Lecture Note 08

• Remark.
  Examples of local homeomorphisms ($\mathbb{R} \rightarrow S^1$ and $\mathbb{R}^2 \rightarrow T^2$). Comments on HW2 and HW3. HW3 is handed back for people to polish/finish.

• Example.
  A constant function between any topological spaces is continuous.

• Proposition 49.
  Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ surjective and continuous. If $(X, \tau_X)$ is connected, then $(Y, \tau_Y)$ is connected.

  Proof.
  This is the same as proving that if $(Y, \tau_Y)$ is disconnected, then $(X, \tau_X)$ is disconnected. Let $A$ be a clopen set in $(Y, \tau_Y)$ that is not $Y$ or $\emptyset$. The preimage of $A$ is clopen in $(X, \tau_X)$ by definition. Since $f$ is surjective, we obtain that $f^{-1}(A)$ cannot be $X$ or $\emptyset$.

  (Note: above is different from the proof in class.)

• Definition 50.
  A topological space $(X, \tau)$ is said to be path-connected if for each pair of distinct points $a$ and $b$ of $X$ there exists a continuous mapping $f : [0, 1] \rightarrow (X, \tau)$, such that $f(0) = a$ and $f(1) = b$. The mapping $f$ is said to be a path joining $a$ to $b$.

• Examples.
  (a) $\mathbb{R}^n$ with the Euclidean topology is path-connected via the linear map $f(t) = (1 - t)a + tb$
  (b) Discrete spaces are path-connected if and only if $X$ has only one point.
  (c) Indiscrete spaces are always path-connected. The function $f([0, 1]) = \{a\}$ and $f(1) = b$ is continuous.
  (d) Moore plane is path-connected since we can build paths in the upper-half-plane. (where the topology coincides with the Euclidean topology.)

• Proposition 51.
  Every path-connected space is connected.

• Remark.
  Converse is not true. The famous example being $y = \sin(1/x)$ along with the $y$-axis. Draw the picture and try to prove this carefully.

• Remark.
  Beforehand, we didn’t have a way to prove that $\mathbb{R}^2$ minus a point is connected. Now we do.