Erratum to: "A Proof of Tsygan's Formality Conjecture for an Arbitrary Smooth Manifold"

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Abstract

Boris Shoikhet noticed that the proof of lemma 1 in section 2.3 of [1] contains an error. In this note I give a correct proof of this lemma which was kindly suggested to me by Dmitry Tamarkin. The correction does not change the results of [1].

1 Introduction

In this note I give a correct proof of lemma 1 from section 2.3 in [1]. This proof was kindly suggested to me by Dmitry Tamarkin and it is based on the interpretation of L_∞-morphisms as Maurer-Cartan elements of an auxiliary L_∞-algebra.

The notion of partial homotopy proposed in section 2.3 in [1] is poorly defined and this note should be used as a replacement of section 2.3 in [1]. The main result of this section (lemma 1) is used in section 5.2 of [1] in the proof of theorem 6. Since the statement of the lemma still holds so does the statement of theorem 6 as well as all other results of [1].

In section 2 of this note I recall the notion of an L_∞-algebra and the notion of a Maurer-Cartan element. In section 3, I give the interpretation of L_∞-morphisms as Maurer-Cartan elements of an auxiliary L_∞-algebra and use it to define homotopies between L_∞-morphisms. Finally, in section 4 I formulate and prove lemma 1 from section 2.3 of [1].

Notation. I use the notation from [1]. The underlying symmetric monoidal is the category of cochain complexes. For this reason I sometimes omit the combination "DG" (differential graded) talking about (co)operads and their algebras. For an operad $\mathcal{O}$ I denote by $\mathbb{F}_{\mathcal{O}}$ the corresponding Schur functor. $s\,K$ denotes the suspension of the complex K. In other words,

$$s\,K = s \otimes K\,,$$

where s is the one-dimensional vector space placed in degree $+1$. Similarly,

$$s^{-1}K = s^{-1} \otimes K\,,$$

where s^{-1} is the one-dimensional vector space placed in degree -1. **cocomm** is the cooperad of cocommutative coalgebras.

By "suspension" of a (co)operad $\mathcal{O}$ I mean the (co)operad $\Lambda(\mathcal{O})$ whose m-th space is

$$\Lambda(\mathcal{O})(m) = \Sigma^{1-m}\mathcal{O}(m) \otimes \mathrm{sgn}_m\,, \tag{1.1}$$

where sgn_m is the sign representation of the symmetric group S_m.

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2 L_∞-algebras and Maurer-Cartan elements

Let me recall from [4] that an L_∞-algebra structure on a graded vector space $\mathcal{L}$ is a degree 1 codifferential Q on the colagebra $\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L})$ cogenerated by $\mathcal{L}$. Following [1] I denote the DG coalgebra $(\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}), Q)$ by $C(\mathcal{L})$:

$$C(\mathcal{L}) = (\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}), Q)\,. \tag{2.1}$$

A morphism F from an L_∞-algebra $(\mathcal{L}, Q)$ to an L_∞-algebra $(\mathcal{L}^\diamond, Q^\diamond)$ is by definition a morphism of (DG) coalgebras

$$F : C(\mathcal{L}) \to C(\mathcal{L}^\diamond)\,. \tag{2.2}$$

Since

$$\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}) = s\,\mathbb{F}_{\mathbf{cocomm}}(s^{-1}\mathcal{L})$$

the vector space of $C(\mathcal{L})$ can be identified with the exterior algebra $\wedge^\bullet\mathcal{L}$ and for a graded vector space V a map

$$f : \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}) \to V$$

of degree $|f|$ can be identified with the infinite collection of maps

$$f_n : \mathcal{L}^{\otimes n} \to V\,, \qquad n \ge 1\,,$$

where each map f_n has degree $|f| + 1 - n$ and

$$f_n(\dots, \gamma, \gamma', \dots) = -(-1)^{|\gamma||\gamma'|} f_n(\dots, \gamma', \gamma, \dots)$$

for every pair of elements $\gamma, \gamma' \in \mathcal{L}$.

Due to proposition 2.14 in [3] every coderivation of $\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L})$ is uniquely determined by its composition with the projection

$$\mathrm{pr}_{\mathcal{L}} : \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}) \to \mathcal{L} \tag{2.3}$$

from $\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L})$ onto cogenerators.

In particular, the codifferential Q of the coalgebra $C(\mathcal{L})$ is uniquely determined by the infinite collection of maps

$$Q_n = \mathrm{pr}_{\mathcal{L}} \circ Q\Big|_{\wedge^n\mathcal{L}} : \wedge^n\mathcal{L} \to \mathcal{L}\,, \tag{2.4}$$

such that Q_n has degree $2 - n$. In [1] Q_n are called structure maps of the L_∞-algebra $\mathcal{L}$.

The equation $Q^2 = 0$ is equivalent to an infinite collection of quadratic equations on the maps (2.4). The precise form of these equations can be found in definition 4.1 in [2].

One of the obvious equations implies that the structure map of the first level Q_1 is a degree 1 differential of $\mathcal{L}$. Thus an L_∞-algebra can be thought of as an algebra over an operad in the category of cochain complexes.

If $\mathcal{L}$ is a pronilpotent L_∞-algebra then it makes sense to speak about its Maurer-Cartan elements:

Definition 1 (Definition 4.3 in [2]) *A Maurer-Cartan π of a pronilpotent L_∞-algebra $(\mathcal{L}, Q)$ is a degree 1 element of $\mathcal{L}$ satisfying the equation*

$$\sum_{n=1}^{\infty} \frac{1}{n!} Q_n(\pi, \pi, \dots, \pi) = 0\,. \tag{2.5}$$

Let me remark that the infinite sum in (2.5) is well defined since $\mathcal{L}$ is pronilpotent.

Every Maurer-Cartan element π of $\mathcal{L}$ can be used to modify the L_∞-algebra structure on $\mathcal{L}$. This modified structure is called the L_∞-structure twisted by the Maurer-Cartan π and its structure maps are given by

$$Q_n^\pi(\gamma_1, \dots, \gamma_n) = \sum_{m=1}^{\infty} \frac{1}{m!} Q_{m+n}(\pi, \dots, \pi, \gamma_1, \dots, \gamma_n)\,, \qquad \gamma_i \in \mathcal{L}\,. \tag{2.6}$$

It is equation (2.5) which implies that the maps (2.6) define an L_∞-algebra structure on $\mathcal{L}$.

Two Maurer-Cartan elements π_0 and π_1 are called equivalent if there is an element $\xi \in \mathcal{L}^0$ such that the solution of the equation

$$\frac{d}{dt}\pi_t = Q_1^{\pi_t}(\xi) \tag{2.7}$$

connects π_0 and π_1:

$$\pi_t\Big|_{t=0} = \pi_0\,, \qquad \pi_t\Big|_{t=1} = \pi_1\,.$$

3 L_∞-morphisms and their homotopies

I will need the following auxiliary statement:

Proposition 1 *Let $\mathcal{O}$ be an operad and A be an algebra over $\mathcal{O}$. If B is a (DG) cocommutative coalgebra then the cochain complex*

$$\mathcal{H}_{B,A} = \mathrm{Hom}(B, A) \tag{3.1}$$

of all linear maps from B to A has a natural structure of an algebra over $\mathcal{O}$.

Proof. The $\mathcal{O}$-algebra structure on A is by definition the map (of complexes)

$$\mu_A : \mathbb{F}_{\mathcal{O}}(A) \to A \tag{3.2}$$

making the following diagrams commutative:

$$\begin{array}{ccc} \mathbb{F}_{\mathcal{O}}(\mathbb{F}_{\mathcal{O}}(A)) & \stackrel{\mathbb{F}_{\mathcal{O}}(\mu_A)}{\longrightarrow} & \mathbb{F}_{\mathcal{O}}(A) \\ \downarrow {\scriptstyle \mu_{\mathcal{O}}(A)} & & \downarrow {\scriptstyle \mu_A} \\ \mathbb{F}_{\mathcal{O}}(A) & \stackrel{\mu_A}{\longrightarrow} & A\,, \end{array} \tag{3.3}$$

$$\begin{array}{ccc} A & \stackrel{u_{\mathcal{O}}(A)}{\longrightarrow} & \mathbb{F}_{\mathcal{O}}(A) \\ & \searrow {\scriptstyle \mathrm{id}} & \downarrow {\scriptstyle \mu_A} \\ & & A \end{array} \tag{3.4}$$

where $\mu_{\mathcal{O}}$ and $u_{\mathcal{O}}$ are the transformation of functors

$$\mu_{\mathcal{O}} : \mathbb{F}_{\mathcal{O}} \circ \mathbb{F}_{\mathcal{O}} \to \mathbb{F}_{\mathcal{O}}\,,$$

$$u_{\mathcal{O}} : \mathrm{Id} \to \mathbb{F}_{\mathcal{O}}$$

defined by the operad structure on $\mathcal{O}$. The map μ_A is called the multiplication.

For every $n > 1$ the comultiplication Δ in B provides me with the following map

$$\Delta^{(n)} : B \to B^{\otimes n}$$

$$\Delta^{(n)} X = (\Delta \otimes 1^{\otimes (n-2)}) \dots (\Delta \otimes 1 \otimes 1)(\Delta \otimes 1)\Delta X \qquad (3.5)$$

Using this map and the $\mathcal{O}$-algebra structure on A I define the $\mathcal{O}$-algebra structure on $\mathcal{H}_{B,A}$ (3.1) by

$$\mu(v, \gamma_1, \dots, \gamma_n; X) = \mu_A(v)[\gamma_1 \otimes \dots \otimes \gamma_n (\Delta^{(n)} X)]\,, \qquad (3.6)$$

where $v \in \mathcal{O}(n)$, $\gamma_i \in \mathrm{Hom}(B, A)$, and $X \in B$.

The equivariance with respect to the action of the symmetric group follows from the cocommutativity of the comultiplication on B.

The commutativity of the diagram

$$\begin{array}{ccc} \mathbb{F}_{\mathcal{O}}(\mathbb{F}_{\mathcal{O}}(\mathcal{H}_{B,A})) & \stackrel{\mathbb{F}_{\mathcal{O}}(\mu)}{\longrightarrow} & \mathbb{F}_{\mathcal{O}}(\mathcal{H}_{B,A}) \\ \downarrow {\scriptstyle \mu_{\mathcal{O}}(\mathcal{H}_{B,A})} & & \downarrow {\scriptstyle \mu} \\ \mathbb{F}_{\mathcal{O}}(\mathcal{H}_{B,A}) & \stackrel{\mu}{\longrightarrow} & \mathcal{H}_{B,A}\,, \end{array} \qquad (3.7)$$

follows from the commutativity of (3.3) and the associativity of the comultiplication in B.

The commutativity of the diagram

$$\begin{array}{ccc} \mathcal{H}_{B,A} & \stackrel{u_{\mathcal{O}}(\mathcal{H}_{B,A})}{\longrightarrow} & \mathbb{F}_{\mathcal{O}}(\mathcal{H}_{B,A}) \\ & \searrow {\scriptstyle \mathrm{id}} & \downarrow {\scriptstyle \mu} \\ & & \mathcal{H}_{B,A} \end{array} \qquad (3.8)$$

and the compatibility of μ (3.6) with the differential are obvious. $\square$

Since

$$\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}) = s\, \mathbb{F}_{\mathbf{cocomm}}(s^{-1}\mathcal{L})$$

for every L_∞-algebra $\mathcal{L}^\diamond$ proposition 1 gives me a L_∞-structure on the cochain complex

$$\mathcal{U} = s\, \mathrm{Hom}(C(\mathcal{L}), \mathcal{L}^\diamond)\,. \qquad (3.9)$$

This algebra $\mathcal{U}$ can be equipped with the following decreasing filtration:

$$\mathcal{U} = \mathcal{F}^1\mathcal{U} \supset \mathcal{F}^2\mathcal{U} \supset \dots \supset \mathcal{F}^k\mathcal{U} \supset \dots$$

$$\mathcal{F}^k\mathcal{U} = \{ f \in \mathrm{Hom}(\wedge^\bullet \mathcal{L}, \mathcal{L}^\diamond) \quad | \quad f\Big|_{\wedge^{<k}\mathcal{L}} = 0 \}\,. \qquad (3.10)$$

It is not hard to see that this filtration is compatible with the L_∞-algebra structure on $\mathcal{U}$. Furthermore, since $\mathcal{U} = \mathcal{F}^1\mathcal{U}$, for every k the L_∞-algebra $\mathcal{U}/\mathcal{F}^k\mathcal{U}$ is nilpotent. On the other hand,

$$\mathcal{U} = \lim_k \mathcal{U}/\mathcal{F}^k\mathcal{U}\,, \qquad (3.11)$$

and hence, the L_∞-algebra $\mathcal{U}$ is pronilpotent and the notion of a Maurer-Cartan element of $\mathcal{U}$ makes sense.

My next purpose is identify the Maurer-Cartan elements of the L_∞-algebra $\mathcal{U}$ (3.9) with L_∞-morphisms from $\mathcal{L}$ to $\mathcal{L}^\diamond$:

Proposition 2 *L_∞-morphisms from $\mathcal{L}$ to $\mathcal{L}^\diamond$ are identified with Maurer-Cartan elements of the L_∞-algebra $\mathcal{U}$ (3.9)*

Proof. Since $C(\mathcal{L}^\diamond)$ is a cofree coalgebra, the map F (2.2) is uniquely determined by its composition $\mathrm{pr}_{\mathcal{L}^\diamond} \circ F$ with the projection $\mathrm{pr}_{\mathcal{L}^\diamond}$ (2.3). This composition is a degree zero element of $\mathrm{Hom}(C(\mathcal{L}), \mathcal{L}^\diamond)$. Thus, since $\mathcal{U}$ (3.9) is obtained from $\mathrm{Hom}(C(\mathcal{L}), \mathcal{L}^\diamond)$ by the suspension, every morphism F (2.2) is identified with a degree 1 element of $\mathcal{U}$.

It remains to prove that the compatibility condition

$$Q^\diamond F = F Q \tag{3.12}$$

of F with the codifferentials Q and $Q^\diamond$ on $C(\mathcal{L})$ and $C(\mathcal{L}^\diamond)$, respectively, is equivalent to the Maurer-Cartan equation (2.5) on $\mathrm{pr}_{\mathcal{L}^\diamond} \circ F$ viewed as an element of $\mathcal{U}$.

It is not hard to see that

$$\mathrm{pr}_{\mathcal{L}^\diamond} \circ (Q^\diamond F - F Q) = 0\,. \tag{3.13}$$

is equivalent to the Maurer-Cartan equation on the composition $\mathrm{pr}_{\mathcal{L}^\diamond} \circ F$ viewed as an element of $\mathcal{U}$ (3.9).

Thus, I have to show that equation (3.13) is equivalent to the compatibility condition (3.12).

For this, I denote by Ψ the difference:

$$\Psi = Q^\diamond F - F Q$$

and remark that

$$\Delta \Psi = -(\Psi \otimes F + F \otimes \Psi)\Delta\,, \tag{3.14}$$

where Δ denotes the coproduct both in $C(\mathcal{L})$ and $C(\mathcal{L}^\diamond)$.

The latter follows from the fact that Q and $Q^\diamond$ are coderivations and F is a morphism of cocommutative coalgebras.

Given a cooperad $\mathcal{C}$, a pair of cochain complexes V, W, a degree zero map

$$f : V \to W$$

and an arbitrary map

$$b : V \to W$$

I denote by $\partial(b, f)$ the following map[1]

$$\begin{gathered}\partial(b,f) : \mathbb{F}_{\mathcal{C}}(V) \to \mathbb{F}_{\mathcal{C}}(W) \\ \partial(b,f)(\gamma, v_1, v_2, \dots, v_n) = \\ (-1)^{|b|(|\gamma|+|v_1|+\dots+|v_{i-1}|)} \sum_{i=1}^{n} (\gamma, f(v_1), \dots, f(v_{i-1}), b(v_i), f(v_{i+1}), \dots, f(v_n))\,, \\ \gamma \in \mathcal{C}(n)\,, \qquad v_i \in V\,,\end{gathered} \tag{3.15}$$

where $|\gamma|$, $|b|$, $|v_j|$ are, respectively, degrees of γ, b, and v_j. The equivariance of (3.15) with respect to permutations is obvious.

[1] A similar construction was introduced at the beginning of section 2.2 in [3].

It is not hard to see that condition (3.14) is equivalent to commutativity of the following diagram

$$\begin{array}{ccc} \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}) & \stackrel{\Psi}{\longrightarrow} & \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}^{\diamond}) \\ \downarrow^{\nu} & & \downarrow^{\nu} \\ \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L})) & \stackrel{\partial(\Psi,F)}{\longrightarrow} & \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}^{\diamond}))\,, \end{array} \qquad (3.16)$$

where ν is the coproduct of the cotriple $\mathbb{F}_{\Lambda\mathbf{cocomm}}$.

Since the functor $\mathbb{F}_{\Lambda\mathbf{cocomm}}$ with the transformations $\nu : \mathbb{F}_{\Lambda\mathbf{cocomm}} \to \mathbb{F}_{\Lambda\mathbf{cocomm}} \circ \mathbb{F}_{\Lambda\mathbf{cocomm}}$ and $\mathrm{pr} : \mathbb{F}_{\Lambda\mathbf{cocomm}} \to \mathrm{Id}$ form a cotriple[2], the following diagram

$$\begin{array}{ccc} \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}^{\diamond}) & & \\ \downarrow^{\nu} & \searrow^{\mathrm{id}} & \\ \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}^{\diamond})) & \stackrel{p}{\longrightarrow} & \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}^{\diamond})\,, \end{array} \qquad (3.17)$$

with p being $\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathrm{pr}_{\mathcal{L}^{\diamond}})$, commutes.

Attaching this diagram to (3.16) I get the commutative diagram

$$\begin{array}{ccccc} \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}) & \stackrel{\Psi}{\longrightarrow} & \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}^{\diamond}) & & \\ \downarrow^{\nu} & & \downarrow^{\nu} & \searrow^{\mathrm{id}} & \\ \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L})) & \stackrel{\partial(\Psi,F)}{\longrightarrow} & \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}^{\diamond})) & \stackrel{p}{\longrightarrow} & \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathcal{L}^{\diamond})\,, \end{array} \qquad (3.18)$$

where, as above, $p = \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathrm{pr}_{\mathcal{L}^{\diamond}})$.

Hence,

$$\Psi = \mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathrm{pr}_{\mathcal{L}^{\diamond}}) \circ \partial(\Psi, F) \circ \nu\,.$$

On the other hand

$$\mathbb{F}_{\Lambda\mathbf{cocomm}}(\mathrm{pr}_{\mathcal{L}^{\diamond}}) \circ \partial(\Psi, F) = \partial(\mathrm{pr}_{\mathcal{L}^{\diamond}} \circ \Psi, \mathrm{pr}_{\mathcal{L}^{\diamond}} \circ F)\,.$$

Therefore,

$$\Psi = \partial(\mathrm{pr}_{\mathcal{L}^{\diamond}} \circ \Psi, \mathrm{pr}_{\mathcal{L}^{\diamond}} \circ F) \circ \nu$$

and Ψ vanishes if and only if so does the composition $\mathrm{pr}_{\mathcal{L}^{\diamond}} \circ \Psi$.

This concludes the proof of the proposition. □

The identification proposed in the above proposition allows me to introduce a notion of homotopy between two L_∞-morphisms. Namely,

Definition 2 *L_∞-morphisms F and $\widetilde{F}$ from $\mathcal{L}$ to $\mathcal{L}^{\diamond}$ are called homotopic if the corresponding Maurer-Cartan elements of the L_∞-algebra $\mathcal{U}$ (3.9) are equivalent.*

4 Lemma 1 from [1]

Let me denote by F_n the components

$$F_n : \wedge^n \mathcal{L} \to \mathcal{L}^{\diamond}$$

[2]See, for example, section 1.7 in [3].

$$F_n = \mathrm{pr}_{\mathcal{L}^\diamond} \circ F \Big|_{\wedge^n \mathcal{L}} \tag{4.1}$$

of the composition $\mathrm{pr}_{\mathcal{L}^\diamond} \circ F$, where $\mathrm{pr}_{\mathcal{L}^\diamond}$ is the projection from $\mathbb{F}_{\mathbf{\Lambda cocomm}}(\mathcal{L}^\diamond)$ onto cogenerators. In [1] the maps (4.1) are called structure maps of the L_∞-morphism (2.2).

The compatibility condition (3.12) implies that the structure map F_1 of the first level is morphism of complexes:

$$F_1 : \mathcal{L} \to \mathcal{L}^\diamond \,, \qquad Q_1^\diamond F_1 = F_1 Q_1 \,.$$

By definition, an L_∞-morphism F is a L_∞-quasi-isomorphism if the map F_1 is a quasi-isomorphism of the corresponding complexes.

I can now prove the following lemma:

Lemma 1 *Let*

$$F : C(\mathcal{L}) \mapsto C(\mathcal{L}^\diamond)$$

be a quasi-isomorphism from an L_∞-algebra $(\mathcal{L}, Q)$ to an L_∞-algebra $(\mathcal{L}^\diamond, Q^\diamond)$. For $n \geq 1$ and any map

$$\widetilde{H} : \wedge^n \mathcal{L} \mapsto \mathcal{L}^\diamond \tag{4.2}$$

of degree $-n$ one can construct a quasi-isomorphism

$$\widetilde{F} : C(\mathcal{L}) \mapsto C(\mathcal{L}^\diamond)$$

such that for any $m < n$

$$\widetilde{F}_m = F_m \ : \ \wedge^m \mathcal{L} \mapsto \mathcal{L}^\diamond \tag{4.3}$$

and

$$\begin{gathered} \widetilde{F}_n(\gamma_1, \dots, \gamma_n) = F_n(\gamma_1, \dots, \gamma_n) + \\ Q_1^\diamond \widetilde{H}(\gamma_1, \dots, \gamma_n) - (-)^n \widetilde{H}(Q_1(\gamma_1), \gamma_2, \dots, \gamma_n) - \dots \\ \dots - (-)^{n+k_1+\dots+k_{n-1}} \widetilde{H}(\gamma_1, \dots, \gamma_{n-1}, Q_1(\gamma_n)) \,, \end{gathered} \tag{4.4}$$

where $\gamma_i \in \mathcal{L}^{k_i}$.

Proof. Let $Q^{\mathcal{U}}$ denote the L_∞-algebra structure on $\mathcal{U}$ (3.9). Let α be the Maurer-Cartan elements of $\mathcal{U}$ corresponding to the L_∞-morphism F.

By setting

$$\xi \Big|_{\wedge^m \mathcal{L}} = \begin{cases} \widetilde{H} \,, & \text{if } m = n \,, \\ 0 \,, & \text{otherwise} \end{cases} \tag{4.5}$$

I define an element $\xi \in \mathcal{U}$ of degree -1. By definition of the filtration (3.10) the element ξ belongs to $\mathcal{F}^n \mathcal{U}$

Let α_t be the unique path of Maurer-Cartan elements defined by

$$\frac{d}{dt} \alpha_t = (Q^{\mathcal{U}})_1^{\alpha_t}(\xi) \,, \qquad \alpha_t \Big|_{t=0} = \alpha \,. \tag{4.6}$$

The unique solution α_t of (4.6) can be found by iterating the following equation in degrees in t

$$\alpha_t = \alpha + \int_0^t (Q^{\mathcal{U}})_1^{\alpha_\tau}(\xi) d\tau \,. \tag{4.7}$$

Since the L_∞-algebra $\mathcal{U}$ is pronilpotent the recurrent procedure (4.7) converges.

It is not hard to see that, since $\xi \in \mathcal{F}^n \mathcal{U}$,

$$\alpha_t - \alpha \in \mathcal{F}^n \mathcal{U} \tag{4.8}$$

and

$$\alpha_t - (\alpha + t Q_1^{\mathcal{U}}(\xi)) \in \mathcal{F}^{n+1} \mathcal{U} \,. \tag{4.9}$$

Let $\widetilde{F}$ be the L_∞-morphism from $\mathcal{L}$ to $\mathcal{L}^\diamond$ corresponding to the Maurer-Cartan element

$$\widetilde{\alpha} = \alpha_t \Big|_{t=1} \,.$$

Equation (4.8) implies (4.3) and equation (4.9) implies (4.4). It is obvious that, since F is a quasi-isomorphism, so is $\widetilde{F}$.

The lemma is proved. □

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