Lecture Note for Differential Geometry & Topology I, Fall 2015

• 8/24/2015
• This course assumes knowledge in linear algebra, real analysis, and topology.
• There is a difference between the field of Differential Geometry and the field of Manifolds Theory.

• Definition 1.
  Suppose $M$ is a topological space (recall the definition of a topological space). We say that $M$ is a **topological manifold of dimension** $n$ if it has the following properties:
  1. $M$ is a **Hausdorff space**. Recall the definition of Hausdorff.
  2. $M$ is **second countable**. Recall the definition of second countable.
  3. $M$ is **locally Euclidean of dimension** $n$. For every point $p$ in $M$, there exists an open neighborhood $N_p$ and a homeomorphism $\varphi$ mapping $N_p$ to an open set in $\mathbb{R}^n$.

• Examples.
  1. Integers $\mathbb{Z}$ is a topological manifold of dimension 0.
  2. The union of the $x$-axis and the $y$-axis is not a topological manifold because the origin does not satisfy the locally Euclidean condition.
  3. $\mathbb{R}^n$, $S^n$, etc.
  4. The M'obius band is not a topological manifold because the boundary points do not satisfy the locally Euclidean condition.

• Theorem 2.
  (Topological Invariance of Dimension) Let $m \neq n$. A topological manifold of dimension $m$ cannot be homeomorphic to a topological manifold of dimension $n$.

• Definition 3.
  A **coordinate chart** is a pair $(U, \varphi)$ such that $U$ is an open subset of $M$ and $\varphi$ is a homeomorphism from $U$ to an open set in $\mathbb{R}^n$. The coordinate chart induces a **local coordinate** at $p$, that is, $\varphi(p) = (x^1(p), \ldots, x^n(p))$.

• Definition 4.
  A function $F : U \to V$, where $U$ open in $\mathbb{R}^n$ and $V$ open in $\mathbb{R}^m$, is called **smooth** if all partial derivatives of $f^j$ are continuous for $F = (f^1, \ldots, f^m)$. We say $F$ is a **diffeomorphism** if $F$ is bijective, smooth, and $F^{-1}$ is smooth.

• There are smooth homeomorphisms that are not diffeomorphisms.

• Definition 5.
  A **smooth manifold** is a pair $(M, \mathcal{A})$. $M$ is a topological manifold and $\mathcal{A}$ is a smooth structure.
  By a **smooth structure** $\mathcal{A}$, we mean a collection of coordinate charts $(U_i, \varphi_i)$ such that
  1. The sets $\{U_i\}$ form an open cover of $M$
  2. For any $U_i$ and $U_j$ such that $U_i \cap U_j \neq \emptyset$, the **transition map** $\varphi_j \circ \varphi_i^{-1}$ is smooth.

• Remark.
For canonical reasons, we often assume that a smooth structure is a maximal smooth atlas.

People also study various different structures. Just replace smooth by other types of functions.

**Proposition 6.**
Every smooth atlas is contained in a unique maximal smooth atlas.

*Proof:* (next time)

**Examples.**
(1) 0-dimensional manifolds: discrete sets
(2) 1-dimensional manifolds: circles and open intervals

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**8/26/2015**

- Recall the definition of a smooth manifold. A topological manifold is a smooth manifold if there exists a smooth atlas. A smooth atlas consists of an open cover along with a map from each open set into \( \mathbb{R}^n \). The maps pairwise satisfy the condition that the transition map is a smooth map from \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

- Proof of Proposition 6.
We want to show that every smooth atlas is contained in a unique maximal smooth atlas. First we need to prove existence of maximal smooth atlases that contains the given atlas. Then we need to show uniqueness to finish the proof.

Let \( A \) be a smooth atlas over \( M \), a topological \( n \)-manifold. We define \( \overline{A} := \{(V, \psi) \mid U \text{ open and } \phi \text{ is smoothly compatible with } A\} \).

We check that \( \overline{A} \) is actually a smooth atlas by considering the transition maps. Let \((V_1, \psi_1)\) and \((V_2, \psi_2)\) be two charts in \( \overline{A} \). If \( V_1 \cap V_2 \neq \emptyset \), then there exists \( p \in V_1 \cap V_2 \). Since \( A \) is an atlas, there exists \((U, \phi) \in A\) such that \( p \in U \). The transition map \( \psi_1 \circ \psi_2^{-1} \) is smooth at \( p \) since it can be written as \((\phi \circ \psi_1^{-1})^{-1} \circ (\phi \circ \psi_2^{-1})\) which is a composition of maps smooth at \( p \) by definition. In fact, \( \overline{A} \) is a maximal smooth atlas because any chart that can be added to the atlas must be smoothly compatible with \( \overline{A} \), hence in the atlas already. This proves existence.

To show uniqueness, we suppose that \( B \) is a maximal smooth atlas containing \( A \). By definition the charts in \( B \) are smoothly compatible with \( A, B \subset \overline{A} \). Now by maximality, we obtain \( \overline{A} = B \). Hence we are done.

- Exercise.
\( A \) and \( B \) determine the same smooth structure if and only if \( A \cup B \) is a smooth atlas.

- Examples.
(1) 0-dimensional manifolds. If \( M \) is a topological manifold of dimension 0, then \( M \) has to be a discrete countable set of points. Any atlas is smooth.
(2) 1-dimensional manifolds. If \( M \) is a connected topological manifold of dimension 1, then \( M \) has to be homeomorphic to either \( \mathbb{R} \) or the circle \( S^1 \). Here we consider two smooth atlases on \( S^1 \).

Angle function: We use two charts on \( S^1 \). One of the open sets maps to \((0, 2\pi)\) and the other one maps to \((-\pi, \pi)\) via angles in the usual sense. The transition maps between the two charts are simply translation functions \( f(x) = x + c \) in \( \mathbb{R} \).
Stereographic projection: By removing the point $N(0, 1)$ or the point $S(0, -1)$, we can project the rest of the circle on to the $x$-axis via stereographic projection. More explicitly, we map $(x, y)$ to $\frac{x}{1-y}$ and $\frac{x}{1+y}$ under the two maps. The stereographic projection extends naturally to $n$-dimensional spheres.

(3) $\mathbb{R}^n$ is a smooth manifold with a single chart where the map is the identity map.

(4) Finite dimensional vector space $V$ is identified with $\mathbb{R}^n$ naturally once a basis is chosen. The transition map between different choices of basis is exactly the change of basis matrix map.

(5) The space of linear maps between two finite dimensional vector spaces $L(V, W)$. Recall from linear algebra that this is identified with $\mathbb{R}^{mn}$, where $m$ and $n$ are the dimensions of $V$ and $W$.

(6) The space of $m \times n$ matrices is also identified with $\mathbb{R}^{mn}$.

(7) Graphs of smooth functions. The graph of a function $F : M \to N$ is the set $\{(x,y) \in M \times N \mid y = F(x)\}$. The graph is homeomorphic to $M$ under the projection map as long as $F$ is continuous. If $F$ is smooth (defined later), then we can obtain a smooth atlas on the graph induced by a smooth atlas on $M$.

Exercise: Think about the cases when $F$ is not smooth, does that imply that the induced atlas is not smooth?

(8) Level sets of smooth functions. Let $F : M \to N$ be a continuous map. Under conditions that will be introduced later, the level set $F^{-1}(c)$ for some $c \in N$ is a topological manifold of codimension 1. The steps to show that the level set is in fact a smooth manifold is more complicated and we will discuss this later.

(9) Real projective space and Grassmannians. We can consider the space of $k$-dimensional vector subspaces inside an $n$-dimensional vector space. When $k = 1$, we call this space the real projective space, which can alternatively be obtained by quotienting the $n$-sphere by the antipodal map. More on these will come later.

(10) Products of smooth manifolds. Similar to how product topology is defined, we can use smooth atlases on each component to obtain a smooth atlas for the product space.

**Lemma 7.** Let $M$ be a set, and suppose we are given a collection $\{U_\alpha\}$ of subsets of $M$ together with maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, such that the following properties are satisfied:

(a) For each $\alpha$, $\varphi_\alpha$ is a bijection between $U_\alpha$ and an open subset of $\mathbb{R}^n$.

(b) For each $\alpha$ and $\beta$, the set $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$.

(c) Whenever $U_\alpha \cap U_\beta \neq \emptyset$, the map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi(U_\alpha \cap U_\beta)$ is smooth.

(d) Countably many of the sets $U_\alpha$ cover $M$.

(e) Whenever $p, q$ are distinct points in $M$, either there exists some $U_\alpha$ containing both $p$ and $q$ or there exist disjoint sets $U_\alpha, U_\beta$ with $p \in U_\alpha$ and $q \in U_\beta$.

Then $M$ has a unique smooth manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

**Definition 8.** An $n$-dimensional topological manifold with boundary is a second countable Hausdorff space $M$ in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a (relatively) open subset of $H^n = \{(x^1, \ldots, x^n) \mid x^n \geq 0\}$.
Recall the definition of a topological manifold with boundary. An $n$-dimensional topological manifold with boundary is a second countable Hausdorff space $M$ in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a (relatively) open subset of $H^n = \{(x^1, \ldots, x^n) \mid x^n \geq 0\}$. We can further specify that in the second case the point is mapped to the origin.

If $p \in M$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$, we say that $p$ is an interior point and denote it by $p \in \text{Int}(M)$.

If $p \in M$ has a neighborhood homeomorphic to an open subset of $H^n$ with $p$ being mapped to the origin, we say that $p$ is a boundary point and denote it by $p \in \partial M$.

Note that a topological manifold with boundary is technically not a topological manifold.

**Theorem 9.** (Topological invariance of the boundary) For a topological manifold with boundary, $\text{Int}(M)$ is disjoint from $\partial M$.

**Proposition 10.** Let $M$ be a topological manifold with boundary of dimension $n$.

(a) The interior of $M$ is open in $M$ and is a topological manifold of dimension $n$.

(b) The boundary of $M$ is closed in $M$ and is a topological manifold of dimension $(n - 1)$.

(c) $M$ is a topological manifold if and only if $\partial M = \emptyset$.

**Definition 11.** Let $M, N$ be smooth manifolds, and let $F : M \to N$ be a map. We say that $F$ is a smooth map if for every $p \in M$, there exist smooth charts $(U, \varphi)$ containing $p$ and $(V, \psi)$ containing $F(p)$ such that $F(U) \subset V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$.

We are particularly interested in the case when $N = \mathbb{R}$. We use $C^\infty(M)$ to denote the space of smooth functions $F : M \to \mathbb{R}$. $C^\infty(M)$ is a vector space over $\mathbb{R}$.

**Proposition 12.** Every smooth map is continuous.

**Proof.**

Let $F : M \to N$ be a smooth map. For $p \in M$, there exist smooth charts $(U, \varphi)$ containing $p$ and $(V, \psi)$ containing $F(p)$. $F$ is continuous at $p$ because it is the composition $\psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi$ of three continuous functions.

**Proposition 13.** Let $M$ and $N$ be smooth manifolds with or without boundary, and let $F : M \to N$ be a map. If every point $p \in M$ has a neighborhood $U$ such that the restriction $F|_{U}$ is smooth, then $F$ is smooth.

**Examples.**

1. Constant maps. Let $F : M \to N$ be a constant map where $F(p) = c$ for all $p \in M$. $F$ is a smooth map since the map $\psi \circ F \circ \varphi^{-1}$ is a constant map.

2. Identity maps. Let $F : M \to M$ be the identity map $F(p) = p$. We choose the same chart in the domain and the range, then the map $\psi \circ F \circ \varphi^{-1}$ is the identity map, hence smooth.

3. The map from $\mathbb{R} \to S^1$ given by $f(x) = e^{2\pi ix}$ is smooth. This map extends to the map from $\mathbb{R}^n$ to the $n$-torus.

4. The inclusion map from $S^1 \to \mathbb{R}^2$ is a smooth map. The same is true for the inclusion map from the $n$-sphere to $\mathbb{R}^{n+1}$. Question: How to determine if the inclusion map of an object in $\mathbb{R}^n$ is smooth?
(5) Projection maps from products of smooth manifold to a component smooth manifold are smooth maps.

- **Remark.** Recall that $F : M \to N$ is a diffeomorphism if $F$ is bijective, smooth, and $F^{-1}$ is smooth.

  **Theorem**.*2. A nonempty smooth manifold of dimension $m$ cannot be diffeomorphic to an $n$-dimensional smooth manifold unless $m = n$.

  **Proof.**

  Consider the derivative of $\psi \circ F \circ \varphi^{-1}$ and its inverse. They have to be matrices inverse of each other, hence $m$ has to be equal to $n$.

- **Proposition 14.** (Properties of diffeomorphisms).
  
  (a) Every composition of diffeomorphisms is a diffeomorphism.
  (b) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.
  (c) Every diffeomorphism is a homeomorphism and an open map.
  (d) The restriction of a diffeomorphism to an open submanifold is a diffeomorphism onto its image.
  (e) “Diffeomorphic” is an equivalence relation on the class of all smooth manifolds.