Lecture Note 14

- **Theorem 79.**
  Let $A$ be a subset of $\mathbb{R}$ with the Euclidean topology. Then $A$ is compact if and only if $A$ is closed and bounded.
  
  *Sketch of proof.*
  
  Step 1: Compact implies closed since $\mathbb{R}$ is a $T_2$-space.
  
  Step 2: Compact implies bounded follows from considering the open cover $\{(−n, n)\}_{n \in \mathbb{N}}$.
  
  Step 3: Closed and bounded implies compactness because closed intervals are compact and closed subsets of compact spaces are compact. □

- **Theorem 80.**
  Let $f : (X, \tau) \to (\mathbb{R}, \text{Euclidean})$ be a continuous function. If $X$ is compact, then there exists $y, z \in X$ such that $f(y) \leq f(x) \leq f(z)$ for all $x \in X$.
  
  *Sketch of proof.*
  
  Step 1: $f(X)$ is compact in $\mathbb{R}$, hence closed and bounded.
  
  Step 2: Bounded subsets in $\mathbb{R}$ have a least upper bound and a greatest lower bound.
  
  Step 3: Closed subsets contain the limit points, which includes the least upper bound and the greatest lower bound. □

- **Definition 81.**
  A subset $A$ in $(X, \tau)$ is said to be countably compact if every countable open covering of $A$ has a finite subcovering. The subset $A$ is said to be pseudocompact if every continuous function $f : A \to \mathbb{R}$ is bounded. The subset $A$ is said to be sequentially compact if every sequence in $A$ has a subsequence and a point $x$ such that every open set containing $x$ also contains the tail of the subsequence.

- **Theorem 82.**
  Let $(X, \tau)$ be a metrizable topological space. The following are equivalent.
  
  (1) $X$ is compact.
  
  (2) $X$ is countably compact.
  
  (3) $X$ is pseudocompact.
  
  (4) $X$ is sequentially compact.
  
  *Remark.*
  
  The last thing to mention before we move on is compactification. There are two common methods. The one-point compactification adds a point with open sets containing that point to be complements of compact closed sets. The Stone-Čech compactification uses continuous functions into $[0, 1]$.
  
  This is the end of Chapter 7. Chapter 8 is on finite products.

- **Definition 83.**
  Let $(X_i, \tau_i)_{i=1,\ldots,n}$ be topological spaces. The product topology $\tau$ on $X_1 \times \cdots \times X_n$ is the topology with the basis $\{O_1 \times \cdots \times O_n\}_{O_i \in \tau_i}$.

- **Definition 90.**
  A topological space $(Y, \tau_1)$ is said to be a quotient space of a topological space $(X, \tau)$ if there exists a surjective mapping $f : X \to Y$ such that $U \in \tau_1 \iff f^{-1}(U) \in \tau$.