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$, we consider pairs

$$\big(m + N_{\mathrm{ord}}\mathbb{Z},\ f\mathsf{N}_{\mathsf{F}_2}\big) \ \in\ \mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}$$

satisfying the hexagon relations

$$\sigma_1 x_{12}^m\, f^{-1}\sigma_2 x_{23}^m f\, \mathsf{N}_{\mathsf{PB}_3} \ =\ f^{-1}\sigma_1\sigma_2(x_{13}x_{23})^m\, \mathsf{N}_{\mathsf{PB}_3}\,,$$

$$f^{-1}\sigma_2 x_{23}^m f\, \sigma_1 x_{12}^m\, \mathsf{N}_{\mathsf{PB}_3} \ =\ \sigma_2\sigma_1(x_{12}x_{13})^m\, f\, \mathsf{N}_{\mathsf{PB}_3}\,,$$

the pentagon relation

$$\varphi_{234}(f)\,\varphi_{1,23,4}(f)\,\varphi_{123}(f)\,\mathsf{N} \ =\ \varphi_{1,2,34}(f)\varphi_{12,3,4}(f)\,\mathsf{N}.$$

For an integer m and $f \in \mathsf{F}_2$, $[m, f]$ denotes the pair $\big(m + N_{\mathrm{ord}}\mathbb{Z},\ f\mathsf{N}_{\mathsf{F}_2}\big)$ in $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}$.

A possible motivation for the hexagon relations

The hexagon relations imply that the formulas

$$T_{m,f}(\sigma_1) := \sigma_1^{2m+1}\, \mathsf{N}_{\mathrm{PB}_3}, \qquad T_{m,f}(\sigma_2) := f^{-1}\sigma_2^{2m+1} f\, \mathsf{N}_{\mathrm{PB}_3}$$

define a homomorphism $T_{m,f} : \mathrm{B}_3 \to \mathrm{B}_3/\mathsf{N}_{\mathrm{PB}_3}$.

Restricting $T_{m,f}$ to PB_3, we get a homomorphism $\mathrm{PB}_3 \to \mathrm{PB}_3/\mathsf{N}_{\mathrm{PB}_3}$. Moreover, we have

$$T_{m,f}(x_{12}) := x_{12}^{2m+1}\, \mathsf{N}_{\mathrm{PB}_3}, \quad T_{m,f}(x_{23}) := f^{-1} x_{23}^{2m+1} f\, \mathsf{N}_{\mathrm{PB}_3},$$

$$T_{m,f}(c) := c^{2m+1}\, \mathsf{N}_{\mathrm{PB}_3}.$$

Similarly, we have the homomorphism $T_{m,f} : \mathrm{B}_4 \to \mathrm{B}_4/\mathsf{N}$ and its restriction to PB_4 gives us the homomorphism $\mathrm{PB}_4 \to \mathrm{PB}_4/\mathsf{N}$.

Well... a GT-shadows is...

Definition

A GT*-shadow with the target* N *is a pair*

$$(m + N_{\mathrm{ord}}\mathbb{Z},\ f\mathrm{N}_{\mathrm{F}_2}) \in \mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathrm{F}_2/\mathrm{N}_{\mathrm{F}_2}$$

that satisfies

- *the hexagon relations (modulo* $\mathrm{N}_{\mathrm{PB}_3}$*),*
- *the pentagon relation (modulo* N*),*
- $2m + 1$ *represents a unit of the ring* $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z}$*, and*
- *the homomorphism* $T_{m,f} : \mathrm{PB}_3 \to \mathrm{PB}_3/\mathrm{N}_{\mathrm{PB}_3}$ *in onto.*

We denote by GT(N), the set of GT-shadows with the target N.

GT-shadows form a groupoid GTSh

Let $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$, $[m, f] \in \mathsf{GT}(\mathsf{N})$ and

$$\mathsf{K} := \ker\left(\mathsf{B}_4 \stackrel{T_{m,f}}{\longrightarrow} \mathsf{B}_4/\mathsf{N}\right).$$

One can show that $\mathsf{K} = \ker\left(\mathsf{PB}_4 \stackrel{T_{m,f}}{\longrightarrow} \mathsf{PB}_4/\mathsf{N}\right)$ and $\mathsf{K} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$.

We say that K is the *source* of $[m, f]$ and consider $[m, f]$ as a *morphism* from K to N in the groupoid GTSh.

In other words, the set of objects of GTSh is the poset $\mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$. For, $\mathsf{N}, \mathsf{K} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$, we set

$$\mathsf{GTSh}(\mathsf{K}, \mathsf{N}) := \left\{[m, f] \in \mathsf{GT}(\mathsf{N}) \mid \ker\left(\mathsf{B}_4 \stackrel{T_{m,f}}{\longrightarrow} \mathsf{B}_4/\mathsf{N}\right) = \mathsf{K}\right\}.$$

Wait! How do we compose such bizarre morphisms?

Well... if $[m_1, f_1] \in \mathrm{GTSh}(\mathrm{N}^{(1)}, \mathrm{N}^{(2)})$, $[m_2, f_2] \in \mathrm{GTSh}(\mathrm{N}^{(2)}, \mathrm{N}^{(3)})$ and

$$m := 2m_1 m_2 + m_1 + m_2 \,,$$
$$f(x,y) := f_2(x,y)\, f_1(x^{2m_2+1}, f_2(x,y)^{-1} y^{2m_2+1} f_2(x,y)),$$

then the pair (m, f) represents the GT-shadows

$$[m_2, f_2] \circ [m_1, f_1] \in \mathrm{GTSh}(\mathrm{N}^{(1)}, \mathrm{N}^{(3)}).$$

Remark. It is natural to consider the operad PaB of parenthesized braids. Then GT-shadows in GT(N) with the source K can be identified with isomorphisms of operads (in the category of finite groupoids):

$$\mathrm{PaB}/\sim_{\mathrm{K}} \;\stackrel{\sim}{\longrightarrow}\; \mathrm{PaB}/\sim_{\mathrm{N}} .$$

The composition of morphisms in GTSh comes from the composition of isomorphisms of operads.

GT-shadows coming from elements of $\widehat{\mathrm{GT}}$

Let $(\hat{m}, \hat{f}) \in \widehat{\mathrm{GT}}$ and $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$. Let $m + N_{\mathrm{ord}}\mathbb{Z}$ (resp. $f\mathsf{N}_{\mathsf{F}_2}$) be the image of $\hat{m}$ (resp. $\hat{f}$) in $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z}$ (resp. in $\mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}$).

Then $[m, f]$ is a GT-shadow with the target N and we say that $[m, f]$ **comes from** the element $(\hat{m}, \hat{f}) \in \widehat{\mathrm{GT}}$.

If a GT-shadow comes from an element of $\widehat{\mathrm{GT}}$, then this GT-shadow is called **genuine**. Otherwise, this GT-shadow is called **fake**.

The approximation functor PR

If a GT-shadow $[m, f] \in \mathsf{GT}(\mathsf{N})$ comes from an element $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}$, then we denote by $\mathsf{N}^{(\hat{m},\hat{f})}$ the source of $[m, f]$. One can show that the assignment $\mathsf{N} \mapsto \mathsf{N}^{(\hat{m},\hat{f})}$ is a *right action* of $\widehat{\mathsf{GT}}$ on the poset $\mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$. We denote by

$$\widehat{\mathsf{GT}}_{\mathsf{NFI}}$$

the corresponding transformation groupoid.

One can show that "passing from elements of $\widehat{\mathsf{GT}}$ to GT-shadows" gives us a functor

$$\mathsf{PR} : \widehat{\mathsf{GT}}_{\mathsf{NFI}} \to \mathsf{GTSh}\,.$$

Informally, we may call it the **approximation functor.**

The action of GT-shadows on child's drawings

Let $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$ and $H \le \mathsf{F}_2$ be a subgroup that represents a child's drawing D. We say that D is **subordinate** to N if $\mathsf{N}_{\mathsf{F}_2} \subset H$.

Dessin(N) denotes the set of child's drawings subordinate to N.

We denote by Dessin the category whose objects are elements of $\mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$. For $\mathsf{N}^{(1)}, \mathsf{N}^{(2)} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$, morphisms from $\mathsf{N}^{(1)}$ to $\mathsf{N}^{(2)}$ are all functions from $\mathsf{Dessin}(\mathsf{N}^{(1)})$ to $\mathsf{Dessin}(\mathsf{N}^{(2)})$.

Theorem (V.D., 2021)

Let $\mathsf{N}^{(1)}, \mathsf{N}^{(2)} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$ *and* $[m, f] \in \mathsf{GTSh}(\mathsf{N}^{(1)}, \mathsf{N}^{(2)})$. *Let* $\varphi : \mathsf{F}_2 \to S_d$ *be a homomorphism that represents* $D \in \mathsf{Dessin}(\mathsf{N}^{(2)})$ *and* $\tilde{\varphi}$ *be a homomorphism* $\mathsf{F}_2 \to S_d$ *defined by*

$$\tilde{\varphi}(x) := \varphi(x^{2m+1}), \qquad \tilde{\varphi}(y) := \varphi(f^{-1} y^{2m+1} f).$$

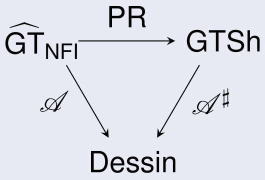
The assignment $\varphi \mapsto \tilde{\varphi}$ *gives us a functor* $\mathscr{A}^{\sharp} : \mathsf{GTSh} \to \mathsf{Dessin}$.

The actions are compatible!

Using the action of $\widehat{\mathsf{GT}}$ on child's drawings, we define a functor $\mathscr{A} : \widehat{\mathsf{GT}}_{\mathsf{NFI}} \to \mathsf{Dessin}$. Wait! We also have the approximation functor $\mathsf{PR} : \widehat{\mathsf{GT}}_{\mathsf{NFI}} \to \mathsf{GTSh}$. And the good news is that...

Theorem (V.D., 2021)

The diagram



commutes.

Hierarchy of orbits

Consider a chain in the poset $\mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$

$$\mathsf{N}^{(1)} \supset \mathsf{N}^{(2)} \supset \mathsf{N}^{(3)} \supset \dots$$

and a child's drawing $D \in \mathsf{Dessin}(\mathsf{N}^{(1)})$. It is clear that D is subordinate to $\mathsf{N}^{(i)}$ for every $\mathsf{N}^{(i)}$ in this chain.
Recall that $\mathsf{GT}(\mathsf{N})$ denotes the set of all GT-shadows with the target N. ($\mathsf{GT}(\mathsf{N})$ is finite!)

For every child's drawing D, we have the following *hierarchy of orbits*:

$$\mathsf{GT}(\mathsf{N}^{(1)})(D) \supset \mathsf{GT}(\mathsf{N}^{(2)})(D) \supset \mathsf{GT}(\mathsf{N}^{(3)})(D) \supset \dots \supset \widehat{\mathsf{GT}}(D) \supset G_{\mathbb{Q}}(D).$$

It is **very hard** to compute $G_{\mathbb{Q}}(D)$; **there are no tools in modern mathematics** to compute orbits $\widehat{\mathsf{GT}}(D)$; **it is relatively easy** to compute orbits $\mathsf{GT}(\mathsf{N})(D)$.

Selected results of computer experiments

Jointly with students, I have been developing a software package 'GT' for working with GT-shadows and their action on child's drawings. The final version of this package is available at `https://math.temple.edu/~vald/PackageGT/PackageGT.zip`

- Let D be a child's drawing for which the $G_{\mathbb{Q}}$-orbit and a GTSh-orbit are computed. Then these orbits coincide!
- There is **no** "Furusho phenomenon" for GT-shadows: there are examples $f \in \mathrm{F}_2$ that satisfy the pentagon relation modulo N but at least one hexagon fails for (m, f) for every $m \in \{0, 1, \dots, N_{\mathrm{ord}} - 1\}$.
- The connected components of GTSh we found have a very small number of objects: ≤ 2. So far, we could **not** find a connected component of GTSh with > 2 objects!
- So far, we did **not** find **any** example of a fake GT-shadow.

Selected open questions

1. Is it possible to find $K, N \in NFI_{PB_4}(B_4)$ such that $K \le N$ and the natural map
$$GT(K) \to GT(N)$$
is **not onto**? In other words, can we produce an example of a (charming) GT-shadow that is also **fake**? Is the situation easier when the quotient N/K is Abelian?

2. Can we produce an example of a **genuine** GT-shadow (different from $[0, 1_{F_2}]$ and $[-1, 1_{F_2}]$) in the non-Abelian setting? Can we get such examples using (non-Abelian) Galois child's drawings?

3. (D. Harbater) Given a child's drawing D, is there $N \in NFI_{PB_4}(B_4)$ such that $D \in \mathrm{Dessin}(N)$ and we can prove that, for every $K \in NFI_N(B_4)$, the orbits $GT(K)(D)$ and $GT(N)(D)$ coincide?

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More References?!... Sure!

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What?!... Even more references?!

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Dear Collaborators! Thank you!

These people worked with me on GT-shadows for the "gentle version" $\widehat{\mathsf{GT}}_{gen}$ of the Grothendieck-Teichmueller group:

- Jacob Guynee (Temple University, undergraduate, then graduate...)
- Jessica Radford (Temple University, undergraduate)
- Jingfeng Xia (Temple University, graduate)
- Hm... your name can be here! :-)
- Are you an "adult mathematician"? Your name can be here too! :-)

In the literature, $\widehat{\mathsf{GT}}_{gen}$ is often denoted by $\widehat{\mathsf{GT}}_0$.

THANK YOU!

GT-shadows in the Abelian setting

Proposition

For $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$, *the following conditions are equivalent:*

- **a)** *the quotient group* PB_4/N *is Abelian;*
- **b)** *the quotient group* $\mathsf{PB}_3/\mathsf{N}_{\mathsf{PB}_3}$ *is Abelian;*
- **c)** *the quotient group* $\mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}$ *is Abelian.*

Theorem

If we are in the Abelian setting, then

$$\mathsf{GT}(\mathsf{N}) = \{(\overline{m}, \underline{1}) \mid 0 \le m \le N_{\mathrm{ord}} - 1, \ \ \gcd(2m+1, N_{\mathrm{ord}}) = 1\},$$

where $\overline{m} := m + N_{\mathrm{ord}}\mathbb{Z}$, $\underline{1}$ *is the identity element of* $\mathsf{PB}_3/\mathsf{N}_{\mathsf{PB}_3}$.
Furthermore, ***every*** GT*-shadow in* $\mathsf{GT}(\mathsf{N})$ *is* ***genuine****.*

Abelian child's drawings

Let $\varphi : \mathsf{F}_2 \to S_d$ be a homomorphism that represents a child's drawing D. Recall that (the conjugacy class of) the permutation group $\varphi(\mathsf{F}_2) \le S_d$ is called the **monodromy group** of D.

A child's drawing D is called **Abelian** if its monodromy group is Abelian. One can show that every Abelian child's drawing is Galois.

Here is what we know about Abelian child's drawings:

Theorem (R. Hidalgo 2012, V.D. 2021)

Let D be an Abelian child's drawing and $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$ such that D is subordinate to N. Then the orbits

$$\mathsf{GT}(\mathsf{N})(D) \supset \widehat{\mathsf{GT}}(D) \supset G_{\mathbb{Q}}(D)$$

*are **singletons**.*