Lecture Note 11

• **Proposition 63.**
  If $x_n \to x$ and $x_n \to y$ in a metric space $(X, d)$, then $x = y$.
  
  *Proof.*
  By the definition of $(X, d)$ being a metric space, it suffices to show that $d(x, y) = 0$.
  If $d(x, y) = \delta \neq 0$, consider $B_{\delta/2}(x)$. By definition of $x_n \to x$, there exists $n_0$ such that $n \geq n_0 \Rightarrow x_n \in B_{\delta/2}(x)$.
  
  Consider $B_{\delta/2}(y)$. By definition of $x_n \to y$, there exists $n'_0$ such that $n \geq n'_0 \Rightarrow x_n \in B_{\delta/2}(y)$.
  
  Let $N = \max(n_0, n'_0)$. Then $x_N \in B_{\delta/2}(x) \cap B_{\delta/2}(y) = \emptyset$ by triangular inequality. Therefore a contradiction to our assumption, $d(x, y) = 0 \Rightarrow x = y$. □

• **Proposition 64.**
  Let $(X, d)$ be a metric space. A subset $A$ of $X$ is closed in $(X, \tau)$ induced by $(X, d)$ if and only if every convergent sequence of points in $A$ converges to a point in $A$.
  
  *Proof.*
  We need to prove both directions.
  
  $(\Rightarrow)$: We have a subset $A$ of $X$ is closed in $(X, \tau)$ induced by a metric space $(X, d)$. Let \{ $x_n$ \} be a sequence in $A$ and $x_n \to x$ with $x \in X$. We want to show that $x$ is in $A$.
  
  Consider any $U \in \tau$ with $x \in U$. By the definition of induced topology from the metric $d$, there exists $\varepsilon > 0$ such that $B_{\varepsilon}(x) \subset U$. By the definition of $x_n \to x$, there exists $n_0$ such that whenever $n \geq n_0$, we have $x_n \in B_{\varepsilon}(x)$. Alternatively we have just shown that any open set containing $x$ contains some point $x_n \in A$. Therefore if $x \notin A$ then $x$ is a limit point of $A$. However, $A$ is closed implies that $A$ contains all limit points of $A$, which leads to $x \in A$.
  
  $(\Leftarrow)$: We are given that every convergent sequence of points in $A$ converges to a point in $A$. We want to show that $A$ is closed.
  
  Suppose that $A$ is not closed. Then there exists $x \notin A$ that is a limit point of $A$. We use the fact that the open balls form a basis of $\tau$. Consider $B_{1/n}(x)$ for all $n \geq 1$.
  
  By the definition of $x$ being a limit point of $A$ and $x \notin A$, there exists $x_n \in B_{1/n}(x)$ for each $n$ with $x_n \in A$. For all $\varepsilon > 0$, we can pick $n_0 \geq 1/\varepsilon$ to satisfy the definition of $x_n \to x$. Since every convergent sequence of points in $A$ converges to a point in $A$, the point $x$ is in $A$ contradicting our assumption. Therefore $A$ is closed. □

• **Proposition 65.**
  Let $(X, d)$ and $(Y, d_1)$ be metric spaces and $f$ a mapping of $X$ into $Y$. Let $\tau$ and $\tau_1$ be the topologies determined by $d$ and $d_1$, respectively. Then $f : (X, \tau) \to (Y, \tau_1)$ is continuous if and only if
  
  $x_n \to x \Rightarrow f(x_n) \to f(x)$.
  
  *Proof.*
  We need to prove both directions.
  
  $(\Rightarrow)$: Let $f : (X, \tau) \to (Y, \tau_1)$ be continuous and $x_n \to x$ in $(X, d)$. We consider $B_{\varepsilon}(f(x))$. By the definition of continuity, the preimage $U = f^{-1}(B_{\varepsilon}(f(x)))$ is an open set and $x \in U$. Since open balls form a basis of $\tau$, there exists $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(x)$
is a subset of $U$. We use the fact that $x_n \to x$ to find $n_0$ such that whenever $n \geq n_0$ implies $x_n \in B_{\varepsilon_1}(x)$. This automatically implies that $f(x_n) \in B_{\varepsilon}(f(x))$, which shows that $f(x_n) \to f(x)$ in $(Y, d_1)$.

$(\Leftarrow)$: We know that for every convergent sequence $x_n \to x$ in $(X, d)$, the sequence \{\f(x_n)\} converges to $f(x)$ in $(Y, d_1)$. Let $A$ be a closed set in $(Y, \tau_1)$. Consider $f^{-1}(A)$ as a subset of $(X, \tau)$. By Proposition 64 we know that if every convergent sequence of points in $f^{-1}(A)$ converges to a point in $f^{-1}(A)$, then $f^{-1}(A)$ is closed. Suppose that $\{x_n\} \subset f^{-1}(A)$ and $x_n \to x$ in $(X, d)$. By our assumption we know that $f(x_n) \to f(x)$ in $(Y, d_1)$. Also since $A$ is closed, $f(x) \in A$. Therefore $x \in f^{-1}(A)$ which implies that $f^{-1}(A)$ is closed. Hence $f$ is continuous. \hfill \Box