Motivating Questions

- Can we reconstruct the metric on a surface with just lengths of curves?
- Can we classify minimal sets of curves whose lengths determine the metric in a specified family?
- What is the obstruction to determining the metric?
- Can lengths of curves provide a local coordinate of the space of metrics?

Background

People have answered these questions in various different settings. We highlight three of the results here.

- (Klein 1890’s) The lengths of a specific finite set of curves determine the hyperbolic metric.
- (Otal [3]) The lengths of all closed curves determine the negatively curved metric.
- (Duchin, Leininger, Rafi [1]) A set of simple closed curves determines the hyperbolic metric.

Objects of Study

- The space of flat metrics is denoted by \( \text{Flat}(S, \alpha) \). \( \alpha_4 = (6\pi), \alpha_B = (4\pi, 4\pi), \alpha_C = (3\pi, 3\pi, 3\pi) \).
- Our goal is to determine metrics in a stratum by lengths of closed curves (or simple ones).

Global Rigidity

**Theorem 1 (F.)** Fix a stratum of flat metrics with \( \dim(\text{Flat}(S, \alpha)) > (6g + 2n - 6) \). The lengths of a set of simple closed curves determines the metric in \( \text{Flat}(S, \alpha) \) if and only if the set is dense in \( \mathcal{P}M(S) \).

The main ingredient is the construction of a family of metrics where the lengths of a large set of simple closed curves are constant. Take a flat metric \( \rho \) in \( \text{Flat}(S, \alpha) \) where \( \rho \) is \( \rho^{(\pi)} \subset \mathcal{M}F(S) \) for each \( \pi \).

We map the flat metric to the circle of straight line foliations in \( \mathcal{M}F(S) \).

Hence there exists a pair of train tracks \( \tau \) and \( \tau' \) that meet efficiently such that \( \tau \) carries the circle \( \rho^{(\pi)} \subset \mathcal{M}F(S) \). For example, pick a suitable pants decomposition and \( \tau, \tau' \) on each pair of pants as below.

Using the homeomorphism between measured foliations carried by \( \tau \) and a set \( W_\tau \) of weight vectors on \( \tau \), we obtain a circle of weight vectors. By taking average \( \frac{1}{2} \int_0^\tau w^\tau d\theta \), we have a map

\[
f : \rho \rightarrow \frac{1}{2} \int_0^\tau w^\tau d\theta \text{ which extends to a neighborhood } f : N_\rho \rightarrow W_\tau.
\]

The fibers \( f^{-1}(w) \) will be families of flat metrics with the length of every simple closed curve carried by \( \tau' \) being a constant. Generically the dimension of the fibers is at least \( \dim(\text{Flat}(S, \alpha)) - (6g + 2n - 6) \). Finally we make use of the North-South dynamics of the action of the mapping class group, this allows us to adjust the choice of \( \rho \) such that \( \tau \) carries the circle of measured foliation and \( \tau' \) carries the non-dense set of simple closed curves \( \Sigma \).

Local Rigidity

**Theorem 2 (F.)** Fix a flat metric \( \rho \in \text{Flat}(S, \alpha) \). There exists a finite set of closed curves \( \Sigma \) and a neighborhood \( N_\rho \subset \text{Flat}(S, \alpha) \) inside the stratum such that the lengths of \( \Sigma \) determine the metric inside the neighborhood.

Triangulate \( S \) by line segments connecting cone points. Then the lengths of the line segments determine the metric locally, so it suffices to find closed curves whose lengths determine them.

Let \( \gamma_0 \) be an oriented line segment from the triangulation and let its initial and terminal points be \( a \) and \( b \). There exists geodesic segments \( \gamma_j \) for \( j = 1, 2, 3, 4 \) from \( a \) to \( b \) with directions as shown in the picture below.

The following closed curves are closed geodesics with respect to \( \rho \).

\[
c_1 = \gamma_0 \cup \gamma_1, \ c_2 = \gamma_0 \cup \gamma_2, \ c_3 = \gamma_1 \cup \gamma_4, \ c_4 = \gamma_2 \cup \gamma_3, \ \text{and } c_5 = \gamma_3 \cup \gamma_4.
\]

The length of \( \gamma_0 \) is

\[
\frac{1}{2} (\ell_1(\rho') + \ell_2(\rho') - \ell_2(\rho') - \ell_3(\rho') + \ell_3(\rho')) \text{ for any } \rho' \text{ near } \rho.
\]

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References


Contact Information

Web: http://www.math.uic.edu/~fu15
Email: fu15@illinois.edu

ILLINOIS UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN