Lecture Note 17

• Remark.
  We begin Chapter 11, the last chapter in Topology Without Tears. There are 3 ways to think about quotient spaces.

• Definition 90.
  A topological space \((Y, \tau_1)\) is said to be a quotient space of a topological space \((X, \tau)\) if there exists a surjective mapping \(f : X \to Y\) such that \(U \in \tau_1 \iff f^{-1}(U) \in \tau\). The map \(f\) is said to be a quotient map.

• Alternative definition.
  If \((X, \tau)\) is a topological space, \(Y\) is a set and \(f : X \to Y\) is a surjective map, then the quotient topology \(\tau_f\) is defined by
  \[
  \tau_f = \{ U \subset Y \mid f^{-1}(U) \in \tau \}.
  \]
  In this sense, the quotient topology is the strongest topology on \(Y\) such that \(f\) is continuous.

• Definition 94.
  Let \((X, \tau)\) be a topological space and \(\sim\) any equivalence relation on \(X\). Let \(Y\) be the set of all equivalence classes of \(\sim\). We denote \(Y = X/\sim\) and \(Y\) is said to be an identification space.

• Examples of quotient spaces.
  (a) \(S^1\) is a quotient space of \([0, 1]\), both with Euclidean topology.
  We choose the function \(f(x) = (\cos 2\pi x, \sin 2\pi x)\). We need to check three conditions for \(f\) being a quotient map.
  (i) Surjective: \(f\) is surjective since the domain contains \([0, 1]\).
  (ii) \((\Rightarrow)\): Consider \(U\) an open set of \(S^1\). Then by continuity \(f^{-1}(U)\) is open in \([0, 1]\).
  (iii) \((\Leftarrow)\): Let \(U\) be a subset of \(S^1\) where \(f^{-1}(U)\) is open. Note that \([0, \varepsilon)\) is an open set in \([0, 1]\) that does not map to an open set in \(S^1\). However, any preimage that contains 0 must also contain 1. Let \(x \in U\). Since \(f(x) \in f^{-1}(U)\), there exists \(B_\delta(f(x)) \subset f^{-1}(U)\). Therefore \(B_\delta(x) \subset U\) which implies that \(U\) is open.
  (b) \(S^1\) is a quotient space of \(\mathbb{R}\), both with Euclidean topology. This follows from a similar argument above.
  (c) The real projective space \(\mathbb{R}P^{n-1}\) is the space of straight lines through the origin in \(\mathbb{R}^n\). It is the quotient space of \(S^{n-1}\) by identifying antipodal points.
  (d) The discrete space \(Z\) is a quotient space of the Sorgenfrey line via the floor function \(f(x) = \lfloor x \rfloor = \text{the largest integer no greater than } x\).
  (i) Surjective: \(f\) is surjective since the domain contains \(Z\).
  (ii) \((\Rightarrow)\): Consider \(U\) an open set of \(Z\). The preimage \(f^{-1}(U)\) is a union of sets \([n, n+1]\) which are open in the Sorgenfrey line.
  (iii) \((\Leftarrow)\): Any subset in \(Z\) is open since it is discrete.
  (e) Real line with two origins. Consider the two horizontal lines in \(\mathbb{R}^2\) described by \(y = 0\) and \(y = 1\). We look at the quotient space obtained by identifying \((x, 0)\) and \((x, 1)\) for all \(x \neq 0\). The space essentially has both \((0, 0)\) and \((0, 1)\) as origin. It is a \(T_1\) space that is not \(T_2\).
• Proposition 91.
  Let \( f \) be a continuous mapping of a compact space \((X, \tau)\) onto a Hausdorff space \((Y, \tau_1)\). Then \( f \) is a quotient mapping.
  
  Proof.
  We make an observation before the proof. A map \( f \) is a quotient mapping if and only if \( f \) is surjective and a subset \( A \subset Y \) is closed \( \iff f^{-1}(A) \) is closed in \( X \).

  Since \( f \) is already continuous and surjective, we only need to check that \( A \subset Y \) with \( f^{-1}(A) \) closed have to be closed. Using previous proposition, \( f^{-1}(A) \) is compact in \( X \).

  Using previous proposition, \( f(f^{-1}(A)) \) is compact in \( Y \). Using previous proposition, \( A = f(f^{-1}(A)) \) is closed in \( Y \).

• Definition 92.
  A mapping \( f : (X, \tau) \to (Y, \tau_1) \) is said to be an open mapping if for every open subset \( A \subset X \), \( f(A) \) is open in \( (Y, \tau_1) \). A mapping \( f : (X, \tau) \to (Y, \tau_1) \) is said to be a closed mapping if for every closed subset \( A \subset X \), \( f(A) \) is closed in \( (Y, \tau_1) \).

• Remark.
  An important exercise is to show the independence between the properties continuity, open mapping, and closed mapping. We will talk about more examples next time.

• Proposition 93.
  If \( f : (X, \tau) \to (Y, \tau_1) \) is a surjective, continuous and open mapping, then it is also a quotient mapping. If \( f : (X, \tau) \to (Y, \tau_1) \) is a surjective, continuous and closed mapping, then it is also a quotient mapping.
  
  Proof.
  Left as exercise.

• Remark.
  We use Definition 94 to introduce the \( T_0 \)-identification space of \((X, \tau)\). We define an equivalence relation \( a \sim b \) for \( a, b \in X \) if and only if every open set that contains either \( a \) or \( b \) contains the other. Then \( Y = X/\sim \) is a \( T_0 \) space. For example, if \((X, \tau)\) is the indiscrete topology, then \( Y \) is a one point set.

• Definition 95.
  The cone space \((CX, \tau_1)\) over \((X, \tau)\) is the quotient space defined by a equivalence relation \((x, 1) \sim (y, 1)\) in \( X \times [0, 1] \) for all \( x, y \in X \). The suspension space \((SX, \tau_1)\) over \((X, \tau)\) is the quotient space defined by a equivalence relation \((x, -1) \sim (y, -1)\) and \((x, 1) \sim (y, 1)\) in \( X \times [-1, 1] \) for all \( x, y \in X \).

• Proposition 96.
  The cone over \( S^n \) is homeomorphic to the closed ball of dimension \( n + 1 \). The suspension over \( S^n \) is homeomorphic to \( S^{n+1} \).