Lecture Note 10

• Definition 56.
Let \(X\) be a non-empty set and \(d\) a real-valued function defined on \(X \times X\) such that for \(a, b \in X\):

(i) \(d(a, b) \geq 0\) and \(d(a, b) = 0\) if and only if \(a = b\);
(ii) \(d(a, b) = d(b, a)\); and
(iii) \(d(a, c) \leq d(a, b) + d(b, c)\), for all \(a, b, c\) in \(X\).

Then \(d\) is said to be a metric on \(X\), \((X, d)\) is called a metric space and \(d(a, b)\) is referred to as the distance between \(a\) and \(b\).

• Examples.
(a) Discrete metric on a space: \(d(a, b) = 1\) if and only if \(a \neq b\).
(b) \(p\)-norm on \(\mathbb{R}^n\).
\[d((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \left(\sum_{j=1}^{n} |a_j - b_j|^p\right)^{1/p}\]
(c) Generalizations: One way to see \(\mathbb{R}^n\) is as real-valued functions over a finite set. We are also interested in the case when the domain is \(\mathbb{N}\) (which are called sequences) and when the domain is \(\mathbb{R}\) (functions).
(d) Metric on graphs.
A graph is a set of vertices and a set of edges. The usual metric given by the graph between two vertices is \(d(v_1, v_2) =\) the least number of edges used in a path from \(v_1\) to \(v_2\). The metric can be further generalized by specifying weights on edges.
(e) Metric on groups.
Given a presentation of a group, we can place a metric on the group. The distance between two elements is given by the length of the word of generators that takes one to the other. We can also generate a Cayley graph first and then use the graph metric.

• Lemma 57.
Let \((X, d)\) be a metric space and \(a, b\) points of \(X\). Further, let \(\delta_1, \delta_2\) be positive real numbers. If \(c \in B_{\delta_1}(a) \cap B_{\delta_2}(b)\), then there exists a \(\delta > 0\) such that \(B_\delta(c)\) is a subset of \(B_{\delta_1}(a) \cap B_{\delta_2}(b)\).

Proof.
Let \(\delta_a = d(a, c)\) and \(\delta_b = d(b, c)\). Since \(c \in B_{\delta_1}(a) \cap B_{\delta_2}(b)\), \(\delta_a < \delta_1\) and \(\delta_b < \delta_2\). Let \(\delta = \min\{\delta_1 - \delta_a, \delta_2 - \delta_b\}\). By triangular inequality, \(B_\delta(c)\) is a subset of \(B_{\delta_1}(a) \cap B_{\delta_2}(b)\). \(\square\)

• Corollary 58.
Let \((X, d)\) be a metric space. Then the collection of open balls in \((X, d)\) is a basis for a topology \(\tau\) on \(X\).

• Definition 59.
Let \((X, d)\) be a metric space. The topology with the collection of open balls in \((X, d)\) as basis is called the topology induced by the metric \(d\) on \(X\).

• Remark.
Let \((X,d)\) be a metric space. Does \(d\) induce a metric space on a subset \(A \subset X\)? Does the induced topology of \(d_A\) correspond to the subspace topology \(\tau_A\)?

- **Definition 60.**
  Metrics on a set \(X\) are said to be **equivalent** if they induce the same topology on \(X\).

- **Proposition 61.**
  Let \((X,d)\) be a metric space and \(\tau\) the topology induced on \(X\) by \(d\). Then \((X,\tau)\) is a \(T_2\)-space (Hausdorff space).

- **Definition 62.**
  Let \((X,d)\) be a metric space and \(\{x_n\}\) a sequence of points in \(X\). Then the sequence is said to **converge to** \(x \in X\) if given any \(\varepsilon > 0\) there exists an integer \(n_0\) such that for all \(n \geq n_0\), \(d(x,x_n) < \varepsilon\). This is denoted by \(x_n \to x\).

- **Remark.**
  A different choice of definition of a converging sequence is to say that every open set around \(x\) contains the tail of a sequence. However, this will raise situations when a sequence converges to more than one point. See the story about sequences and nets for more details.

- **Proposition 63.**
  If \(x_n \to x\) and \(x_n \to y\) in a metric space \((X,d)\), then \(x = y\).

- **Proposition 64.**
  Let \((X,d)\) be a metric space. A subset \(A\) of \(X\) is closed in \((X,\tau)\) induced by \((X,d)\) if and only if every convergent sequence of points in \(A\) converges to a point in \(A\).

- **Proposition 65.**
  Let \((X,d)\) and \((Y,d_1)\) be metric spaces and \(f\) a mapping of \(X\) into \(Y\). Let \(\tau\) and \(\tau_1\) be the topologies determined by \(d\) and \(d_1\), respectively. Then \(f : (X,\tau) \to (Y,\tau_1)\) is continuous if and only if
  \[
  x_n \to x \Rightarrow f(x_n) \to f(x).
  \]