Lecture Note 06

• Remark.
Today we continue to talk about Chapter 4 with more details.

• Definition 36. Topological properties.
(1) Connected: Let \((X, \tau)\) be a topological space. Then it is said to be connected if the only clopen subsets of \(X\) are \(X\) and \(\emptyset\).

\textit{Proof.} Recall that a homeomorphism \(f\) maps open sets to open sets. In fact it also maps closed sets to closed sets. Therefore if \((X, \tau_X)\) and \((Y, \tau_Y)\) are homeomorphic, then the clopen sets of \((X, \tau_X)\) are mapped bijectively to the clopen sets of \((Y, \tau_Y)\).

(2) \(T_0\)-space: A topological space \((X, \tau)\) is said to be a \(T_0\)-space if for each pair of distinct points \(a, b\) in \(X\), either there exists an open set containing \(a\) and not \(b\), or there exists an open set containing \(b\) and not \(a\).

(3) \(T_1\)-space: A topological space \((X, \tau)\) is said to be a \(T_1\)-space if every singleton set \(\{x\}\) is closed in \((X, \tau)\).

An alternative definition of a \(T_1\)-space is: for any pair of distinct points \(a, b\) in \(X\) there exist open sets \(U\) and \(V\) such that \(a \in U\), \(b \notin U\), \(a \notin V\), and \(b \in V\).

(4) \(T_2\)-space: A topological space \((X, \tau)\) is said to be Hausdorff or a \(T_2\)-space if given any pair of distinct points \(a, b\) in \(X\) there exist open sets \(U\) and \(V\) such that \(a \in U\), \(b \notin U\), \(a \notin V\), and \(b \in V\).

(5) \(T_3\)-space: A topological space \((X, \tau)\) is said to be a regular space if for any closed subset \(A\) of \(X\) and any point \(x \in X \setminus A\), there exist open sets \(U\) and \(V\) such that \(x \in U\), \(A \subset V\), and \(U \cap V = \emptyset\). If \((X, \tau)\) is regular and a \(T_1\)-space, then it is said to be a \(T_3\)-space.

(6) \(T_4\)-space: A topological space \((X, \tau)\) is said to be a normal space if for each pair of disjoint closed sets \(A\) and \(B\), there exist open sets \(U\) and \(V\) such that \(A \subset U\), \(B \subset V\), and \(U \cap V = \emptyset\). If \((X, \tau)\) is normal and a \(T_1\)-space, then it is said to be a \(T_4\)-space.

(7) Second countable: A topological space \((X, \tau)\) is said to satisfy the second axiom of countability or to be second countable if there exists a basis \(B\) for \(\tau\), where \(B\) consists of only a countable number of sets.

(8) First countable: A topological space \((X, \tau)\) is said to satisfy the first axiom of countability or to be first countable if for each \(x \in X\) there exists a countable family \(\{U_i(x)\}\) of open sets containing \(x\) with the property that every open set \(V\) containing \(x\) has one of the \(U_i(x)\) as a subset.

(9) Separable: A topological space \((X, \tau)\) is said to be separable if it has a dense subset which is countable.

(10) Door space: A topological space \((X, \tau)\) is said to be a door space if every subset of \(X\) is either an open set or a closed set.

• Remark.
Some of these properties are preserved in the subspace topology, but not all. Most of these properties have their own history and we will just introduce them here for future discussion.

• Proposition 37.

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Any topological space homeomorphic to a connected space is connected.

- **Definition 38.**
  A subset $S$ of $\mathbb{R}$ is said to be an interval if it has the following property: if $x \in S, z \in S,$ and $y \in \mathbb{R}$ are such that $x < y < z,$ then $y \in S.$

  The definition of an interval can be used for any subset of an well-ordered set. Any well-ordered space has the initial segment topology, terminal segment topology, and the open interval topology.

- **Proposition 39.**
  A subspace $S$ of $\mathbb{R}$ is connected if and only if it is an interval.

  *Proof.*
  
  $(\Rightarrow)$: Suppose that $S$ is not an interval. There exists $y \in \mathbb{R}$ that is between some $x, z \in S.$ Then the nonempty set $(-\infty, y) \cap S$ is a clopen set of $S$ that is not $S, \emptyset.$ Hence $S$ is not connected and that is a contradiction.

  $(\Leftarrow)$: Suppose that $S$ is not connected. There exists a subset $A \subset S$ that is clopen in the subspace topology $(S, \tau_S).$ Consider $x \in A$ and $z \in S \setminus A.$ Without loss of generality, let $x < z.$ Let $y$ be the least upper bound of the set $[x, z] \cap A.$ Then by Lemma 29 and by the fact that $A$ is clopen, $y \in A$ and $y \not\in A$ simultaneously, a contradiction.

\qed