Lecture Note 13

• Definition 73.
  Let $A$ be a subset of a topological space $(X, \tau)$. Then $A$ is said to be compact if for every open cover of $A$ there exists a finite subcover of $A$.

• Example.
  (a) Any finite set $A$ is compact in $(X, \tau)$.
  \begin{proof}
  Let $\{O_j\}_{j \in J}$ be an open cover of $A$. For each $x \in A$ we can pair it up with an open set in the open cover that contains $x$. We name the open set $O_x$. Observe that $\{O_x\}_{x \in A}$ is an open cover of $A$. Since $A$ is finite and $O_x$’s are open sets in the open cover we began with, $\{O_x\}_{x \in A}$ is a finite subcover of $A$.
  \end{proof}

  (b) Any subset $A \subset X$ with the finite-closed topology is compact.
  \begin{proof}
  Let $\{O_j\}_{j \in J}$ be an open cover of $A$. Let $O'$ be a nonempty open set in the open cover. By the definition of finite-closed topology, $X \setminus O'$ is a finite set. Therefore $A \setminus O'$ is a finite set. Let $A \setminus O' = \{x_1, \ldots, x_n\}$. Since $\{O_j\}_{j \in J}$ is an open cover of $A$, there exists $O_1, \ldots, O_n$ in the open cover such that $x_i \in O_i$ for all $i = 1, \ldots, n$. The collection of open sets $O', O_1, \ldots, O_n$ is a finite subcover of $A$.
  \end{proof}

• Remark.
  Let $A$ be a subset of $X$ in a topological space $(X, \tau)$. Then $A$ is compact if and only if $(A, \tau_A)$ as a subspace with the subspace topology $\tau_A$ is compact. This allows us to always work with the subspace topology if we want to show a subset is compact.

• Proposition 74.
  The closed interval $[0, 1]$ is compact in $\mathbb{R}$ with the Euclidean topology.
  \begin{proof}
  Let $\{O_j\}_{j \in J}$ be an open cover of $[0, 1]$. We can obtain a subcover $\{O_x\}_{x \in [0, 1]}$ such that $O_x$ contains $x$. Furthermore, there exist open intervals $U_x$ such that $x \in U_x \subseteq O_x$. Observe that if there exists a finite subcollection $U_{x_1}, \ldots, U_{x_n}$ that covers $[0, 1]$, then $O_{x_1}, \ldots, O_{x_n}$ will be a finite subcover of $[0, 1]$.
  Consider the set
  \[ Z = \{y \in [0, 1] \mid [0, y] \text{ can be covered by a finite collection of } \{U_x\}_{x \in [0, 1]}\}. \]
  Since the singleton $\{0\}$ is covered by $U_0$, $0 \in Z$ and $Z$ is nonempty. For an arbitrary $y \in Z$, there exists $\varepsilon > 0$ such that $B_\varepsilon(y) \subseteq U_y$. Adding $U_y$ to the finite cover of $[0, y]$ implies that $y + \varepsilon/2 \in Z$. Hence $Z$ is open. For an arbitrary $y \in [0, 1] \setminus Z$, there exists $\delta > 0$ such that $B_\delta(y) \subseteq U_y$. Therefore $y - \delta/2$ cannot be in $Z$ otherwise adding $U_y$ would result in $y \in Z$. Hence $[0, 1] \setminus Z$ is also open. We have disjoint open sets whose union covers $[0, 1]$, by connectedness, $Z = [0, 1]$.
  \end{proof}

• Proposition 75.
  Let $f : (X, \tau) \to (Y, \tau_1)$ be a continuous surjective map. If $(X, \tau)$ is compact, then $(Y, \tau_1)$ is compact.
  \begin{proof}
  
  \end{proof}
Let \( \{O_j\}_{j \in J} \) be an open cover of \( Y \). By continuity, \( \{f^{-1}(O_j)\}_{j \in J} \) is an open cover of \( X \). By compactness, there exists a finite subcover \( \{f^{-1}(O_{j_k})\}_{k=1,\ldots,n} \). Since \( f \) is surjective, \( \{O_{j_k}\}_{k=1,\ldots,n} \) is a finite subcover of \( Y \).

**Corollary 76.**
Compactness is a topological property.

**Proposition 77.**
Let \((X, \tau)\) be a compact topological space. If \( A \subset X \) is closed, then \( A \) is compact.

*Proof.*
Let \( \{O_j\}_{j \in J} \) be an open cover of \( A \). Then \( \{O_j\}_{j \in J} \) along with the open set \( X \setminus A \) is an open cover of \( X \). By compactness, there exists a finite subcover. The finite subcover covers \( A \), but it might include \( X \setminus A \). The open sets in the finite subcover that is not disjoint from \( A \) is a finite subcover that does not include \( X \setminus A \). \( \square \)

**Proposition 78.**
Let \((X, \tau)\) be a \( T_2 \) topological space. If \( A \subset X \) is compact, then \( A \) is closed.

*Proof.*
Let \( x \) be an arbitrary point in \( X \setminus A \). For each point \( a \in A \), by the property of \( T_2 \), there exists disjoint open sets \( U_a \) and \( V_a \) such that \( a \in U_a \) and \( x \in V_a \). The collection of open sets \( \{U_a\}_{a \in A} \) is an open cover of \( A \). By compactness, there exists \( a_1, \ldots, a_n \) such that \( \{U_{a_i}\}_{i=1,\ldots,n} \) is a finite subcover of \( A \). The finite intersection of the open sets \( \{V_{a_i}\}_{i=1,\ldots,n} \) is an open set that contains \( x \) while being disjoint from \( A \). Therefore \( X \setminus A \) is open and \( A \) is closed. \( \square \)

**Theorem 79.**
Let \( A \) be a subset of \( \mathbb{R} \) with the Euclidean topology. Then \( A \) is compact if and only if \( A \) is closed and bounded.

**Theorem 80.**
Let \( f : (X, \tau) \to (\mathbb{R}, \text{Euclidean}) \) be a continuous function. If \( X \) is compact, then there exists \( y, z \in X \) such that \( f(y) \leq f(x) \leq f(z) \) for all \( x \in X \).