Research statement
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1 Introduction

My research interests centers around hyperbolic geometry, geometric group theory, low-dimensional topology, and dynamical systems. These topics are tied together in my work through the study of closed curves on a surface. The utility of this approach to geometry and dynamics is famously illustrated by Thurston's work on hyperbolic geometry, Teichmüller theory, and mapping class groups which is based heavily on his analysis of simple closed curves, see [13,28,30].

The classical work of Fricke and Klein proves that a hyperbolic metric is determined (up to isotopy) by the lengths of a finite set of simple closed curves. In my thesis, the motivating question was the extent to which the Fricke-Klein result holds for certain family of flat metrics. A flat metric is the metric determined by a holomorphic quadratic differential. Duchin-Leininger-Rafi in [8] proved that there is no such finite set of curves for the space of all flat metrics. I extended the result to “strata” of flat metrics, in a natural stratification, with sufficiently large dimension. On the other hand, I proved that the obstruction to finding such a finite set is a global one. Specifically, I proved that every point in a stratum has a neighborhood and a finite set of curves so that the lengths determine any metrics within the neighborhood. One of my current projects involves extending those results in a variety of ways, see Section 2.

I am also currently working on a number of other projects surrounding curves and surfaces in low-dimensional geometry and dynamics. Working with Jayadev Athreya, I am investigating a logarithm law for the systole with respect to the earthquake flow over the Teichmüller space, which will be outlined in Section 3. There is also a project with Jonah Gaster studying mapping class invariant functions on curve systems in Section 4. We find minimizers for each function and describe the asymptotics as the size of the curve system grow. In Section 5, I will describe a 1-parameter family of deformations of a quadratic differential using a new construction I call flat grafting. The goal of that project is to analyze, by way of analogy, the flat grafting rays comparing them with grafting and earthquake rays in Teichmüller space.

2 Length spectral rigidity

For a closed surface $S$, let $C(S)$ denote the set of (homotopy classes of) closed curves on $S$. Given a metric $m$ on $S$, let $\ell_{C(S)}(m)$ denote the point of $\mathbb{R}^{C(S)}$ that records the minimal length of every homotopy class. The length spectral rigidity question for a family of (isotopy classes of) metrics $\mathcal{G}$ asks whether the length spectrum $\ell_{C(S)}$ thought of as a function $\mathcal{G} \to \mathbb{R}^{C(S)}$ is injective. In 1990, Otal [26] gave a positive answer for the class of negatively curved metrics using geodesic currents introduced by Bonahon [5]. This has since been extended and refined by a number of authors [6,7,10,11,15].

Motivated by the results of [8], I am interested in length spectral rigidity results either restricting to simple closed curves or a finite set of closed curves when considering the space of flat metrics $\text{Flat}(S)$. We say that $\Sigma \subset C(S)$ is length spectrally rigid over $\mathcal{G} \subset \text{Flat}(S)$ if $\ell_{\Sigma}$ is an injective map over $\mathcal{G}$. The set $\Sigma$ is said to be locally length spectrally rigid at $\rho$ over $\mathcal{G}$ if there exists a neighborhood $N_\rho \subset \mathcal{G}$ such that $\ell_{\Sigma}$ is injective over $N_\rho$. In my case, $\mathcal{G}$ is a stratum of the space of flat metrics and I will use the notation $\text{Flat}(S, \alpha)$ to denote the stratum determined by $\alpha$ which encodes the cone angles and the holonomy.

To describe my results, we recall that Thurston described a topology on the set of isotopy classes of simple closed curves, and proved that the closure in this topology is naturally his space of projective measured foliations $\mathcal{PMF}(S)$. Duchin-Leininger-Rafi [8] proved that for a surface $S_{g,n}$ of genus $g$ with $n$ punctures in
which $3g + n - 3 \geq 2$ has the following property. A set of simple closed curves $\Sigma$ is spectrally rigid for $\text{Flat}(S)$ if and only if $\Sigma = PMF(S)$. In particular, for any non-dense set of simple closed curves, they constructed a family of flat metrics on which $\ell_\Sigma : \text{Flat}(S) \to \mathbb{R}^\Sigma$ is constant. My first result in [12] extends and refines this result of [8].

**Theorem 1.** If $(3g + n - 3) \geq 2$ and $\dim(\text{Flat}(S, \alpha)) > (6g + 2n - 6)$, then a set of simple closed curves $\Sigma$ is length spectrally rigid over $\text{Flat}(S, \alpha)$ if and only if $\Sigma = PMF(S)$.

The lower bound on the dimension of $\text{Flat}(S, \alpha)$ in this theorem is roughly half the dimension of the entire space $\text{Flat}(S)$, and hence applies to a large number of strata. The main ingredient of the proof is a construction of positive dimensional families of flat metrics on which $\ell_\Sigma$ is constant. We note that our construction is considerably more flexible than that of [8], producing families in all of $\text{Flat}(S)$ of dimension roughly three times that of [8]. Theorem 1 is at first somewhat surprising as $\text{Flat}(S, \alpha)$ is finite dimensional, which suggests that one should be able to find a finite set of curves which is spectrally rigid. In both [8] and in my construction, the families of metrics constructed can potentially come from very far away, suggesting that the difficulty is a global one, rather than a local one. I show in [12] that this is indeed the case by proving that at least locally, one can find a finite rigid set.

**Theorem 2.** For any $\rho \in \text{Flat}(S, \alpha)$, there exists a finite set of closed curves $\Sigma \subset C(S)$ such that $\Sigma$ is locally spectrally rigid at $\rho$ over $\text{Flat}(S, \alpha)$.

There are still various directions to extend my results for all strata, without imposing any restrictions on the dimensions. Furthermore, I would like to be able to answer the length spectral rigidity problem for all strata. On the other hand, I would like to answer the local length spectral rigidity problem over $\text{Flat}(S)$ instead of inside a stratum. The main question that I am currently working on is the following.

**Question 1.** When $\mathcal{G} = \text{Flat}(S)$ or $\text{Flat}(S, \alpha)$, can we find a set of closed curves $\Sigma$ for $\rho \in \mathcal{G}$ such that $\Sigma$ is locally length spectrally rigid at $\rho$ over $\mathcal{G}$ and $|\Sigma| = \dim(\mathcal{G})$?

### 3 Logarithm law for systole along earthquake flows

Mumford’s compactness criterion tells us that to tend to infinity in moduli space, the length of some curve must tend to zero. The length of the shortest curve in a hyperbolic metric, or the “systole” function, is thus a natural function on Teichmüller space. In particular, the behavior of this function along various flows provides useful geometrical and dynamical information. The non-divergence property of horocyclic flows and earthquake flows proved by Minsky-Weiss [24] shows that the orbits of these flows spend only a small amount of time in the thin-part (i.e. the place where the systole is small). However, the ergodicity of the two flows (see [19, 23]) implies that the systole goes to zero (along a subsequence), i.e., infinity is a limit point of almost every flow-line. These contrasting properties motivate further investigations.

In [2], Athreya proved logarithm laws for horocycle flows on hyperbolic surfaces and moduli spaces of flat surfaces. This type of quantitative approach to behaviors of the systole function is motivated by [20, 27]. Given the close relation between the horocycle flow and the earthquake flow, we are motivated to develop an earthquake flow analogue. The logarithm law we propose considers the quantity

$$C(X, \lambda) := \limsup_{t \to \infty} \frac{-\log \ell_{sys}(E_t(X, \lambda))}{\sqrt{t}},$$

where $X$ is a point in the Teichmüller space $T(S)$, $\lambda \in MF(S)$ is a measured foliation on $S$, $E_t$ describes the time $t$ earthquake flow on $T(S) \times MF(S)$, and $\ell_{sys} : T(S) \to \mathbb{R}_+$ is the systole with respect to the hyperbolic metric $X$. See [17, 29] for more details on the earthquake map. Our goal is to compute values of $C(X, \lambda)$. We observe that $C(X, \lambda) = 0$ whenever $\lambda$ happens to be a weighted multicurve on $S$. On the other hand, ergodicity of the earthquake flow implies that $C(X, \lambda)$ is constant almost everywhere. We answer the question in the case of the once-punctured torus.
Theorem-in-progress 1. When $S$ is the once-punctured torus or the four-punctured sphere, for almost every pair $(X, \lambda) \in \mathcal{T}(S) \times \mathcal{MF}(S)$ with respect to the Weil-Petersson measure, $C(X, \lambda) = 1/2$.

The upper bound follows from an argument using the Borel-Cantelli lemma in [27] and the lower bound is closely related to the continued fraction approximation. We are interested in extending the construction to general surfaces.

4 Complexity of curve systems

On a closed surface $S$, a curve system is a set of distinct homotopic classes of simple closed curves. In [1, 16, 18], the authors studied the size of $k$-systems, defined to be curve systems such that the geometric intersection number of pairs of curves in the system is no larger than $k$. Working with Jonah Gaster, we consider $k$-sum-systems which are systems with total intersection number of all pairs no larger than $k$. In fact, we are interested in the dual problem that fixes the number of curves and asks for the curve system that minimizes the sum of the geometric intersection numbers.

For an ordered set $\Sigma = \{\gamma_1, \ldots, \gamma_N\}$ of simple closed curves, we define the intersection matrix $M(\Sigma)$ as the $N \times N$ matrix with entries

$$a_{j,k}(\Sigma) = \{i(\gamma_j, \gamma_k)\},$$

where $i(\cdot, \cdot)$ is the geometric intersection number. By definition, $M(\Sigma)$ is invariant under mapping class actions. We consider functions on $M(\Sigma)$ that are independent of the ordering of $\Sigma$. The function corresponding to the study of $k$-systems is the $\ell^\infty$-norm of $M(\Sigma)$ and $k$-sum-systems correspond to the $\ell^1$-norm. The case when the surface is the torus has been thoroughly studied. We analyze the quantity

$$F_p(N) := \min_{|\Sigma| = N} ||M(\Sigma)||_p, \text{ with } \Sigma \text{ a curve system on the torus},$$

for $p = 1, \infty$ and prove asymptotics in each case. The $\ell^1$-norm has a surprising relation to the classical problem regarding convex lattice polygons studied in [3], whereas the study of the $\ell^\infty$-norm turns into a problem of counting primitive lattice points. We are working on understanding the quantity for other $p$ values and extending our results to general surfaces.

Since the intersection matrix is invariant under mapping class actions, we say the norm measures the “complexity” of the curve system. In this direction, we also study other functions on curve systems that have similar properties and compare configurations of curve sets that achieve minima. For example, the linear algebra aspect of $M(\Sigma)$ also contains geometric information. We prove a nonzero lower bound for the determinant of $M(\Sigma)$ in the case of the torus. Currently we are studying the top eigenvalue of $M(\Sigma)$ and its eigenvector.

Theorem-in-progress 2. Let $\Sigma$ be a curve system on the torus with $|\Sigma| = N \geq 2$.

$$\det M(\Sigma) \geq 2^{N-2}$$

with equality if and only if the cyclic ordering of $\Sigma$ by slopes satisfy $i(\gamma_j, \gamma_{j+1}) = 1$ for all $j \mod N$.

There are also combinatoric approaches to the study of curve systems via the curve complex. We expect to find interesting correlations between different viewpoints.

5 Flat grafting

The space of quadratic differentials $QD(S)$, seen as the cotangent bundle over the Teichmüller space $\mathcal{F}(S)$, is a very fascinating space. While working on the length spectral rigidity problem described in Section 2, I searched for a description of local deformations near a point in $QD(S)$ and found no geometric version that can interact well with lengths. Studying the properties of the earthquake flow and the grafting ray in
... motivated me to construct deformations of $QD(S)$ that I call flat graftings. The constructions are related to Masur and Zorich’s hole transports in [25] and Wright’s cylinder deformations in [31].

A horizontal flat grafting along a simple closed curve $\gamma$ is a function $HF_\gamma : QD(S) \rightarrow QD(S)$ that takes $q \in QD(S)$ and replace the geodesic representative of $\gamma$, which is a sequence of saddle connections, by a sequence of parallelograms of width 1, see Figure 1 for examples. Topologically, we cut along $\gamma$ to get two boundary components, and then we glue in a strip. The construction degenerates when horizontal saddle connections occur and special care is required. Note that I am not using the convention of quadratic differentials being unit area and $HF_\gamma$ is not a globally continuous function.

We extend to weighted curves by using parallelograms of width $t$ and obtain $HF_{t\gamma}$ that describes a 1-parameter family of deformations. In fact, the following statement is true.

**Theorem-in-progress 3.** The function $HF_{t\gamma} : QD(S) \rightarrow QD(S)$ with $t \geq 0$ defines a semi-flow for any simple closed curve $\gamma$.

When the geodesic representative goes through a cone point, the flat grafting could break the cone point into two cone points with smaller cone angles. A specifically interesting property is that the horizontal flat grafting preserves the horizontal foliation and changes the vertical foliation only.

We extend horizontal flat grafting to flat grafting in any direction by conjugating $HF_\gamma$ by multiplication of $QD(S)$ by a unit complex number. For example, a vertical flat grafting is obtained by conjugating the rotation of $\pi/2$ and it will preserve the vertical foliation. The space $QD(S)$ can be described as filling pairs of measured foliations, namely the horizontal and the vertical. We have the following observation which motivates the question of whether we can use a horizontal flat grafting and a vertical flat grafting to describe a neighborhood in $QD(S)$.

**Theorem-in-progress 4.** Consider the set $H_q = \{ HF_{t\gamma}(q) \mid 0 < t < \varepsilon, \text{simple closed curve } \gamma \}$ for $q \in QD(S)$. The horizontal measured foliations of $H_q$ lie in the same projective class as the horizontal measured foliation of $q$. The vertical measured foliations of $H_q$ form a dense set projectively in a neighborhood of the vertical measured foliation of $q$.

Furthermore, when $\gamma$ is a cylinder curve for some $q \in QD(S)$, meaning it has a Euclidean cylinder neighborhood in the metric from $q$, then we can define flat grafting for both positive and small negative values. Then composition of horizontal flat grafting with a holonomy coordinate (see [4,21,32]) is a homeomorphism onto its image. Using this I am currently working to find a finite set of cylinder curves whose flat grafting defines a parameterization of the neighborhood. This will provide a new direction of attack for the local length spectral rigidity problem in Section 2.
References


