1 Introduction

My research interests centers around hyperbolic geometry, geometric group theory, low-dimensional topology, and dynamical systems. These topics are tied together in my work through the study of closed curves on a surface. The utility of this approach to geometry and dynamics is famously illustrated by Thurston’s work on hyperbolic geometry, Teichmüller theory, and mapping class groups which is based heavily on his analysis of simple closed curves, see [13,31,33].

The classical work of Fricke and Klein proves that a hyperbolic metric is determined (up to isotopy) by the lengths of a finite set of simple closed curves. In my thesis, the motivating question was the extent to which the Fricke-Klein result holds for certain family of flat metrics. A flat metric is the metric determined by a holomorphic quadratic differential. Duchin-Leininger-Rafi in [8] proved that there is no such finite set of curves for the space of all flat metrics. I extended the result to “strata” of flat metrics, in a natural stratification, with sufficiently large dimension. On the other hand, I proved that the obstruction to finding such a finite set is a global one. Specifically, I proved that every point in a stratum has a neighborhood and a finite set of curves so that the lengths determine any metrics within the neighborhood. One of my current projects involves extending those results in a variety of ways, see Section 2.

I am also currently working on a number of other projects surrounding surfaces and low-dimensional geometry and dynamics. In Section 3, I will describe a 1-parameter family of deformations of a quadratic differential using a new construction I call flat grafting. The goal of that project is to analyze, by way of analogy, the flat grafting rays comparing them with grafting and earthquake rays in Teichmüller space. In other directions, I am working with Jayadev Athreya on proving a logarithm law for the systole with respect to the earthquake flow over the Teichmüller space, which will be outlined in Section 4. I am also just beginning a third project with Chris Leininger comparing convex cocompactness with various notions of convergence actions for subgroups of the mapping class group, see Section 5.

2 Length spectral rigidity

For a closed surface $S$, let $C(S)$ denote the set of (homotopy classes of) closed curves on $S$. Given a metric $m$ on $S$, let $\ell_{C(S)}(m)$ denote the point of $\mathbb{R}^{C(S)}$ that records the minimal length of every homotopy class. The length spectral rigidity question for a family of (isotopy classes of) metrics $\mathcal{G}$ asks whether the length spectrum $\ell_{C(S)}$ thought of as a function $\mathcal{G} \to \mathbb{R}^{C(S)}$ is injective. In 1990, Otal [29] gave a positive answer for the class of negatively curved metrics using geodesic currents introduced by Bonahon [3]. This has since been extended and refined by a number of authors [6,7,10,11,18].

Motivated by the results of [8], I am interested in length spectral rigidity results either restricting to simple closed curves or a finite set of closed curves when considering the space of flat metrics $\text{Flat}(S)$. We say that $\Sigma \subset C(S)$ is length spectrally rigid over $\mathcal{G} \subset \text{Flat}(S)$ if $\ell_\Sigma$ is an injective map over $\mathcal{G}$. The set $\Sigma$ is said to be locally length spectrally rigid at $\rho$ over $\mathcal{G}$ if there exists a neighborhood $N_\rho \subset \mathcal{G}$ such that $\ell_\Sigma$ is injective over $N_\rho$. In my case, $\mathcal{G}$ is a stratum of the space of flat metrics and I will use the notation $\text{Flat}(S, \alpha)$ to denote the stratum determined by $\alpha$ which encodes the cone angles and the holonomy.

To describe my results, we recall that Thurston described a topology on the set of isotopy classes of simple closed curves, and proved that the closure in this topology is naturally his space of projective measured foliations $\mathcal{PMF}(S)$. Duchin-Leininger-Rafi [8] proved that for a surface $S_{g,n}$ of genus $g$ with $n$ punctures in
which $3g + n - 3 \geq 2$ has the following property. A set of simple closed curves $\Sigma$ is spectrally rigid for $\text{Flat}(S)$ if and only if $\Sigma = PMF(S)$. In particular, for any non-dense set of simple closed curves, they constructed a family of flat metrics on which $\ell_\Sigma : \text{Flat}(S) \to \mathbb{R}^2$ is constant. My first result in [12] extends and refines this result of [8].

**Theorem 1.** If $(3g + n - 3) \geq 2$ and $\dim(\text{Flat}(S, \alpha)) > (6g + 2n - 6)$, then a set of simple closed curves $\Sigma$ is length spectrally rigid over $\text{Flat}(S, \alpha)$ if and only if $\Sigma = PMF(S)$.

The lower bound on the dimension of $\text{Flat}(S, \alpha)$ in this theorem is roughly half the dimension of the entire space $\text{Flat}(S)$, and hence applies to a large number of strata. The main ingredient of the proof is a construction of positive dimensional families of flat metrics on which $\ell_\Sigma$ is constant. We note that our construction is considerably more flexible than that of [8], producing families in all of $\text{Flat}(S)$ of dimension roughly three times that of [8]. Theorem 1 is at first somewhat surprising as a construction of positive dimensional families of flat metrics on which $\ell_\Sigma$ is constant is considerably more flexible than that of [8], producing families in all of $\text{Flat}(S)$ of dimension three times that of [8].

In [8] and in my construction, the families of metrics constructed can potentially come from very far away, suggesting that the difficulty is a global one, rather than a local one. I show in [12] that this is indeed the case by proving that at least locally, one can find a finite rigid set.

**Theorem 2.** For any $\rho \in \text{Flat}(S, \alpha)$, there exists a finite set of closed curves $\Sigma \subset C(S)$ such that $\Sigma$ is locally spectrally rigid at $\rho$ over $\text{Flat}(S, \alpha)$.

There are still various directions to extend my results for all strata, without imposing any restrictions on the dimensions. Furthermore, I would like to be able to answer the length spectral rigidity problem for all strata. On the other hand, I would like to answer the local length spectral rigidity problem over $\text{Flat}(S)$ instead of inside a stratum. The main question that I am currently working on is the following.

**Question 1.** When $G = \text{Flat}(S)$ or $\text{Flat}(S, \alpha)$, can we find a set of closed curves $\Sigma$ for $\rho \in G$ such that $\Sigma$ is locally length spectrally rigid at $\rho$ over $G$ and $|\Sigma| = \dim(G)$?

3 Flat grafting

The space of quadratic differentials $QD(S)$, seen as the cotangent bundle over the Teichmüller space $T(S)$, is a very fascinating space. While working on the length spectral rigidity problem described in Section 2, I searched for a description of local deformations near a point in $QD(S)$ and found no geometric version that can interact well with lengths. Studying the properties of the earthquake flow and the grafting ray in [9, 19, 25] motivated me to construct deformations of $QD(S)$ that I call flat graftings. The constructions are related to Masur and Zorich’s hole transports in [28], Wright’s cylinder deformations in [34], and the Euclidean earthquakes considered in [5].

A horizontal flat grafting along a simple closed curve $\gamma$ is a function $HF_\gamma : QD(S) \to QD(S)$ that takes $q \in QD(S)$ and replace the geodesic representative of $\gamma$, which is a sequence of saddle connections, by a sequence of parallelograms of width 1, see Figure 1 for examples. Topologically, we cut along $\gamma$ to get two boundary components, and then we glue in a strip. The construction degenerates when horizontal saddle connections occur and special care is required. Note that I am not using the convention of quadratic differentials being unit area and that the difficulty is a global one, rather than a local one. I show in [12] that this is indeed the case by proving that at least locally, one can find a finite rigid set.

**Theorem-in-progress 1.** The function $HF_{t, \gamma} : QD(S) \to QD(S)$ with $t \geq 0$ describes a well-defined semi-flow for any simple closed curve $\gamma$.

When the geodesic representative goes through a cone point, the flat grafting could break the cone point into two cone points with smaller cone angles. A specifically interesting property is that the horizontal flat grafting preserves the horizontal foliation and changes the vertical foliation only.
We extend horizontal flat grafting to flat grafting in any direction by conjugating $H_t \gamma$ by multiplication of $QD(S)$ by a unit complex number. For example, a vertical flat grafting is obtained by conjugating the rotation of $\pi/2$ and it will preserve the vertical foliation. Using Hubbard and Masur’s result in [17], $QD(S)$ can be described as filling pairs of measured foliations, namely the horizontal and the vertical. Hence whether we can use a horizontal flat grafting and a vertical flat grafting to describe a neighborhood in $QD(S)$ becomes an interesting question.

I will briefly describe an application of flat graftings. When $\gamma$ is a cylinder curve for some $q \in QD(S)$, meaning it has a Euclidean cylinder neighborhood in the metric from $q$, then we can define flat grafting for both positive and small negative values. Then composition of horizontal flat grafting with a holonomy coordinate (see [2, 24, 35]) is a homeomorphism onto its image. Using this I am currently working to find a finite set of cylinder curves whose flat grafting defines a parameterization of the neighborhood. This will provide a new direction of attack for the local length spectral rigidity problem in Section 2.

4 Logarithm law for systole along earthquake flows

Mumford’s compactness criterion tells us that to tend to infinity in moduli space, the length of some curve must tend to zero. The length of the shortest curve in a hyperbolic metric, or the “systole” function, is thus a natural function on Teichmüller space. In particular, the behavior of this function along various flows provides useful geometrical and dynamical information. The non-divergence property of horocyclic flows and earthquake flows proved by Minsky-Weiss [27] shows that the orbits of these flows spend only a small amount of time in the thin-part (i.e. the place where the systole is small). However, the ergodicity of the two flows (see [22, 26]) implies that the systole goes to zero (along a subsequence), i.e., infinity is a limit point of almost every flow-line. These contrasting properties motivate further investigations.

In [1], Athreya proved logarithm laws for horocycle flows on hyperbolic surfaces and moduli spaces of flat surfaces. This type of quantitative approach to behaviors of the systole function is motivated by [23, 30]. Given the close relation between the horocycle flow and the earthquake flow, we are motivated to develop an earthquake flow analogue. The logarithm law that we propose considers the quantity

\[ C(X, \lambda) := \limsup_{t \to \infty} \frac{-\log \ell_{sys}(E_t(X, \lambda))}{\log t}, \]

where $X$ is a point in the Teichmüller space $T(S)$, $\lambda \in \mathcal{MF}(S)$ is a measured foliation on $S$, $E_t$ describes the time $t$ earthquake flow on $T(S) \times \mathcal{MF}(S)$, and $\ell_{sys} : T(S) \to \mathbb{R}_+$ is the systole with respect to the
hyperbolic metric $X$. See [19,32] for more details on the earthquake map. Our goal is to characterize values of $C(X,\lambda)$. We observe that $C(X,\lambda) = 0$ whenever $\lambda$ happens to be a weighted multicurve on $S$. On the other hand, ergodicity of the earthquake flow implies that $C(X,\lambda)$ is constant almost everywhere.

**Question 2.** What is the typical value of $C(X,\lambda)$?

To answer the question we need to find an upper bound and construct a corresponding lower bound. We can obtain an upper bound by combining the argument using the Borel-Cantelli lemma in [30] and the earthquake invariant measure on $T(S) \times \mathcal{MF}(S)$ in [26]. We have also done some investigation regarding the construction of sequences for the lower bound. It is closely related to diophantine approximation of measured foliations and it provides more quantitative information on earthquake flows.

5 Dynamics on projective measured foliations

The action of the mapping class group on the Thurston compactification of Teichmüller space is analogous in many ways to the action of a Kleinian group on the visual compactification of hyperbolic space. This has provided motivation for numerous advances in the study of mapping class groups.

One example of this is the notion of convex cocompactness as described by Farb and Mosher [14]. This is defined using the above analogy, and in [21] Kent and Leininger expanded on this analogy. On the other hand, combining the work of Farb and Mosher [14] with that of Hamenstädt [16] proves that convex cocompactness for a subgroup of the mapping class group is intimately related to the coarse hyperbolic geometry of surface bundles over general spaces (not necessarily the circle).

In [20], Kent and Leininger characterize convex cocompactness in terms of a certain *uniform convergence* type property analogous to the uniform convergence action of a Kleinian group on its limit set as in the work of Gehring and Martin [15] (see also [4]). However, Kent and Leininger needed to work on an enlargement of the limit set called the zero locus (they proved that the limit set alone is insufficient). On the other hand, the zero locus of the limit set is defined in terms of geometry on the surface. It turns out that every characterization involves some reference to the surface, which leads to the following question.

**Question 3.** Is there a characterization of convex cocompactness just in terms of the topology of the action of the Thurston compactification?

We propose to use convergence properties to study this problem. Our goal is to show that a subgroup of the mapping class group is convex cocompact if and only if it acts as a convergence group on the Thurston boundary of Teichmüller space and every limit point is conical *in the dynamical sense*. Here we recall that in [21] Kent and Leininger characterized convex cocompactness as groups for which every limit point is conical *in the geometric sense*, meaning that there is a Teichmüller geodesic ray limiting to the point and having infinitely many orbit points a bounded distance from the ray. The dynamical notion of conical limit point agrees with this in the Kleinian group case, but it is unclear whether this is true in the mapping class group. Thus, one of the key problems we are working on is to prove that these two notions agree in the mapping class group.

References

[32] ______, *Earthquakes in two-dimensional hyperbolic geometry*, Low-dimensional topology and Kleinian groups (Coven-

