• Proposition 15.
Let $\mathcal{B}$ be a basis for a topology on a set $X$. Then a subset $U$ of $X$ is open if and only if for each $x \in U$ there exists a $B \in \mathcal{B}$ such that $x \in B \subset U$.

Proof.
$(\Rightarrow): U$ is open. Hence $U$ is the union of members $B_j$ of $\mathcal{B}$. For any $x \in U$, there exists $B_j$ such that $x \in B_j \subset U$.
$(\Leftarrow)$: For each $x \in U$ there exists a $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. Hence $U$ is the union of $B_x$'s which by definition shows that $U$ is open. □

• Proposition 16.
Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for topologies $\tau_1$ and $\tau_2$, respectively, on a non-empty set $X$. Then $\tau_1 = \tau_2$ if and only if
(I) for each $x \in B$ for $B \in \mathcal{B}_1$, there exists $B' \in \mathcal{B}_2$ such that $x \in B' \subset B$, and
(II) for each $y \in B$ for $B \in \mathcal{B}_2$, there exists $B' \in \mathcal{B}_1$ such that $y \in B' \subset B$.

Proof.
$(\Rightarrow):$ This follows immediately from Proposition 15.
$(\Leftarrow):$ We show that if (I) is true, then $\tau_1 \subset \tau_2$. Then the symmetry proves $\tau_1 = \tau_2$. Let $U \in \tau_1$. By definition $U$ is a union of members of $\mathcal{B}_1$. Let $U$ be the union of $B_j$'s for $j \in J$. For each $x \in B_j$ there exists $B_j' \in \mathcal{B}_2$ and the union of all $B_j'$'s is equal to $U$. Hence $U \in \tau_2$. □

• Corollary 17.
Two topologies $\tau_1$ and $\tau_2$ are equal if and only if
(I) for each $x \in U$ for $U \in \tau_1$, there exists $V \in \tau_2$ such that $x \in V \subset U$, and
(II) for each $y \in V$ for $V \in \tau_2$, there exists $U \in \tau_1$ such that $y \in U \subset V$.

• Remark.
The Euclidean topology on $\mathbb{R}^n$ can use open balls as basis or use open rectangles as basis.

• Definition 18.
Let $(X, \tau)$ be a topological space. A non-empty collection $S$ of open subsets of $X$ is said to be a subbasis for $\tau$ if the collection of all finite intersections of members of $S$ forms a basis for $\tau$.

• Examples.
(a) $\tau$ is a subbasis. $\mathcal{B}$ is a subbasis.
(b) $(-\infty, a)$ and $(b, \infty)$ form a subbasis for the Euclidean topology on $\mathbb{R}$.
(c) $X = \{1, 2, 3\}$, $S = \{\emptyset, X, \{1, 2\}, \{1, 3\}\}$ is a subbasis.
(d) Mistake in class! Every collection of subsets is a subbasis of some topology! Terribly sorry!

• Remark.
A motivating space $X$ that people introduced topology to study is the space $C[0, 1]$, the space of continuous functions on $[0, 1]$. See Exercise 2.3.4.

• Definition 19.
Let $A$ be a subset of a topological space $(X, \tau)$. A point $x \in X$ is said to be a limit point of $A$ if every open set, $U$, containing $x$ contains a point of $A$ different from $x$.

• Examples.
(a) For $\mathbb{R}$ with Euclidean topology, the set of limit points of $(a, b]$ is $[a, b]$.

(b) For $\mathbb{R}$ with Euclidean topology, the set of limit points of $\mathbb{Z}$ is $\emptyset$.

(c) For the discrete topology on any space $X$, the set of limit points of any set $A$ is $\emptyset$.

(d) For the indiscrete topology on any space $X$, the set of limit points of any set $A$ with more than one elements is $X$. The set of limit points of $A = \{x\}$ is the set $X \setminus \{x\}$.

(e) For the finite-closed topology on $\mathbb{Z}$, the set of limit points of the set of all even numbers is $\mathbb{X}$.

• Proposition 20.

Let $A$ be a subset of a topological space $(X, \tau)$. Then $A$ is closed in $(X, \tau)$ if and only if $A$ contains all of its limit points.

Proof. 

($\Rightarrow$): every point in the complement of $A$ is not a limit point because the complement of $A$ is an open set that does not intersect $A$.

($\Leftarrow$): the complement of $A$ is the union of all open sets that arise from the definition of limit points, hence open. Therefore $A$ is closed.

• Proposition 21.

Let $A$ be a subset of a topological space $(X, \tau)$ and $A'$ the set of all limit points of $A$. Then $A \cup A'$ is a closed set. This set is called the closure of $A$ and is denoted $\overline{A}$.

• Proposition 22.

Let $A$ be a subset of a topological space $(X, \tau)$. Then $\overline{A} = X$ if and only if every non-empty open subset of $X$ intersects $A$ non-trivially. $A$ is said to be dense in $X$. 

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