(1) For each of the pairs given, check whether the pair is a topological space.
   (a) $X_1 = \{1, 2, 3, 4\}$, $\tau_1$ consists of all sets whose elements total to an even number.
   (b) $X_2 = \mathbb{Z}$ (the set of integers), $\tau_2$ consists of $\mathbb{Z}$, $\emptyset$, and every set $\{n, n+1, n+2, \ldots\}$ for all $n \in \mathbb{Z}$.
   (c) $X_3 = \mathbb{N}$ (the set of natural numbers), $\tau_3$ consists of $\emptyset$ and every set that contains all positive even numbers.
   (d) $X_4 = \mathbb{Q}$ (the set of rational numbers), $\tau_4$ consists of $\mathbb{Q}$, $\emptyset$, and every set $\{\frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ for all nonzero integer $n$.

(2) Find a topological space with the property that every open set is clopen but the topology is neither discrete nor indiscrete.

(3) If $(X, \tau)$ is a topological space, the interior of $S \subset X$ is the set

$$S^\circ = \bigcup\{G \subset X \mid G \text{ is open and } G \subset S\}.$$ 

In other words, $S^\circ$ is the largest open set contained by $S$. Prove the following properties.
   (a) $A^\circ \subset A$
   (b) If $A \subset B$, then $A^\circ \subset B^\circ$
   (c) $(A \cap B)^\circ = A^\circ \cap B^\circ$
   (d) $A$ is open if and only if $A = A^\circ$

(4) Let $\tau_1$ and $\tau_2$ be two topologies on $X$. Prove the following.
   (a) If $\tau_3$ is defined by $\tau_1 \cap \tau_2$, then $\tau_3$ is a topology on $X$.
   (b) If $\tau_4$ is defined by $\tau_1 \cup \tau_2$, then $\tau_4$ is not necessarily a topology on $X$. (Justify this by finding an example.)
   (c) Describe the smallest topology that contains $\tau_4$ for your example above.

(5) Prove the following statements with respect to the Euclidean topology.
   (a) The set $\mathbb{N}$ is a closed set in $X = \mathbb{R}$.
   (b) The set $\{(x, y, z) \mid x^2 + y^2 + z^2 < 1\}$ is open in $X = \mathbb{R}^3$.
   (c) The set $\{(x, y) \mid x \in \mathbb{Q}, y \in \mathbb{Z}\}$ is neither open nor closed in $X = \mathbb{R}^2$.
   (d) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then the set $\{x \mid f(x) = 0\}$ is closed in $X = \mathbb{R}$.