Topology Homework 04

• Remember to reference all resources used. Due end of class on Thursday, 2/12.

1) Recall that a space \((X, \tau)\) is said to be a \(T_1\)-space if every singleton set \(\{x\}\) is closed. Also recall that a space \((X, \tau)\) is said to be a \(T_2\)-space if given any pair of distinct points \(a, b\) in \(X\) there exist disjoint open sets \(U, V\) such that \(a \in U\) and \(b \in V\).

(a) Show that any \(T_2\)-space is also a \(T_1\)-space.
(b) Show that \(\mathbb{Z}\) with the finite-closed topology is a \(T_1\)-space but is not a \(T_2\)-space.
(c) Show that any subspace of a \(T_1\)-space is a \(T_1\)-space.
(d) Show that any subspace of a \(T_2\)-space is a \(T_2\)-space.
(e) Show that \(\mathbb{R}^n\) with the Euclidean topology is a \(T_2\)-space.
(f) If \((X, \tau)\) is a \(T_1\)-space and every infinite subset of \(X\) is dense, show that \(\tau\) is the finite-closed topology.

2) Prove that the following pairs of topological spaces are homeomorphic.

(a) \(\mathbb{Z}\) and \(\mathbb{N}\) as subspaces of Euclidean \(\mathbb{R}\).
(b) \((0, 1]\) and \([a, \infty)\) as subspaces of Euclidean \(\mathbb{R}\).
(c) \(\{(x, y) \mid x^2 + y^2 = 1, x \neq 1\}\) and \(\{(x, y) \mid y = x\}\) as subspaces of Euclidean \(\mathbb{R}^2\).
(d) \(\{(x, y) \mid x^2 + y^2 \leq 1\}\) and \(\{(x, y) \mid |x| \leq 1, |y| \leq 1\}\) as subspaces of Euclidean \(\mathbb{R}^2\).

3) Let \((X, \tau)\) be any topological space and \(G\) the set of all homeomorphisms of \(X\) into itself.

(a) Show that \(G\) is a group under the operation of composition of functions.
(b) If \(X = [0, 1]\), show that \(G\) is infinite.
(c) If \(X = [0, 1]\), is \(G\) an abelian group?
(d) If \(X = [0, 1]\), how many finite order elements of \(G\) can you find?
(e) Find a space \(X\) such that \(G\) contains finite order elements of every order \(n \in \mathbb{N}\).

4) Show that the following four spaces are pairwise non-homeomorphic.

(i) \(\mathbb{R}\) with the Euclidean topology.
(ii) The Sorgenfrey line. \(\mathbb{R}\) with basis \([a, b)\) for all \(a < b\).
(iii) \(\{(x, y) \mid x^2 + y^2 = 1\} \cup \{(x, y) \mid (x - 2)^2 + y^2 = 1\}\).
(iv) \(\{(x, y) \mid x^2 + y^2 = 1\} \cup \{(x, y) \mid (x - 1)^2 + y^2 = 1\}\).

5) Let \((X, \tau_X)\) and \((Y, \tau_Y)\) be topological spaces. Prove that if \(f : X \to Y\) is a bijection with \(f(A) = f(A)\) for all subsets of \(X\), then \(f\) is a homeomorphism.