Lecture Note 19

- **Definition 98.**
  Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces and \(A \subset X\) a closed set. A continuous map \(f : A \to Y\) attaches \(X\) to \(Y\) via the equivalence relation \(p \sim f(p)\) for every \(p \in A\). We denote the quotient space \(X + f Y\).

- **Remark.**
  Today we describe a general method to construct topological spaces.

- **Definition 99.**
  A space constructed via steps below is called a cell complex or CW complex.
  1. Start with a discrete set \(X^0\), whose points are regarded as 0-cells.
  2. Inductively, form the \(n\)-skeleton \(X^n\) from \(X^{n-1}\) by attaching \(n\)-cells (or open \(n\)-disks) \(e^n_\alpha\) via maps \(\varphi_\alpha : S^{n-1} \to X^{n-1}\). This means that \(X^n\) is the quotient space of the disjoint union of \(X^{n-1}\) with a collection of \(n\)-disks \(D^n_\alpha\) under the identifications \(x \sim \varphi_\alpha(x)\) for \(x \in S^{n-1}_\alpha\), boundary of \(D^n_\alpha\).
  3. A set \(A \subset X = \bigcup X^i\) is open if and only if \(A \cap X^n\) is open in \(X^n\) for each \(n\). This is a weak topology.

- **Remark.**
  The letters used in CW complex stands for closure-finiteness and weak topology.

- **Examples.**
  (a) A discrete topological space is a CW complex with just the 0-skeleton \(X^0\).
  (b) A graph is a CW complex with the vertices being \(X^0\) and the edges connecting vertices understood as closed intervals attached to \(X^0\). Reminder: when \(n = 1\) in step (2), \(S^0\) is the boundary of 1-cells, which is a two point set.
  (c) A 2-dimensional sphere is a CW complex constructed by 2 0-cells, 2 1-cells, and 2 2-cells. \(X^1\) is the circle and the 2-cells are attached to form hemispheres. Similarly, \(S^k\) is a CW complex with 2 \(n\)-cells for each \(n = 0, 1, \ldots, k\).
  (d) By attaching an \(n\)-cell to \(S^{n-1}\), a closed \(n\)-ball is a CW complex following (c).
  (e) All examples obtained by gluing sides of the square last time are CW complexes. The square is a CW complexes with 4 0-cells, 4 1-cells, and 1 2-cell. The cylinder is a CW complex with 2 0-cells, 3 1-cells, and 1 2-cell. The torus is a CW complex with 1 0-cell, 2 1-cells, and 1 2-cell. These can be defined by quotient maps that respect the CW complex structure.
  (f) Finite products of CW complexes are CW complexes. The cone of a CW complex is a CW complex. The suspension of a CW complex is a CW complex.

- **Remark.**
  A topological space can have many distinct CW complex descriptions. For example, any polygon is homeomorphic to the circle. A sphere can be obtained by gluing 8 triangles together (octahedron). That uses 6 0-cells, 12 1-cells, and 8 2-cells.

- **Theorem 100.**
  The **Euler characteristic** is a topological invariant that does not depend on the CW complex description. The Euler characteristic is defined by
  \[
  \chi(X) = \#(0 - \text{cells}) - \#(1 - \text{cells}) + \#(2 - \text{cells}) - \#(3 - \text{cells}) + \cdots .
  \]

- **Corollary 101.**
The sphere and the torus are not homeomorphic.

Proof.

\[ 2 - 2 + 2 \neq 1 - 2 + 1. \]

\[ \square \]

• Remark.

CW complex is the natural method to construct many topological spaces using our understanding of \( \mathbb{R}^n \). We will now return to the topic of deciding whether two spaces are homeomorphic.

• Definition 102.

A deformation retract from \( X \) to \( A \subset X \) is a map \( F : X \times I \to X \) that satisfies the following properties.

(1) \( F(x, 0) = x \) for all \( x \in X \);
(2) \( F(x, 1) \in A \) for all \( x \in X \);
(3) \( F(a, t) = a \) for all \( a \in A, t \in [0, 1] \).