# Research Statement

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My research interests are in the area of low-dimensional geometry and topology, with a specialization in problems involving 3-dimensional manifolds. A central interest of mine is the idea of effectivization, which is of wide interest since it revolves around producing explicit examples and constructions. For example, Agol [1] recently proved the famous virtual Haken conjecture, i.e., that every closed hyperbolic 3-manifold M has a finite cover containing a properly embedded surface that cannot be compressed. However, little to nothing is known about how large a cover one must take (e.g., in terms of the topological complexity of M). My current research program is separated into two projects focused on effectivization. First I discuss finding the minimal degree non-cyclic covers of a knot complement in the 3-sphere, which relates to finding smallest degree coverings having other interesting properties like betti number growth. Second I discuss investigation into dehn filling and its classical relationship to character varieties, in particular giving topological interpretation to intersections in character varieties, and large-scale computation related to a 1990 conjecture of Gordon [16].

# 1 Non-Cylic Covers of Knot Complements

## 1.1 Work Done on Non-Cyclic Covers

My thesis work lies in the area of knot theory and low dimensional topology, specifically the study of the 3-manifold $\mathbf{S}^3 \smallsetminus K$ for $K$ a knot. A critical problem involves the existence of covering spaces of knot complements. A consequence of the seminal work of Hempel [17] on residual finiteness of the fundamental groups of geometric 3-manifolds and Thurston's proof of geometrization of Haken manifolds, is that every knot complement has a rich family of finite coverings. It is natural to ask for the minimal degree of a finite covering of $\mathbf{S}^3 \smallsetminus K$ that does not come from the obvious cyclic covers arising from $H_1\left(\mathbf{S}^3 \smallsetminus K\right)$. The first systematic study of this problem was done by Broaddus [4] who proved that there exists a non-cyclic cover $X$ of $\mathbf{S}^3 \smallsetminus K$ which has degree bounded above by a computable function $\mathfrak{b}(c_K)$, here $c_K$ is the crossing number of the knot. Kuperberg [22] then shows that, assuming the Generalized Reimann hypothesis that there is a non-abelian cover of $\mathbf{S}^3 \smallsetminus K$ which has degree bounded by $\exp\left(\mathrm{pol}(c_K)\right)$, here pol is a polynomial in the variable $c_K$. This leads to the following question.

**General Problem 1.1.** *Find the minimal degree non-cyclic covers of $\mathbf{S}^3 \smallsetminus K$ and investigate the degree as a function of topological properties of $\mathbf{S}^3 \smallsetminus K$ and combinatorial invariants of $K$.*

The main result of my thesis uses the finite field analogue of the Alexander polynomial of $K$ to construct a non-cyclic cover of $\mathbf{S}^3 \smallsetminus K$. Generalizing a result of de Rham [10], I construct a representation $\rho$ of the knot group to an affine subgroup of $\mathrm{GL}_2(\mathbb{F}_{p^d})$. Let $X_\rho$ denote the cover corresponding to the kernel of $\rho$. The non-cyclic cover $Y_\rho$ is the cover corresponding to the inverse image of a subgroup in the image of $\rho$. In particular I have proved,

**Theorem 1.2** (M.)**.** *If $K$ is a knot with crossing number $c_K$ and non-trivial Alexander polynomial, then there exists a regular non-abelian cover $X_\rho$ of $\mathbf{S}^3 \smallsetminus K$ with*

$$[\mathbf{S}^3 \smallsetminus K : X_\rho] \leq (2^{2c_K-3}-2)^{2c_K-2},$$

*and there exists an irregular non-cyclic cover $Y_\rho$ of $\mathbf{S}^3 \smallsetminus K$ with*

$$[\mathbf{S}^3 \smallsetminus K : Y_\rho] \leq (2^{2c_K-3}-2)^{c_K-1}.$$

The bounds in Theorem 1.2 can be improved by restricting to certain families of knots.

**Theorem 1.3** (M.)**.** *Let $Y_\rho$ be the non-cyclic cover constructed above.*

1. *If $K$ is a twist knot with $n$ half twists, then*

$$[\mathbf{S}^3 \smallsetminus K : Y_\rho] \leq 4n^2 - 8n + 4$$

2. *If $K$ is a fibered knot of genus $g$, then*

$$[\mathbf{S}^3 \smallsetminus K : Y_\rho] \leq 2^{2g}.$$

3. *If $K$ is a knot with Alexander polynomial of degree $n$, then*

$$[\mathbf{S}^3 \smallsetminus K : Y_\rho] \leq (2^{2n-2} - 2)^n.$$

The quantity $(2^{2c_K - 3} - 2)^{c_K - 1}$ is in the class $\exp(\mathrm{pol}(c_K))$, but does not require the generalized reimann hypothesis, for all $K$ with non-trivial Alexander polynomial. Furthermore in many cases the cover $Y_\rho$ is the minimal degree non-cyclic cover of $\mathbf{S}^3 \smallsetminus K$, so the methods in my thesis are in certain cases optimal. In my thesis, I provide extensive computations demonstrating this minimality and quantify many other topological properties of $Y_\rho$.

Using computer programs such as Sagemath, SnapPy, and Magma, I was able to compute the first homology groups of the $X_\rho$ for many knots and many primes $p$. Using a combination of fundamental topology of covering spaces and results of Hironaka [18] generalizing the classical computation of homology groups of cyclic branched covers of knot complements due to Fox [12], I prove the following.

**Theorem 1.4** (M.). *If $\rho : \pi_1\left(\mathbf{S}^3 \smallsetminus K\right) \to \mathrm{GL}_2(\mathbb{F}_{p^d})$ is the homomorphism described above, then there exists an explicit computable positive integer $n$ so that the $n$-fold cyclic cover of the knot complement, $Z_n$, is subordinate to the the cover $X_\rho$, and*

$$p^d \leq \beta_1(X_\rho) \leq (\mathrm{rank}\,(\pi_1(Z_n)) - 1)(p^d - 1) + \beta_1\left(Z_n\right).$$

In Theorems 1.4 and 1.5 the positive integer $n$ is the multiplicative order of an element in $\mathbb{F}_{p^d}^*$, that is vital to the construction of $\rho$.

This provides an alternate constructive and effective proof of a result of Cooper, Long, and Reid [8, Corollary 1.4] in the case of a knot complement $\mathbf{S}^3 \smallsetminus K$ for $K$ a knot with non-trivial Alexander polynomial. More specifically Theorem 1.4 provides concrete constructions for covers containing non-peripheral elements in its first homology groups. Furthermore, I demonstrate that in the case of fibered knots the bound in Theorem 1.4 is sharp. For fibered knots I can prove the following theorem that refines the conclusion of Theorem 1.4.

**Theorem 1.5** (M.). *If $K$ is a fibered knot with 3-genus $g$, let $\rho : \pi_1\left(\mathbf{S}^3 \smallsetminus K\right) \to \mathrm{GL}_2(\mathbb{F}_{p^d})$ be the homomorphism described above, then there exists an explicit computable positive integer $n$ so that the $n$-fold cyclic cover of the knot complement, $Z_n$, is subordinate to the the cover $X_\rho$, and*

$$p^d \leq \beta_1(X_\rho) \leq (2g - 1)(p^d - 1) + \beta_1\left(Z_n\right).$$

One can see from Theorem 1.5 for the figure 8 knot complement the lower bound is equal to the upper bound, allowing one to compute the first betti number of $X_\rho$ for the figure 8 explicitly. This is equally effective for many other knots.

## 1.2 Future Work Involving Non-Cyclic covers of Knot Complements

The central open question involving the non-cyclic covers of knot complements remains unsolved: find the minimal degree non-cyclic cover for a knot complement. A long-term goal of mine is to answer this question and provide a concrete construction of this cover. My research uses the Alexander polynomial of a knot to construct 2-dimensional affine linear representations over $\mathbb{F}_{p^d}$, showing that the existence of representations is closely related to the $\mathbb{F}_p$ analogue of the Alexander polynomial.

**Problem 1.6.** *Can one construct $\rho : \pi_1\left(\mathbf{S}^3 \smallsetminus K\right) \to \mathrm{GL}_n\left(\mathbb{F}_{p^d}\right)$ which are not affine for all knots with trivial Alexander polynomials?*

The theory of twisted Alexander polynomials provides a promising direction to resolve this problem. Denote $\Delta^{\alpha}(t)$ the Alexander polynomial twisted by a representation $\alpha$. Friedl and Vidussi [13] prove that every knot has a non-trivial Alexander polynomial. Furthermore Wada [29] shows that the Kinoshita-Terasaka knot and the Conway knot are distinguished via different representations to $\mathrm{GL}_2(\mathbb{F}_5)$, these representations result in different twisted Alexander polynomials. These two knots have many of the same invariants such as hyperbolic volume, Jones polynomial, HOMFLY polynomial, and Vassiliev invariants up to order 10. Twisted alexander polynomials also have combinatorial definitions via seifert surfaces, these surfaces are critical in the construction of representation to finite affine group described in Theorem 1.1. I plan to combine the ideas in my thesis with those of Friedl and Vidussi [13] and Wada [29] to construct representations to $\mathrm{GL}_n(\mathbb{F}_{p^d})$ which are not affine and distinguish knots with trivial Alexander polynomial.

Yet another direction to explore involving non-cyclic covers of knot complements involves the computation of $H_1(X_\rho)$. A long term goal of mine is to compute $H_1(X_\rho)$ explicitly, using a combination of Hironaka's and Fox's results on the abelianization of finite index subgroups of $\pi_1(\mathbf{S} \setminus K)$. However, more precise bounds are a more immediate short term goal. For instance, recall Theorem 1.5, as a consequence, using results of Fox [12], I prove.

**Theorem 1.7** (M.). *If $K$ is a fibered knot, then for all primes $p$ there exists a regular non-abelian cover $X_\rho$ satisfying the following*

$$p \leq \beta_1(X_\rho) \leq (2c_K - 3)(p^{c_K - 1} - 1) + (c_K - 1). \tag{1}$$

Theorem 1.7 is important because it relates the first betti number of $X_\rho$ to only the crossing number. Furthermore for many fibered knots the upper bound is the first betti number, exactly. I hope to explicitly decribe the growth rates first described by Deblois, Friedl, and Vidussi [11] for the ranks of the cyclic covers of non-fibered knots. More specifically I hope to describe these growth rates in terms of the crossing number of the knot. It will lead to a result analogous to Theorem 1.7 for non-fibered knot complements, resolving the following problem.

**Problem 1.8.** *If $K$ is a non-fibered knot, describe the the growth rates of [11] in terms of combinatorial or topological invariants of $K$, establishing an analogue to Theorem* 1.8.

### Relationship of non-abelian covers to $L$-space knots

A very popular area of study is that of $L$-spaces, rational homology three spheres $M$ for which the dimension of the Heegaard Floer homology [25] is equal to the order of $H_1(M, \mathbb{Z})$ [26]. Furthermore a knot complement is said to admit $L$-space surgeries if there is a positive integer surgery that results in an $L$-space. The Alexander polynomials of knots admitting $L$ space surgeries are relatively well understood [26], they are the Alexander polynomials with all $\pm 1$'s for coefficients, with alternating sign as degree decreases. Preliminary computations involving the homology groups of knots that have Alexander polynomials with all $\pm 1$'s alternating in sign as degree decreases has led to the following conjecture.

**Conjecture.** *Let $K$ be a knot with Alexander polynomial $\Delta(t)$ and* 3*-genus $g$, if the coefficients of $\Delta(t)$ are all $\pm 1$ and alternate in sign as degree decreases, then for all but finitely many primes $p$.*

$$\beta_1(X_\rho) = (2g - 1)(p^d - 1) + \beta_1(X_n) \tag{2}$$

I will pursue this conjecture in my future work as it provides sharpness of Theorem 1.5 and may shed some light on topological properties of $L$-spaces.

# 2 Character Varieties, Intersection Points, and Dehn Filling

## 2.1 Previous Work on Intersection Points of Character Varieties

Another research interest of mine involves the character varieties of knot manifolds. Character varieties of 3-manifolds combine ideas from many fields like algebraic geometry, topology, geometry, number theory, and representation theory to provide concrete results about the geometry and topology of 3-manifolds. The most famous result in this direction is the classic paper of Culler and Shalen [9] in which they used character varieties to produce essential surfaces in 3-manifolds.

The essential surfaces constructed via Culler-Shalen Theory often have boundary which lies in the boundary of the knot manifold, and the boundary of the surface in the boundary of the of the 3-manifold is called a *boundary slope*. Boundary slopes of knot manifolds are of particular importance since they provide important algebraic and topological information. These boundary slopes are crucial to developing Haken hierarchies, leading to deep results in the theory of 3-manifolds. Boundary slopes are strongly related to a powerful invariant of the manifold, called the $A$-polynomial [7] which in some sense is a bridge between the topological properties of the manifold and $\mathrm{SL}_2(\mathbb{C})$ representation theory of the manifold's fundamental group. The relationship between character varieties and boundary slopes has been extensively studied ([3], [5], [7], [20], [21]).

Suppose $M$ is a connected compact 3-manifold with a single torus boundary and $\Gamma = \pi_1(M)$. The character variety of $M$ is $\mathrm{X}(M) = \mathrm{Hom}\,(\Gamma, \mathrm{SL}_2(\mathbb{C})) /\!\!/ \mathrm{SL}_2(\mathbb{C})$, where the action of $\mathrm{SL}_2(\mathbb{C})$ is by conjugation. The space $\mathrm{X}(M)$ is an algebraic subset in $\mathbb{C}^N$. If $M$ is a hyperbolic 3-manifold, there exists a discrete faithful representation $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ and by Thurston [9, Proposition 3.1.1] there is a lift $\rho_0$ of $\rho$ in $\mathrm{X}(M)$. The *canonical component* is the irreducible subset $\mathrm{X}_0(M) \subset \mathrm{X}(M)$ containing $\rho_0$.

The twist knots are a well known family of knots that result from $1/n$ Dehn filling the unknotted component of the complement of the Whitehead link, denoted $W$. Let $K_n$ and $K_m$ denote the $n$th and $m$th twist knot complements, perfoming $1/n$ and $1/m$ dehn filling on $W$ induces a quotient between the fundmental groups $\pi_1(W) \to \pi_1(K_n)$ and $\pi_1(W) \to \pi_1(K_m)$. On the level of character varieties this induces embeddings $\mathrm{X}_0(K_n) \hookrightarrow \mathrm{X}_0(W)$ and $\mathrm{X}_0(K_m) \hookrightarrow \mathrm{X}_0(W)$. For simplicity let $\mathrm{X}_0(K_n)$ and $\mathrm{X}_0(K_m)$ also denote the Zariski closure of these images in $\mathrm{X}_0(W)$. For different values $n$ and $m$ these embeddings often contain affine intersection points. I computed the defining equations of the intersection points for specific $m$ and $n$. Some of these characters corresponded to algebraic non-integral representations of both $\pi_1(K_n)$ and $\pi_1(K_m)$. Algebraic non-integral representations can be used to construct essential surfaces much like the essential surfaces produced using ideal points of the character variety. These few examples and work of Schanuel and Zhang [27], Long and Reid [23], and Chu [6] led me to believe that this is a general phenomenon of topological significance.

**General Problem 2.1.** *For a knot complement $K$, describe the algebraic non-integral representations found in $\mathrm{X}_0(K)$ in terms of topological or combinatorial invariants of the knot.*

Schanuel and Zhang [27] originally addressed boundary slope detection in knot manifolds, using Culler and Shalen [9] and a contruction of Serre [28], proving that algebraic non-integral characters can be used to detect boundary slopes in $\mathbf{S}^3 \setminus K$. My collaborator Charles Katerba and I were able characterize these algebraic non-integral characters in the case of intersections of twist knot character varieties inside the Whitehead link character variety in terms of $n$ and $m$. Futhermore we are able to use results of [24] and [20] to characterize the boundary slopes corresponding to these characters. Let $\{\mu_1, \lambda_1\}$ be a homological basis for the unknotted boundary component (the meridian and longitude of this specific boundary component) of the Whitehead link.

**Theorem 2.2** (Katerba, M.)**.** *Fix integers $n$ and $k \geq 3$, and suppose $\chi \in \mathrm{X}_0(K_n) \cap \mathrm{X}_0(K_{n+k})$ is the character of a representation $\rho : \pi_1(K_j) \to \mathrm{SL}_2(\mathbb{C})$ for both $j = n$ and $j = n + k$ with $|\rho(\lambda_1)| = k$. The representation $\rho$ is algebraic non-integral if and only if*

*1) $k \nmid 2n + 1$, or*

*2)* $k \nmid 4n+1$.

**Theorem 2.3** (Katerba, M.)**.** *If* $\chi \in \mathrm{X}_0(K_n) \cap \mathrm{X}_0(K_{n+k})$ *is algebraic non-integral, then the same slope is detected by* $\chi$ *in both* $K_n$ *and* $K_{n+k}$*, and furthermore it is either the boundary slope* $0/1$ *or* $-4/1$.

This theorem leads to the following definition; suppose $M_1$ and $M_2$ are knot manifolds; a slope $\gamma$ contained in both $M_1$ and $M_2$ is a *simultaneously detected* if $\gamma$ is a boundary slope detected[1] by a representation of both $M_1$ and $M_2$ with isomorphic image. The slopes described in Theorem 2.3 are simultaneously detected. The study of simultaneously detected slopes is, an interesting open area of the theory of character varieties and boundary slope detection, which could shed new light on existing problems involving characterizing slopes, Dehn filling, and hyperbolic cosmetic surgery.

## 2.2 Previous Work on Dehn Filling Points in Character Varieties

Dehn filling points on the character variety and deformations of such points have been an important area of study for many years, beginning with Thurston's Hyperbolic Dehn Surgery Theorem. There are still many open questions involving the Dehn filling points on the character variety. One of the most famous open questions is the Cosmetic Surgery conjecture. The cosmetic surgery conjecture is a problem which generalizes the famous Gordon-Luecke theorem [15], which states that $1/0$ surgery on a knot in $\mathbf{S}^3$ is the only surgery which produces $\mathbf{S}^3$. Originally posed by Gordon [16] in his 1990 ICM address it states,

**Cosmetic Surgery Conjecture.** *If* $M$ *is a knot manifold, and* $r_1$, $r_2$ *are slopes on* $\partial M$ *then the filled manifolds* $M(r_1)$ *and* $M(r_2)$ *are homeomorphic if and only if* $r_1 \doteq r_2$ *where* $\doteq$ *is equivalence up to symmetry of* $M$ *which extends to a symmetry of* $\partial M$.

An immediate corollary of this conjecture would be that knots have property $P$, which is to say that $1/0$ surgery is the only surgery in which the resulting manifold is simply connected.

Restricting to the case that $M$ is hyperbolic and the filled manifolds $M(r_1)$ and $M(r_2)$ are also hyperbolic, one gains a large amount of rigidity and an analogue of the character variety can be used to address the resolution of this conjecture effectively. The bridge between geometry and character varieties comes from *hyperbolic Dehn surgery space* denoted $\mathcal{HDS}$. Hyperbolic Dehn surgery space is an open subset of the universal cover of a maximal cusp endowed with the euclidean metric which is the restriction of hyperbolic metric to this cusp. For each point in $\mathcal{HDS}$ Thurston describes a way to produce manifolds with cone metrics as a generalization of Dehn filling. In other words $\mathcal{HDS}$ is a moduli space of cone metrics coming from a "filling" construction on $M$. Hodgson and Kerckhoff [19] describe a smooth local parameterization of the canonical component of the character variety by $\mathcal{HDS}$. In short this idea allows one to control representations via the metric structure on the maximal cusp. Let $\mathrm{len}_{\mathbb{E}}(x)$ be the euclidean length of the point $x \in \mathcal{HDS}$. Furthermore suppose $s$ is the hyperbolic length of the systole of $M$, using the results of [14] and [2], there exists an explicit computable function $c : \mathbb{R} \to \mathbb{R}$, such that if $\mathrm{len}_{\mathbb{E}}(r_1) \geq c(s)$ and $\mathrm{len}_{\mathbb{E}}(r_2) \geq c(s)$ then $M(r_1)$ is isometric to $M(r_2)$ if and only if $r_1 \doteq r_2$. Note that pairs of slopes satisfying the above, are hyperbolic since $c(s)$ is strictly larger than the bound described by the seminal work of Hodgson and Kerckhoff [19] for exceptional surgeries. The importance of the results by Futer, Purcell, and Schleimer [14] is that it allowed me to enumerate the remaining slopes and systematically verify the Cosmetic surgery conjecture, for any hyperbolic $M$. When restricted to the SnapPy census of orientable knot manifolds, to date I can experimentally show.

**Experimental Theorem 2.4.** *For the first* 17300 *manifolds in SnapPy's orientable one cusped census the cosmetic surgery conjecture experimentally holds for pairs of slopes which are longer than the exceptional hyperbolic bound due to Hodgson and Kerckhoff.*

[1] Either by ideal points or algebraic non-integral points

### 2.3 Further Work For Intersections of Character Varieties

Theorem 2.2 gives a way to find $\mathrm{SL}_2(\mathbb{C})$ representations where the image is contained in a ring of algebraic integers. First investigated by Long and Reid [23] in the case of the twist knots $K_n$ and $K_{n+k}$ we hope to characterize these representations in terms of $n$ and $k$. More specifically for integral characters contained in $\mathrm{X}_0(K_n) \cap \mathrm{X}_0(K_{n+k})$ the image of the the representation coming from the these characters is either fuchsian or finite. We will establish a relationship between $n$ and $k$ and the image of the representation, since these numbers determine the orders of meridian elements of the image.

The twist knots result from $(1, n)$ Dehn filling of the unknotted component of the Whitehead link. A natural direction is to explore the manifolds which result from filling along different slopes of this component of the Whitehead link. Using the generalizations of the techniques of [20] and [24] we will define the intersection points recursively for different families of fillings. The goal will be to provide a characterization of the affine intersection points in which are a result of these fillings, and to characterize the boundary slopes corresponding to algebraic non-integral slopes in the intersection. A similar direction to take this research is to explore the families of knots which are a result of surgeries on a fixed link, for example pretzel knots, generalized twist knots, etc.

**Problem 2.5.** *Determine the integrality of intersection points of character varieties of knot manifolds resulting from Dehn filling a fixed link. Relate the topology of the knot manifold to the filling coefficients.*

### 2.4 Future Work on Cosmetic Surgery

This is a long term computer computation. The algorithm is very slow for a few reasons, however the largest amount of time is spent computing the systole of the knot manifold $M$. Currently SnapPy does not easily compute the systole of a knot manifold, since it involves the construction of the Dirichlet domain of the knot manifold. The next problem to address is the verification of the Dehn filled manifolds. Currently SnapPy is unable to verify the hyperbolicity of many Dehn filled manifolds, for the filling slopes of length less than the theoretic bound of Agol and Lackenby-Meyerhoff and this needs to be done. An immediate consequence of this will be that we determined all of the exceptional slope for knot manifolds in SnapPy's census.

**Problem 2.6.** *Generalize* SnapPy′s *hyperbolic verification process for small Dehn fillings of knot manifolds in the census.*

An answer to this problem is the first step in completely verifying the the cosmetic surgery conjecture computationally. This along with a positive volume computation for fillings will allow for the verification of the cosmetic surgery conjecture in SnapPy.

Lastly the verification process has returned some interesting examples, there are a manifolds which SnapPy believes to have cosmetic surgeries. I believe that this is not the case, these pairs of fillings are most likely related by equivalent meridians. I plan to show that this is indeed the case, and expand the algorithm to include this as a check.

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