Lecture Note 20

• Remark.
We should pinpoint the amount of material we plan to cover in Algebraic Topology by Allen Hatcher. In Chapter 0: Some Underlying Geometric Notions, we will cover the three topics: Homotopy and Homotopy Types, Cell Complexes, and Two Criteria for Homotopy Equivalence. In Chapter 1: The Fundamental Group, we will cover 1.1: Basic Constructions and introduce Covering Spaces.

• Definition 98.
Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces and \(A \subset X\) a closed set. A continuous map \(f : A \to Y\) is said to be an attaching map for the topological space denoted by \(X +_f Y\). Let \(\sim\) be the equivalence relation given by \(p \sim f(p)\) for every \(p \in A\). We say that \((X, \tau)\) is attached to \((Y, \tau')\) via \(f\) and the quotient space \(X +_f Y\) is the space \(X \cup Y/\sim\).

• Definition 99.
A space constructed via steps below is called a cell complex or CW complex.
(1) Start with a discrete set \(X^0\), whose points are regarded as 0-cells.
(2) Inductively, form the \(n\)-skeleton \(X^n\) from \(X^{n-1}\) by attaching \(n\)-cells (or open \(n\)-disks) \(e^n_\alpha\) via maps \(\varphi_\alpha : S^{n-1} \to X^{n-1}\). This means that \(X^n\) is the quotient space of the disjoint union of \(X^{n-1}\) with a collection of \(n\)-disks \(D^n_\alpha\) under the identifications \(x \sim \varphi_\alpha(x)\) for \(x \in S^{n-1}_\alpha\), boundary of \(D^n_\alpha\).
(3) A set \(A \subset X = \bigcup X^i\) is open if and only if \(A \cap X^n\) is open in \(X^n\) for each \(n\). This is a weak topology.

• Remark.
A subcomplex is a closed subspace \(A\) that is a union of cells of \(X\). The quotient of a cell complex by its subcomplex is identifying the subcomplex into a point.

• Definition 102.
A deformation retract from \(X\) to \(A \subset X\) is a map \(F : X \times I \to X\) that satisfies the following properties.
(1) \(F(x, 0) = x\) for all \(x \in X\);
(2) \(F(x, 1) \in A\) for all \(x \in X\);
(3) \(F(a, t) = a\) for all \(a \in A, t \in [0, 1]\).

• Proposition 103.
Let \((X, \tau)\) and \((Y, \tau')\) be homeomorphic topological spaces. If there exists a deformation retract from \(X\) to a point, then there exists a deformation retract from \(Y\) to a point.

• Definition 104.
Two continuous functions \(f_0, f_1 : X \to Y\) are said to be homotopic if there exists a homotopy \(F : X \times I \to Y\), which is a continuous function with \(F(x, 0) = f_0(x)\) and \(F(x, 1) = f_1(x)\).

• Remark.
In these terms, a deformation retraction of \(X\) onto a subspace \(A\) is a homotopy from the identity map of \(X\) to a “projection” of \(X\) onto \(A\).

• Example.
We can visualize how two functions with domain $S^1$ are homotopic by drawing a “rectangle”. There are ways to map the circle into the torus to obtain non-homotopic maps. Note: If the domain is $[0,1]$ and the range is path-connected, then every pair of functions are homotopic.

- **Definition 105.**
  
  Two topological spaces $(X, \tau)$ and $(Y, \tau')$ are said to be homotopy equivalent or to have the same homotopy type if there exists two continuous functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are both homotopic to the identity map.

- **Proposition 106.**
  
  Two spaces $X$ and $Y$ are homotopy equivalent if and only if there exists a third space $Z$ containing both $X$ and $Y$ as deformation retracts.

- **Definition 107.**
  
  A space having the homotopy type of a point is called contractible.

- **Proposition 108.**
  
  Let $X$ be a CW complex and $A$ a subcomplex. If $A$ is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.

- **Proposition 109.**
  
  Let $X$ be a CW complex and $A$ a subcomplex. If $f, g : A \to Y$ are homotopic, then $X + f Y$ is homotopic equivalent to $X + g Y$. 