Lecture Note 22

- **Remark.**
  This is the last lecture note. We recap some known fundamental groups and briefly say what follows in a more rigorous course in Algebraic topology.

- **Examples.**
  (a) The fundamental group of the circle is $\mathbb{Z}$. It is closely related to the concept of “winding numbers” in complex analysis.
  (b) The fundamental group of the wedge of two circles ($\infty$-shape) is the free group of two generators. This extends to computing fundamental groups of gluing two spaces at a single point.
  (c) The fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$. This extends to computing fundamental groups of product spaces.
  (d) The general method for computing fundamental groups of new spaces is called the Van Kampen’s Theorem, see Section 1.2.
  (e) There is a way to construct spaces whose fundamental groups are subgroups of known ones, see Section 1.3 Covering spaces.

- **Definition 121.**
  A manifold of dimension $n$ is a topological space $(X, \tau)$ that satisfies:
  (a) $(X, \tau)$ is a second countable $T_2$-space, and
  (b) For every $x \in X$, there exists an open set containing $x$ that is homeomorphic to $B^n(0,1)$.

- **Proposition 122.** Any connected manifolds of dimension 1 is homeomorphic to either $S^1$ or $(0,1)$.

- **Proposition 123.** Any connected, closed, and bounded manifolds of dimension 2 is homeomorphic to $S_g$ (orientable) or $S_g\#\mathbb{RP}^2$ (non-orientable) for some nonnegative integer $g$. Furthermore, the Euler characteristic determines the value of $g$.

- **Remark.**
  The above classification can also be determined by their fundamental group, since all objects listed have different fundamental groups.