Practice Midterm

- Work as a group. Each student writes the solution to one problem. Solutions will be collected, graded, and emailed to everyone.

1. Let $\tau$ be a topology on the set of integers $\mathbb{Z}$. Determine whether $\tau_1$ defined below is a topology on $\mathbb{Z} \times \mathbb{Z}$.

   $\tau_1 = \{ U \subset \mathbb{Z} \times \mathbb{Z} \mid \forall n \in \mathbb{Z}, U \cap \{n\} \times \mathbb{Z} = \{n\} \times V \text{ for some } V \in \tau \}$.

2. Let $\mathcal{B}$, described below, be a collection of subsets of $\mathbb{R}$. Show that $\mathcal{B}$ is not a topology. Determine whether $\mathcal{B}$ is a basis for some topology on $\mathbb{R}$.

   $\mathcal{B} = \{ [a, b) \mid a \in \mathbb{Z}, b \in \mathbb{R}, a < b \}$.

3. Let $S$ and $T$ be non-empty subsets of a topological space $(X, \tau)$ with $S \subset T$. Show that if $S$ is dense in $X$, then $T$ is dense in $X$.

4. Prove that the $T_2$-space property is a topological property. As a result, give an example of a pair of non-homeomorphic topological spaces.

5. Let $(X, \tau)$ and $(Y, \tau_1)$ be topological spaces and $f : (X, \tau) \to (Y, \tau_1)$ a continuous map. If $f$ is injective, prove that

   (i) $(Y, \tau_1)$ is a $T_2$-space implies that $(X, \tau)$ is a $T_2$-space.
   (ii) $(Y, \tau_1)$ is a $T_1$-space implies that $(X, \tau)$ is a $T_1$-space.