Lecture Note for Differential Geometry & Topology I, Fall 2015

• 8/24/2015
• This course assumes knowledge in linear algebra, real analysis, and topology.
• There is a difference between the field of Differential Geometry and the field of Manifolds Theory.

Definition 1.
Suppose $M$ is a topological space (recall the definition of a topological space). We say that $M$ is a topological manifold of dimension $n$ if it has the following properties:

1. $M$ is a Hausdorff space. Recall the definition of Hausdorff.
2. $M$ is second countable. Recall the definition of second countable.
3. $M$ is locally Euclidean of dimension $n$. For every point $p$ in $M$, there exists an open neighborhood $N_p$ and a homeomorphism $\varphi$ mapping $N_p$ to an open set in $\mathbb{R}^n$.

Examples.
1. Integers $\mathbb{Z}$ is a topological manifold of dimension 0.
2. The union of the $x$-axis and the $y$-axis is not a topological manifold because the origin does not satisfy the locally Euclidean condition.
3. $\mathbb{R}^n$, $S^n$, etc.
4. The M"obius band is not a topological manifold because the boundary points do not satisfy the locally Euclidean condition.

Theorem 2.
(Topological Invariance of Dimension) Let $m \neq n$. A topological manifold of dimension $m$ cannot be homeomorphic to a topological manifold of dimension $n$.

Definition 3.
A coordinate chart is a pair $(U, \varphi)$ such that $U$ is an open subset of $M$ and $\varphi$ is a homeomorphism from $U$ to an open set in $\mathbb{R}^n$. The coordinate chart induces a local coordinate at $p$, that is, $\varphi(p) = (x^1(p), \ldots, x^n(p))$.

Definition 4.
A function $F : U \to V$, where $U$ open in $\mathbb{R}^n$ and $V$ open in $\mathbb{R}^m$, is called smooth if all partial derivatives of $f^j$ are continuous for $F = (f^1, \ldots, f^m)$. We say $F$ is a diffeomorphism if $F$ is bijective, smooth, and $F^{-1}$ is smooth.

There are smooth homeomorphisms that are not diffeomorphisms.

Definition 5.
A smooth manifold is a pair $(M, \mathcal{A})$. $M$ is a topological manifold and $\mathcal{A}$ is a smooth structure.

By a smooth structure $\mathcal{A}$, we mean a collection of coordinate charts $(U_i, \varphi_i)$ such that
1. The sets $\{U_i\}$ form an open cover of $M$
2. For any $U_i$ and $U_j$ such that $U_i \cap U_j \neq \emptyset$, the transition map $\varphi_j \circ \varphi_i^{-1}$ is smooth.

Remark.
For canonical reasons, we often assume that a smooth structure is a maximal smooth atlas.

People also study various different structures. Just replace smooth by other types of functions.

**Proposition 6.**

Every smooth atlas is contained in a unique maximal smooth atlas.

*Proof:* (next time)

**Examples.**

1. 0-dimensional manifolds: discrete sets
2. 1-dimensional manifolds: circles and open intervals

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**8/26/2015**

**Recall** the definition of a smooth manifold. A topological manifold is a smooth manifold if there exists a smooth atlas. A smooth atlas consists of an open cover along with a map from each open set into \( \mathbb{R}^n \). The maps pairwise satisfy the condition that the transition map is a smooth map from \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

**Proof of Proposition 6.**

We want to show that every smooth atlas is contained in a unique maximal smooth atlas. First we need to prove existence of maximal smooth atlases that contains the given atlas. Then we need to show uniqueness to finish the proof.

Let \( \mathcal{A} \) be a smooth atlas over \( M \), a topological \( n \)-manifold. We define

\[
\overline{\mathcal{A}} := \{ (V,\psi) \mid U \text{ open and } \phi \text{ is smoothly compatible with } \mathcal{A} \}.
\]

We check that \( \overline{\mathcal{A}} \) is actually a smooth atlas by considering the transition maps. Let \( (V_1,\psi_1) \) and \( (V_2,\psi_2) \) be two charts in \( \overline{\mathcal{A}} \). If \( V_1 \cap V_2 \neq \emptyset \), then there exists \( p \in V_1 \cap V_2 \). Since \( \mathcal{A} \) is an atlas, there exists \( (U,\phi) \in \mathcal{A} \) such that \( p \in U \). The transition map \( \psi_1 \circ \psi_2^{-1} \) is smooth at \( p \) since it can be written as \( (\phi \circ \psi_1^{-1})^{-1} \circ (\phi \circ \psi_2^{-1}) \) which is a composition of maps smooth at \( p \) by definition. In fact, \( \overline{\mathcal{A}} \) is a maximal smooth atlas because any chart that can be added to the atlas must be smoothly compatible with \( \mathcal{A} \), hence in the atlas already. This proves existence.

To show uniqueness, we suppose that \( \mathcal{B} \) is a maximal smooth atlas containing \( \mathcal{A} \). By definition the charts in \( \mathcal{B} \) are smoothly compatible with \( \mathcal{A} \), \( \mathcal{B} \subset \overline{\mathcal{A}} \). Now by maximality, we obtain \( \overline{\mathcal{A}} = \mathcal{B} \). Hence we are done.

**Exercise.**

\( \mathcal{A} \) and \( \mathcal{B} \) determine the same smooth structure if and only if \( \mathcal{A} \cup \mathcal{B} \) is a smooth atlas.

**Examples.**

1. 0-dimensional manifolds. If \( M \) is a topological manifold of dimension 0, then \( M \) has to be a discrete countable set of points. Any atlas is smooth.

2. 1-dimensional manifolds. If \( M \) is a connected topological manifold of dimension 1, then \( M \) has to be homeomorphic to either \( \mathbb{R} \) or the circle \( S^1 \). Here we consider two smooth atlases on \( S^1 \).

Angle function: We use two charts on \( S^1 \). One of the open sets maps to \((0,2\pi)\) and the other one maps to \((−\pi,\pi)\) via angles in the usual sense. The transition maps between the two charts are simply translation functions \( f(x) = x + c \) in \( \mathbb{R} \).
Stereographic projection: By removing the point \( N(0, 1) \) or the point \( S(0, -1) \), we can project the rest of the circle on to the \( x \)-axis via stereographic projection. More explicitly, we map \((x, y)\) to \(\frac{x}{1-y} \) and \(\frac{x}{1+y} \) under the two maps. The stereographic projection extends naturally to \( n \)-dimensional spheres.

(3) \( \mathbb{R}^n \) is a smooth manifold with a single chart where the map is the identity map.

(4) Finite dimensional vector space \( V \) is identified with \( \mathbb{R}^n \) naturally once a basis is chosen. The transition map between different choices of basis is exactly the change of basis matrix map.

(5) The space of linear maps between two finite dimensional vector spaces \( L(V, W) \).
Recall from linear algebra that this is identified with \( \mathbb{R}^{mn} \), where \( m \) and \( n \) are the dimensions of \( V \) and \( W \).

(6) The space of \( m \times n \) matrices is also identified with \( \mathbb{R}^{mn} \).

(7) Graphs of smooth functions. The graph of a function \( F : M \to N \) is the set \( \{(x, y) \in M \times N \mid y = F(x)\} \). The graph is homeomorphic to \( M \) under the projection map as long as \( F \) is continuous. If \( F \) is smooth (defined later), then we can obtain a smooth atlas on the graph induced by a smooth atlas on \( M \).
Exercise: Think about the cases when \( F \) is not smooth, does that imply that the induced atlas is not smooth?

(8) Level sets of smooth functions. Let \( F : M \to N \) be a continuous map. Under conditions that will be introduced later, the level set \( F^{-1}(c) \) for some \( c \in N \) is a topological manifold of codimension 1. The steps to show that the level set is in fact a smooth manifold is more complicated and we will discuss this later.

(9) Real projective space and Grassmannians. We can consider the space of \( k \)-dimensional vector subspaces inside an \( n \)-dimensional vector space. When \( k = 1 \), we call this space the real projective space, which can alternatively be obtained by quotienting the \( n \)-sphere by the antipodal map. More on these will come later.

(10) Products of smooth manifolds. Similar to how product topology is defined, we can use smooth atlases on each component to obtain a smooth atlas for the product space.

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**Lemma 7.** Let \( M \) be a set, and suppose we are given a collection \( \{U_\alpha\} \) of subsets of \( M \) together with maps \( \varphi_\alpha : U_\alpha \to \mathbb{R}^n \), such that the following properties are satisfied:

(a) For each \( \alpha \), \( \varphi_\alpha \) is a bijection between \( U_\alpha \) and an open subset of \( \mathbb{R}^n \).
(b) For each \( \alpha \) and \( \beta \), the set \( \varphi_\alpha(U_\alpha \cap U_\beta) \) is open in \( \mathbb{R}^n \).
(c) Whenever \( U_\alpha \cap U_\beta \neq \emptyset \), the map \( \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi(U_\alpha \cap U_\beta) \) is smooth.
(d) Countably many of the sets \( U_\alpha \) cover \( M \).
(e) Whenever \( p, q \) are distinct points in \( M \), either there exists some \( U_\alpha \) containing both \( p \) and \( q \) or there exist disjoint sets \( U_\alpha, U_\beta \) with \( p \in U_\alpha \) and \( q \in U_\beta \).

Then \( M \) has a unique smooth manifold structure such that each \( (U_\alpha, \varphi_\alpha) \) is a smooth chart.

**Definition 8.** An \( n \)-dimensional topological manifold with boundary is a second countable Hausdorff space \( M \) in which every point has a neighborhood homeomorphic either to an open subset of \( \mathbb{R}^n \) or to a (relatively) open subset of \( H^n = \{(x^1, \ldots, x^n) \mid x^n \geq 0\} \).
Recall the definition of a topological manifold with boundary. An $n$-dimensional topological manifold with boundary is a second countable Hausdorff space $M$ in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a (relatively) open subset of $H^n = \{(x^1, \ldots, x^n) \mid x^n \geq 0\}$. We can further specify that in the second case the point is mapped to the origin.

If $p \in M$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$, we say that $p$ is an interior point and denote it by $p \in \text{Int}(M)$.

If $p \in M$ has a neighborhood homeomorphic to an open subset of $H^n$ with $p$ being mapped to the origin, we say that $p$ is a boundary point and denote it by $p \in \partial M$.

Note that a topological manifold with boundary is technically not a topological manifold.

**Theorem 9.** (Topological invariance of the boundary) For a topological manifold with boundary, $\text{Int}(M)$ is disjoint from $\partial M$.

**Proposition 10.** Let $M$ be a topological manifold with boundary of dimension $n$.

(a) The interior of $M$ is open in $M$ and is a topological manifold of dimension $n$.

(b) The boundary of $M$ is closed in $M$ and is a topological manifold of dimension $(n - 1)$.

(c) $M$ is a topological manifold if and only if $\partial M = \emptyset$.

**Definition 11.** Let $M, N$ be smooth manifolds, and let $F : M \to N$ be a map. We say that $F$ is a smooth map if for every $p \in M$, there exist smooth charts $(U, \varphi)$ containing $p$ and $(V, \psi)$ containing $F(p)$ such that $F(U) \subset V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$.

We are particularly interested in the case when $N = \mathbb{R}$. We use $C^\infty(M)$ to denote the space of smooth functions $F : M \to \mathbb{R}$. $C^\infty(M)$ is a vector space over $\mathbb{R}$.

**Proposition 12.** Every smooth map is continuous.

*Proof.*

Let $F : M \to N$ be a smooth map. For $p \in M$, there exist smooth charts $(U, \varphi)$ containing $p$ and $(V, \psi)$ containing $F(p)$ such that $F(U) \subset V$. $F$ is continuous at $p$ because it is the composition $\psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi$ of three continuous functions.

**Proposition 13.** Let $M$ and $N$ be smooth manifolds with or without boundary, and let $F : M \to N$ be a map. If every point $p \in M$ has a neighborhood $U$ such that the restriction $F|_U$ is smooth, then $F$ is smooth.

**Examples.**

1. Constant maps. Let $F : M \to N$ be a constant map where $F(p) = c$ for all $p \in M$. $F$ is a smooth map since the map $\psi \circ F \circ \varphi^{-1}$ is a constant map.

2. Identity maps. Let $F : M \to M$ be the identity map $F(p) = p$. We choose the same chart in the domain and the range, then the map $\psi \circ F \circ \varphi^{-1}$ is the identity map, hence smooth.

3. The map from $\mathbb{R} \to S^1$ given by $f(x) = e^{2\pi ix}$ is smooth. This map extends to the map from $\mathbb{R}^n$ to the $n$-torus.

4. The inclusion map from $S^1 \to \mathbb{R}^2$ is a smooth map. The same is true for the inclusion map from the $n$-sphere to $\mathbb{R}^{n+1}$. Question: How to determine if the inclusion map of an object in $\mathbb{R}^n$ is smooth?
Projection maps from products of smooth manifolds to a component smooth manifold are smooth maps.

- **Remark.** Recall that $F : M \to N$ is a diffeomorphism if $F$ is bijective, smooth, and $F^{-1}$ is smooth.

**Theorem.** A nonempty smooth manifold of dimension $m$ cannot be diffeomorphic to an $n$-dimensional smooth manifold unless $m = n$.

**Proof.**
Consider the derivative of $\psi \circ F \circ \varphi^{-1}$ and its inverse. They have to be matrices inverse of each other, hence $m$ has to be equal to $n$.

- **Proposition 14.** (Properties of diffeomorphisms).
  (a) Every composition of diffeomorphisms is a diffeomorphism.
  (b) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.
  (c) Every diffeomorphism is a homeomorphism and an open map.
  (d) The restriction of a diffeomorphism to an open submanifold is a diffeomorphism onto its image.
  (e) “Diffeomorphic” is an equivalence relation on the class of all smooth manifolds.

**9/2/2015**

- **Lemma 15.** The function $f : \mathbb{R} \to \mathbb{R}$ defined by $e^{-1/t}$ for $t > 0$ and $0$ for $t \leq 0$ is smooth.

- **Lemma 16.** There exists a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ such that $h(x) = 1$ for all $\|x\| \leq 1$ and $h(x) = 0$ for all $\|x\| > 2$.

  Hint: (1) Define $h$ radially. (2) Consider the fraction $\frac{f(t)}{f(t) + f(1-t)}$ for $f$ defined in Lemma 15.

- **Definition 17.** Let $f : M \to \mathbb{R}$ be a nonnegative function. The support of the function $f$ denoted by supp$(f)$ is the closed set $\{p \in M \mid f(p) \neq 0\}$.

- **Proposition 18.** Let $M$ be a smooth manifold and $(U, \varphi)$ be a chart with $p \in U$. There exists a smooth function $F \in C^\infty(M)$ such that $F(M) \subset [0,1]$, supp$(F) \subset U$, and on a neighborhood of $p$ we have $F|_{N_p} = 1$.

- **Definition 19.** Let $M$ be a smooth manifold and $\{U_\alpha\}$ be an open cover of $M$. A partition of unity subordinate to $\{U_\alpha\}$ is the indexed set of smooth functions $\{f_\alpha\}$ such that the following properties hold.
  (a) The support supp$(f_\alpha) \subset U_\alpha$ for all $\alpha$.
  (b) For any point $p \in M$, $f_\alpha(p) \in [0,1]$ for all $\alpha$.
  (c) For any point $p \in M$, there exists a neighborhood of $p$ such that $N_p \cap$ supp$(f_\alpha)$ is nonempty for only a finite number of $\alpha$.
  (d) The sum $\sum f_\alpha$ is well-defined due to (c). The sum is equal to 1 at every $p$.

- **Theorem 20.** Let $M$ be a smooth manifold. Given any open cover $\{U_\alpha\}$, there exists a partition of unity subordinate to $\{U_\alpha\}$.

  **Proof.** (Next time.)

- **Looking ahead.**
In order to apply calculus to smooth functions over $M$, we need a sense of directions at any point $p$ on $M$. There are two common ways to define a tangent space at $p$. 

The tangent space at a point \( p \in M \), denoted \( T_pM \), is the set of all derivation at \( p \). A derivation at \( p \) is a linear map \( v : C^\infty(M) \rightarrow \mathbb{R} \) that satisfies
\[
v(fg) = f(p)v(g) + g(p)v(f) \quad \text{for all } f, g \in C^\infty(M).
\]
The derivation is essentially an analogue of directional derivatives.

The tangent space at a point \( p \in M \) is also the set of equivalence classes of smooth arcs through \( p \) (in the sense that \( \gamma : (-\varepsilon, \varepsilon) \rightarrow M \) with \( \gamma(0) = p \)). Two arcs \( \gamma_1 \) and \( \gamma_2 \) are equivalent if for any chart \((U, \varphi)\) at \( p \), \( \gamma_1 \circ \varphi^{-1} \) and \( \gamma_2 \circ \varphi^{-1} \) have the same first derivative at 0. This definition is more geometric but harder to work with due to the equivalence relation.

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**Theorem 20.** Let \( M \) be a smooth manifold. Given any open cover \( \{U_\alpha\} \), there exists a partition of unity subordinate to \( \{U_\alpha\} \).

*Proof.*

Recall the definition of partition of unity first.

A partition of unity subordinate to \( \{U_\alpha\} \) is the indexed set of smooth functions \( \{f_\alpha\} \) such that the following properties hold.

(a) The support \( \text{supp}(f_\alpha) \subset U_\alpha \) for all \( \alpha \).

(b) For any point \( p \in M \), \( f_\alpha(p) \in [0, 1] \) for all \( \alpha \).

(c) For any point \( p \in M \), there exists a neighborhood of \( p \) such that \( N_p \cap \text{supp}(f_\alpha) \) is nonempty for only a finite number of \( \alpha \).

(d) The sum \( \sum f_\alpha \) is well-defined due to (c). The sum is equal to 1 at every \( p \).

We will use a result from topology (proof can be found in Lee, Introduction to Smooth Manifolds).

Theorem: Given a topological manifold \( M \), open cover \( \mathcal{X} \), and a basis \( \mathcal{B} \), there exists a countable locally finite open refinement of \( \mathcal{X} \) using only sets in \( \mathcal{B} \).

An open refinement of \( \mathcal{X} \) is an open cover \( U_\alpha \) such that for each \( U_\alpha \) there exists \( V \in \mathcal{X} \) such that \( U_\alpha \subset V \). An open cover is locally finite if any point has an open neighborhood that intersects finitely many sets in the open cover.

Let \( \{B_\alpha\} \) be the set of open subsets of \( M \) such that for each \( B_\alpha \) there exists a homeomorphism \( \varphi_\alpha : B_\alpha \rightarrow \mathbb{R}^n \) such that the image of \( \varphi_\alpha \) is \( B_\alpha \circ \varphi(0) \), the open ball of radius \( r \), for some \( r \). This is a basis for the topology of \( M \).

By theorem, there exists \( \{B_\alpha\} \subset \{B_\alpha\} \) that is a countable locally finite open refinement of \( \{U_\alpha\} \). We make a further refinement such that each \( B_\alpha \) lies in some \( U_\alpha \). For each \( B_i \) we can construct a smooth function \( f_i : M \rightarrow \mathbb{R} \) such that \( \text{supp}(f_i) = B_i \) with \( f_i \) positive in \( B_i \). Now consider \( g_i(p) = f_i(p) / \sum f_i(p) \) for each \( p \in M \). The sum \( \sum f_i \) is well-defined and smooth since \( \{B_i\} \) is locally finite and each \( f_i \) is smooth.

Finally we define \( \alpha(i) \) be the choice of \( U_\alpha \) such that \( B_i \subset U_\alpha \). If we consider \( f_\alpha \) to be the sum of \( g_i \)'s over all \( i \)'s with \( \alpha(i) = \alpha \), then by definition the set \( \{f_\alpha\} \) is a partition of unity subordinate to \( \{U_\alpha\} \).

**Proposition 21.** Let \( M \) be a smooth manifold, \( U \) an open subset, and \( A \) a closed subset contained in \( U \). There exists a smooth **bump function** with respect to \( A \subset U \), that is, there exists a smooth positive function \( \varphi : M \rightarrow \mathbb{R} \) such that \( \varphi(p) = 1 \) for all \( p \in A \) and \( \text{supp}(\varphi) \subset U \).
Proof.
Consider the open cover of \( M \) with the sets \( U, M \setminus A \). There exist two functions \( \varphi, \psi \) such that they form a partition of unity subordinate to \( U \) and \( M \setminus A \) respectively. Then \( \varphi \) satisfies the condition of a smooth bump function.

- **Lemma 22.** Let \( M \) be a smooth \( n \)-manifold, \( A \subset M \) be a closed subset, and \( f : A \to \mathbb{R}^k \) be a smooth function. For any open subset \( U \) containing \( A \), there exists a smooth function \( \tilde{f} : M \to \mathbb{R}^k \) such that \( \tilde{f}|_A = f \) and \( \text{supp}(\tilde{f}) \subset U \).

**Proof.**
For each \( p \in A \), there exists an open neighborhood \( N_p \) and a smooth function \( f_p : N_p \to \mathbb{R}^k \) such that \( N_p \subset U \) and \( f_p = f|_{N_p} \). Such a pair exists since \( f \) is smooth at \( p \). The collection \( \{N_p\} \) along with \( M \setminus A \) is an open cover of \( M \). If \( \{\phi_p\} \) and \( \phi \) are the partition of unity subordinate to the open cover, then the smooth function \( \sum f_p \phi_p \) is a choice of \( \tilde{f} \).

- **Exercise.**
The lemma is not true if we replace the closed subset \( A \) by a non-closed subset. Give an example.

The lemma is not true if we replace \( \mathbb{R}^k \) by a smooth manifold. Give an example.

- **Proposition 23.** Every smooth manifold admits a smooth positive exhaustion function. An exhaustion function is a continuous function \( f \) such that \( f^{-1}((-\infty, c]) \) is a compact set for all \( c \).

- **Theorem 24.** Let \( M \) be a smooth manifold. For all closed set \( K \subset M \) there exists a smooth nonnegative function \( f : M \to \mathbb{R} \) such that \( K = f^{-1}(0) \).

- **Definition 25.** The tangent space of a smooth manifold \( M \) at the point \( p \) is the collection of derivations \( v : C^\infty(M) \to \mathbb{R} \) that satisfy \( v(fg) = v(f)g(p) + v(g)f(p) \). Derivations are linear functionals on the space \( C^\infty(M) \).

- **Proposition 26.** Directional derivatives \( D_v|_p \) with \( v \) vector in \( \mathbb{R}^n \) is a derivation in \( T_p\mathbb{R}^n \).

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**9/14/2015**

**Homework 1 remarks.**

Let \( M \) be a topological \( n \)-manifold with boundary. When showing that \( \partial M \) is a topological \((n-1)\)-manifold, you need to use the topological invariance of the boundary. More specifically, if \( p \in \partial M \), then there exists a chart \((U, \varphi)\) about \( p \). It is not clear that every point that is mapped to \( \partial H^n \) via \( \varphi \) is a boundary point of \( M \).

**Recap.**

So far in this course we have covered topological manifolds, smooth manifolds, smooth maps, and tangent spaces. We will now consider how a smooth map induces a map between tangent spaces.

**Definition 27.** Let \( M, N \) be smooth manifolds and \( F : M \to N \) be a smooth map. The differential of \( F \) at the point \( p \) is the map \( dF_p : T_pM \to T_{F(p)}N \) defined by \( dF_p(v)(f) = v(f \circ F) \) for all \( v \in T_pM \) and \( f \in C^\infty(N) \).

We check whether this definition is well-defined to familiarize with derivations.

We need to check if \( dF_p(v) \) is indeed a linear functional over the space \( C^\infty(N) \) that satisfies the “product rule”.

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Consider $dF_p(v)(cf + g)$ and $dF_p(v)(fg)$ for $c \in \mathbb{R}$, $f, g \in C^\infty(N)$.

$dF_p(v)(cf + g) = v((cf + g) \circ F) = v(cf \circ F + g \circ F) = cv(f \circ F) + v(g \circ F) = cdF_p(f) + dF_p(g)$.

d$F_p(v)(fg) = v((fg) \circ F) = v((f \circ F)(g \circ F)) = v(f \circ F)(g \circ F)(p) + v(g \circ F)(f \circ F)(p) = dF_p(v)(f)g(f(p)) + dF_p(v)(g)f(F(p))$.

- **Proposition 28.** Properties of differentials. Let $P$ be a smooth manifold, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.

  (a) $dF_p : T_pM \to T_{F(p)}N$ is linear.
  
  (b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G(F(p))}P$.
  
  (c) $d(Id_M)_p = Id_{T_pM} : T_pM \to T_pM$.
  
  (d) If $F$ is a diffeomorphism, then $dF_p$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

  **Proof.**
  
  (a) $dF_p(v + w)(f) = (v + w)(f \circ F) = v(f \circ F) + w(f \circ F) = (dF_p(v) + dF_p(w))(f)$ for all $v \in C^\infty(N)$
  
  (b) $d(G \circ F)_p(v)(f) = v(f \circ G \circ F) = dF_p(v)(f \circ G) = dG_{F(p)}(dF_p(v))(f)$ for all $v \in T_pM$ and $f \in C^\infty(P)$
  
  (c) $d(Id_M)_p(v)(f) = v(f \circ Id_M) = v(f)$ for all $v \in T_pM$ and $f \in C^\infty(M)$
  
  (d) Since $F$ is a diffeomorphism, we use (b) on $d(F \circ F^{-1})_p$ and (c) to obtain the result.

- **Definition 29.** Let $M$ be a smooth manifold. The tangent space $T_pM$ at $p \in M$ is the set of equivalence classes of smooth curves $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, where $\gamma_1$ is equivalent to $\gamma_2$ if for every smooth chart $(U, \varphi)$, $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$.

- **Proposition 30.** The definition of $T_pM$ given in Definition 29 is equivalent to the definition given in Definition 25.

- **Corollary 31.** With Definition 29, $dF_p : T_pM \to T_{F(p)}N$ is equivalently defined by $dF_p([\gamma]) = [F \circ \gamma]$, where $[\gamma]$ is the equivalence class of $\gamma$.

- **Proposition 32.** Let $M$ be a smooth $n$-manifold. The tangent space $T_pM$ is an $n$-dimensional vector space.

  **Proof**
  
  The alternative definition of $T_pM$ shows immediately that $T_pM = T_pU$ for any smooth chart $(U, \varphi)$ about $p$.
  
  Choose a smooth chart where $\varphi$ is a diffeomorphism between $U$ and $\varphi(U)$, then Proposition 28 (d) shows that $T_pU$ is isomorphic to $T_{\varphi(p)}(\varphi(U))$, which is also equal to $T_{\varphi(p)}\mathbb{R}^n$.
  
  Finally, tangent spaces of $\mathbb{R}^n$ are equivalent to spaces of directional derivatives, which form an $n$-dimensional vector space.

- **Remark.**
  
  Using charts, we can express tangent spaces by an explicit basis and differentials by matrices in coordinates.

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- **9/16/2015**

- **Proposition 33.** Let $M$ be a smooth $n$-manifold and $(U, \varphi)$ be a chart about $p \in M$. If $\{x^i\}$ is the set of coordinates for $\mathbb{R}^n$, then $\left\{ \frac{\partial}{\partial x^j} \right|_p \right\}$ form a basis for $T_pM$.

  If $F : M \to N$ is a smooth map with $(U, \varphi)$ a chart about $p \in M$ and $(V, \psi)$ a chart
about \( F(p) \in N \), then the differential \( dF_p \) is the linear map expressed by the matrix
\[
\begin{bmatrix}
\frac{\partial F^i}{\partial x^j}(p)
\end{bmatrix},
\]
where \( F^i \) is the component of \( F \) in the coordinates with respect to \((V, \psi)\).

- **Definition 34.** The tangent bundle of a smooth manifold \( M \) is the set \( TM \) defined by the disjoint union of tangent spaces.

\[
TM = \bigsqcup_p T_p M.
\]

We want to further understand the tangent bundle by describing the topology on the space. However, the topology follows from the smooth structure given below.

- **Proposition 35.** There exists a smooth structure on \( TM \) to make it into a \( 2n \)-dimensional smooth manifold.

**Proof.**

Let \((U, \varphi)\) be a smooth chart of \( M \) with \( U \) open. Consider the basis of \( T_p M \) for every \( p \in U \) given by \( \{ \frac{\partial}{\partial x^j} \mid p \} \). We take the set \( TU \subset TM \) with map \( \phi_U : TU \to \mathbb{R}^{2n} \) by mapping \((p, a_j \frac{\partial}{\partial x^j})\) to \((\varphi(p), a_j)\). We claim that the collection \( \{TU_\alpha\} \) with the maps \( \{\phi_U\} \) satisfy Lemma 7.

Recall Lemma 7.

**Lemma 7.** Let \( M \) be a set, and suppose we are given a collection \( \{U_\alpha\} \) of subsets of \( M \) together with maps \( \varphi_\alpha : U_\alpha \to \mathbb{R}^n \), such that the following properties are satisfied:

(a) For each \( \alpha \), \( \varphi_\alpha \) is a bijection between \( U_\alpha \) and an open subset of \( \mathbb{R}^n \).

(b) For each \( \alpha \) and \( \beta \), the set \( \varphi_\alpha(U_\alpha \cap U_\beta) \) is open in \( \mathbb{R}^n \).

(c) Whenever \( U_\alpha \cap U_\beta \neq \emptyset \), the map \( \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi(U_\alpha \cap U_\beta) \) is smooth.

(d) Countably many of the sets \( U_\alpha \) cover \( M \).

(e) Whenever \( p, q \) are distinct points in \( M \), either there exists some \( U_\alpha \) containing both \( p \) and \( q \) or there exist disjoint sets \( U_\alpha, U_\beta \) with \( p \in U_\alpha \) and \( q \in U_\beta \).

Then \( M \) has a unique smooth manifold structure such that each \( (U_\alpha, \varphi_\alpha) \) is a smooth chart.

The image of each \( TU_\alpha \) is \( \varphi_\alpha(U_\alpha) \times \mathbb{R}^n \). The conditions all follow from the definition along with \( M \) being a smooth manifold. Hence \( TM \) is a \( 2n \)-dimensional smooth manifold.

- **Remark.**

If we consider \( T(S^1) \), the tangent bundle of the circle, it is diffeomorphic to \( S^1 \times \mathbb{R} \).

(Exercise)

If \((U, \varphi)\) is a smooth chart, then \( TU \) is diffeomorphic to \( U \times \mathbb{R}^n \). But most of the time \( TM \) is not diffeomorphic to \( M \times \mathbb{R}^n \).

- **Definition 36.** The rank of a smooth map \( F : M \to N \) between smooth manifolds at the point \( p \) is the rank of \( dF_p \) written as a matrix in coordinates. Note that this is well-defined only if this is independent of the choice of coordinates about \( p \).

We say that \( F \) is a constant rank map if the rank of \( F \) is a constant \( r \) at every point. We say that \( F \) is a full rank map if the rank of \( F \) is equal to either the dimension of \( M \) or \( N \) at every point.

The map \( F : M \to N \) is a smooth submersion if the differential \( dF_p \) is surjective for all \( p \in M \), in other words, the rank of \( F \) is equal to the dimension of \( N \).
The map $F : M \to N$ is a smooth immersion if the differential $dF_p$ is injective for all $p \in M$, in other words, the rank of $F$ is equal to the dimension of $M$.

**Proposition 37.** Let $M$ be a smooth manifold. If $dF_p$ is surjective for some smooth map $F : M \to N$ and $p \in M$, then there exists a neighborhood $U$ of $p$ such that $F$ is a smooth submersion.

Let $M$ be a smooth manifold. If $dF_p$ is injective for some smooth map $F : M \to N$ and $p \in M$, then there exists a neighborhood $U$ of $p$ such that $F$ is a smooth immersion.

**Examples.**
(a) Let $m \geq n$. The projection map from $\mathbb{R}^m$ to $\mathbb{R}^n$ is a smooth submersion.
(b) Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve. The map $\gamma$ is a smooth immersion.
(c) Let $\Phi(s,t)$ be a smooth parametrized surface in $\mathbb{R}^3$. Then $\Phi$ is a smooth immersion.
(d) The projection map from $TM$ to $M$ defined by $\pi(p,v) = p$ is a smooth submersion.
(e) The map from $\mathbb{R}^n$ to $T^n$ defined by $F(x^1, \ldots, x^n) = (e^{ix^1}, \ldots, e^{ix^n})$ is both a smooth submersion and a smooth immersion.

**Definition 38.** Let $F : M \to N$ be a smooth map. We say that $F$ is a local diffeomorphism if for each point $p \in M$ there exists $U \subset M$ and $V \subset N$ such that $F : U \to V$ is a diffeomorphism.

**Proposition 39.** Let $F : M \to N$ be a smooth map.
(a) $F$ is a local diffeomorphism if and only if $F$ is both a smooth submersion and a smooth immersion.
(b) If $\dim(M) = \dim(N)$, then $F$ being either a smooth submersion or a smooth immersion implies that $F$ is a local diffeomorphism.

**Proof.**
(a) The direction $(\Rightarrow)$ follows from the fact that a diffeomorphism from $U$ to $V$ has $dF_p$ for any $p \in U$ as an isomorphism between $T_pM$ and $T_{F(p)}N$, which implies surjective and injective.

The direction $(\Leftarrow)$ uses the inverse function theorem from real analysis. The inverse function states that if you have a map $G$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that the total derivative $DG_x$ at a point $x$ is invertible, then there exists a neighborhood of $x$ where the map $G$ maps diffeomorphically to the image. Hence $F$ is a local diffeomorphism if it is both a smooth submersion and a smooth immersion.

(b) Straightforward. A square matrix that is either an injective or a surjective map is invertible. Then use (a).

**Remark.**
Proposition 39 actually fails in some cases of manifolds with boundaries. It is important to look up the counterexamples and understand the difference compared to manifolds without boundary.

**Theorem 40.** (The Rank Theorem) If a smooth map $F : M \to N$ has constant rank $r$, then for each point $p \in M$, there exists smooth charts $(U, \varphi)$ about $p$ and $(V, \psi)$
about \( F(p) \) such that
\[
\psi \circ F \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).
\]
In particular, the result of the rank theorem for a smooth submersion is
\[
\psi \circ F \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^n)
\]
and the result of the rank theorem for a smooth immersion is
\[
\psi \circ F \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0).
\]

**Sketch of Proof.**

Step 1: Consider the matrix given by \( D(\psi \circ F \circ \varphi^{-1}) \varphi(p) \) for some smooth charts \((U, \varphi)\) and \((V, \psi)\). Since the rank is \( r \), there exists an \( r \times r \) submatrix that has full rank. Reorder to make it the upper left \( r \times r \) submatrix. The inverse function theorem guarantees a map \( \Phi \) such that \( D(\psi \circ F \circ \varphi^{-1} \circ \Phi^{-1}) \Phi(\varphi(p)) \) has \( I_r \) as the upper left submatrix.

Step 2: Observe that since the rank is \( r \), the image of \((\psi \circ F \circ \varphi^{-1} \circ \Phi^{-1})\) must be a graph. In other words, the coordinates after the \( r \)-th coordinate can be expressed as functions of \( x^1, \ldots, x^r \).

Step 3: Let \( \Psi \) be the projection map from the graph to the domain and verify that \((\Psi \circ \psi \circ F \circ \varphi^{-1} \circ \Phi^{-1})\) satisfies the result of the theorem.

An alternative way to state Theorem 40 is the following.

Let \( M \) and \( N \) be smooth manifolds, let \( F: M \to N \) be a smooth map, and suppose \( M \) is connected. Then \( F \) has constant rank if and only if for each \( p \in M \) there exist smooth charts containing \( p \) and \( F(p) \) in which the coordinate representation of \( F \) is linear.

- **Theorem 41.** (The Global Rank Theorem) Let \( M \) and \( N \) be smooth manifolds, and suppose \( F: M \to N \) is a smooth map of constant rank.
  - (a) If \( F \) is surjective, then it is a smooth submersion.
  - (b) If \( F \) is injective, then it is a smooth immersion.
  - (c) If \( F \) is bijective, then it is a diffeomorphism.

**Sketch of Proof.**

(a) Suppose that \( F \) is not a smooth submersion, then the rank theorem shows that image of open sets in \( M \) are measure zero in \( N \). Therefore cannot be surjective.

(b) Suppose that \( F \) is not a smooth immersion, then the rank theorem shows that there is a family of points in any open set being mapped to the same point. Therefore not injective.

(c) Combine (a), (b), and the fact that a bijective local diffeomorphism map is an actual diffeomorphism.

- **Definition 42.** A smooth embedding of \( M \) into \( N \) is a smooth immersion \( F: M \to N \) that is also a homeomorphism onto its image \( F(M) \) in the subspace topology.

- **Examples.**
  - (a) Let \( U \) be an open set in a smooth manifold \( M \), then the inclusion map \( \iota: U \to M \) is a smooth embedding.
  - (b) The quotient map from \( \mathbb{R}^2 \) to \( T^2 = S^1 \times S^1 \) the torus in \( \mathbb{R}^3 \) is not a smooth embedding since the map is not injective. The map from \( S^1 \times S^1 \) to an embedded torus in \( \mathbb{R}^3 \) is a smooth embedding.
(c) The curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (t^3, 0)$ is not a smooth embedding since it is not a smooth immersion at $t = 0$.

(d) The curve given by $\beta(t) = (\sin 2t, \sin t)$ is not a smooth embedding since it self-intersects.

(e) The curve given by $\alpha(t) = (t \text{ (mod 1)}, st \text{ (mod 1)})$ where $s \notin \mathbb{Q}$ is a dense curve on the torus. $\alpha$ is not a smooth embedding since it does not map homeomorphically onto its image in the subspace topology.

• Proposition 43. Suppose $M$ and $N$ are smooth manifolds and $F : M \to N$ is an injective smooth immersion. If any of the following holds, then $F$ is a smooth embedding.

(a) $F$ is an open map. (b) $F$ is a closed map. (c) $F$ is a proper map. (d) $M$ is compact.