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Decomposition solution of multidimensional groundwater equations

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SUMMARY

The method of decomposition of Adomian is an approximate analytical series to solve linear or nonlinear differential equations. An important limitation is that a decomposition expansion in a given coordinate explicitly uses the boundary conditions in such axis only, but not necessarily those on the others. This paper presents improvements of the method that permit the practical consideration of all of the conditions imposed on multidimensional initial-value and boundary-value problems governed by (nonlinear) groundwater equations, and the analytical modeling of irregularly-shaped heterogeneous aquifers subject to sources and sinks. The method yields simple solutions of dependent variables that are continuous in space and time, which easily permit the derivation of heads, gradients, seepage velocities and fluxes, thus minimizing instability. It could be valuable in preliminary analysis prior to more elaborate numerical analysis.

1. Introduction

A mathematical model of groundwater flow or groundwater transport is constituted by a solution of a governing partial differential equation subject to a set of boundary conditions, and an initial condition if the problem is time dependent. Solutions to partial differential equations appearing in groundwater modeling are obtained by applying a variety of mathematical and numerical procedures. Analytical solutions provide values of the dependent variable continuously in space and time. They require the application of traditional mathematical methods. For continuous spectrum operator equations such as those encountered in infinite or semi-infinite aquifer domains, these include domain transform methods, such as Laplace transform, Fourier transform, wavelets, and Hankel transform (e.g., Jeffrey, 2003; Myint-U and Debnath, 1987; Zauderer, 1983; Powers, 1979). For discrete spectrum operator equations, such as those encountered in groundwater flow in finite domains, they include Fourier series (e.g., Jeffrey, 2003; Zauderer, 1983; Powers, 1979). An advantage of traditional analytical solutions is that the computational implementation is simple (He, 1999; Strack, 1989; De Marsily, 1986; Bear, 1979; Hunt, 1983). Many traditional analytical solutions are restricted to regular geometrical domains, aquifer homogeneity, and linearity in the differential equation. In essence, analytical solutions constitute a suitable approach for preliminary analyses or for studies with scarce hydrologic information.

Numerical solutions of groundwater flow equations provide values of the dependent variable at fixed discrete locations and fixed discrete times. They require the application of numerical methods, including finite differences (e.g., Anderson, 1992; Aral, 1989a,b; Walton, 1989; Bear and Verruijt, 1987), finite elements (Huyakorn and Pinder, 1983; Pinder and Gray, 1979), and boundary elements (Ligget and Liu, 1983). Nobel approaches in this arena are exemplified in Miller et al. (2006). Numerical methods allow the consideration of irregular aquifer shapes, heterogeneity in the hydraulic conductivity, and discrete linearization. Numerical solutions usually require large amounts of information, computer memory, and computer time for its execution. Since numerical solutions are based on space and time discretization and on the numerical approximation of the derivatives in a equation, numerical instability may arise. Numerical solutions are at the heart of the most commonly used computer simulation models today. They are suitable in complex groundwater modeling scenarios enjoying the availability of large data sets. For descriptions on popular groundwater software, such as the USGS MODFLOW program, see Waterloo Hydrogeologic (2004) and Kresic (1997).

Within the class of analytical methods, new developments in the solutions of nonlinear equations worth mentioning are the method of variational iteration (He and Wu, 2006; He, 1999), in which the problems are initially approximated with possible

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unknowns. Then a correction functional is constructed by a general Lagrange multiplier, which can be identified optimally via the variational theory. Recently, an interesting method for solving analytically nonlinear equations is the homotopy perturbation method (He, 2006), which consists in a series expansion based on a homotopy parameter that generate a series of recursive linear partial differential equations whose solution converge to the exact solution to the original problem. One of the best approaches to solve nonlinear equations is the method of decomposition (Adomian, 1994). The "method of decomposition" of Adomian should not be confused with "domain decomposition" techniques, which are a set of numerical algorithms that divide a problem domain into smaller sub-domains that can be solved independently to overcome the memory limitations of mixed finite element methods (Su, 1994; Beckie et al., 1993; Glowinski and Wheeler, 1988; Haimes and Dreizin, 1977). On the other hand, Adomian's method of decomposition consists in deriving an infinite series, much like Fourier series, that in many cases converge to an exact solution (see Appendix A). For a simple introduction to the method with applications in surface and subsurface hydrology, engineering analysis, and stochastic methods the reader is referred to Serrano (2010), Serrano (2001). For nonlinear equations in particular, decomposition is one of the few *systematic* solution procedures available. With the concepts of partial decomposition and of double decomposition (Adomian, 1994), the process of obtaining an approximate solution is simplified.

A drawback of the method of decomposition limits the use of a set of boundary conditions to a one-dimensional axis only. Thus, a partial decomposition expansion in a given coordinate explicitly uses the boundary conditions in such axis only; applications in two-dimensional or three-dimensional domains do not necessarily incorporate the boundary conditions imposed on the other dimensions. Wazwaz (2000a) has shown that a decomposition series can be expressed in terms of the boundary condition not used explicitly and that by equating coefficients of like powers the series can be shown to converge to the exact solution satisfying all of the boundary conditions. For boundary-value problems with simple boundaries (e.g., boundaries at infinity and one-dimensional problems), the procedure is straight forward (Shidfar and Reihani, 2010; Serrano, 1997). However, this procedure could be elaborate in many cases of practical applications. In this article we present several improvements of the method of decomposition for the practical modeling of groundwater equations: A combination of a partial decomposition expansion in each coordinate in conjunction with successive approximation that permits the consideration of boundary conditions imposed on all of the axes of a multidimensional problem; the analysis of regional flow in irregularly-shaped domains whose geometry can be described functionally; the modeling of multiple wells; the inclusion of aquifer heterogeneity when conductivity spatial variability is described in functional form; transient flow in higher-dimensional domains; and the solution of nonlinear problems in higher dimensions. Convergence rate is shown to illustrate the fact that in many dissipative systems the first few terms in the series constitute a reasonably accurate approximation. Verification is done when an exact solution is available.

2. Regional steady flow

2.1. Modeling the effect of regional recharge

Let us consider the two-dimensional regional groundwater flow equation with Dupuit assumptions, with x and y horizontal planar coordinates [L]. An x-partial solution satisfies the differential equation and the x boundary conditions, but not necessarily those in y. Similarly, a y-partial solution satisfies the y boundary conditions, but not necessarily those in x. Although each partial expansion constitutes a general solution to the differential equation (Adomian, 1994), a particular solution (i.e., the one needed in actual modeling calculations) requires the evaluation of the constants of integration such that the boundary conditions in all dimensions are satisfied. Our objective is to develop a decomposition solution of regional groundwater flow that includes all available boundary conditions by combining a decomposition expansion with successive approximation. The governing differential equation is

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=-\frac{R_g}{T}\quad 0\leqslant x\leqslant l_x,\quad 0\leqslant y\leqslant l_y \tag{1}$$

where h is the hydraulic head [L]; R_g is mean monthly recharge from rainfall [LT^{-1}]; T is the mean aquifer transmissivity [L^{2}T^{-1}]; and l_x and l_y are the aquifer horizontal dimensions in the x and y direction, respectively [L]. The following set of boundary conditions are imposed in (1):

$$h(0,y)=f_1(y),\quad h(l_x,y)=f_2(y),\quad h(x,0)=f_3(x),\quad h(x,l_y)=f_4(x) \tag{2}$$

Eqs. (1) and (2) describe the elevation of the water table in a long, thin, mildly sloping aquifer, bounded on all sides by streams. The functions f_1, f_2, f_3 and f_4 represent the mean elevation of the water at the boundaries.

We define the operators $L_x = \partial^2/\partial x^2$ and $L_y = \partial^2/\partial y^2$. The inverse operators L_y^{-1} and L_x^{-1} are the corresponding two-fold indefinite integrals with respect to x and y, respectively. Eq. (1) reduces to

$$L_x h+L_y h=-\frac{R_g}{T} \tag{3}$$

The x-partial solution, h_x, results from operating with L_x^{-1} in Eq. (3) and re-arranging:

$$h\approx h_x=k_1(y)+k_2(y)x-L_x^{-1}\frac{R_g}{T}-L_x^{-1}L_y h \tag{4}$$

where the integration "constants" k_1 and k_2 are to be found from the x boundary conditions. Expanding h in the right side as an infinite series $h = h_0 + h_1 + h_2 + \cdots$, Eq. (4) becomes

$$h_x=k_1(y)+k_2(y)x-L_x^{-1}\frac{R_g}{T}-L_x^{-1}L_y(h_0+h_1+h_2+\cdots) \tag{5}$$

The choice of h_0 often determines the level of difficulty in calculating subsequent decomposition terms and the rate of convergence (Adomian, 1994; Wazwaz, 2000b). A simple choice is to set h_0 as equal to the first three terms in the right side of Eq. (5). Thus, the first approximation to the solutions is

$$h_x\approx h_0=k_1(y)+k_2(y)x-L_x^{-1}\frac{R_g}{T}=k_1(y)+k_2(y)x-\frac{R_g x^2}{2T} \tag{6}$$

Applying the x boundary conditions from Eq. (2)

$$h_x\approx h_0=f_1(y)+\left(\frac{f_2(y)-f_1(y)}{l_x}+\frac{R_g l_x}{2T}\right)x-\frac{R_g x^2}{2T} \tag{7}$$

Eq. (7) satisfies Eq. (3) and the x boundary conditions, but not necessarily those in y. Similarly, the y-partial solution to Eq. (3), h_y, results from operating with L_y^{-1} on Eq. (3), applying the boundary conditions in y, and re-arranging:

$$h_y\approx h_0=f_3(x)+\left(\frac{f_4(x)-f_3(x)}{l_y}+\frac{R_g l_y}{2}\right)y-\frac{R_g y^2}{2T} \tag{8}$$

We now have two partial solutions to Eq. (3): the x-partial solution Eq. (7), and the y-partial solution Eq. (8). Since both are approximations to h, a combination of the two partial solutions yields the improved first decomposition term h_0:

$$h_0(x,y) = \left(\frac{h_x(x,y) + h_y(x,y)}{2}\right) \tag{9}$$

Eq. (9) constitutes an approximate solution to Eq. (1) and its boundary conditions Eq. (2). Eq. (9) can now be used to derive the second term in the series, h_1. This may be accomplished via the x-partial expansion Eq. (5), or the y-partial expansion. From Eq. (5), the second term in the solutions is given by

$$h_1 = k_5(y) + k_6(y)x - L_x^{-1}L_y h_0 \tag{10}$$

where k_5 and k_6 are such that homogeneous boundary conditions in the x direction are satisfied. Similarly, a y-partial expansion the second term in the solution is given by

$$h_1 = k_7(x) + k_8(x)y - L_y^{-1}L_x h_0 \tag{11}$$

where k_7 and k_8 are such that homogeneous boundary conditions in the y direction are satisfied. Since both Eq. (10) are expressions of h_1, then a combination of the two will yield the second decomposition term.

The above procedure may be repeated to obtain higher-order terms, which are easily derived since all calculations involve differentiation and integration of simple functions. Based on the combined $(i-1)$th-order term, h_{i-1}, the ith-order term may be subsequently derived from the x-partial expansion Eq. (5)

$$h_i = k_{4i+1}(y) + k_{4i+2}(y)x - L_x^{-1}L_y h_{i-1}, \quad i > 0 \tag{12}$$

where k_{4i+1} and k_{4i+2} are such that homogeneous x-boundary conditions are satisfied, and from the y-partial expansion

$$h_i = k_{4i+3}(x) + k_{4i+4}(x)y - L_y^{-1}L_x h_{i-1}, \quad i > 0 \tag{13}$$

where k_{4i+3} and k_{4i+4} are such that homogeneous y-boundary conditions are satisfied. A combination of Eqs. (12) and (13) will yield h_i. In this manner, by successively adding more terms to the series $h = h_0 + h_1 + h_2 + \cdots$, it is possible to observe a convergence that satisfies the differential equation and all of its boundary conditions. However, studies indicate that the rate of convergence is usually high for dissipative systems and that only one or two terms yield a good approximation (Adomian, 1994, 1991, 1983; Serrano, 2010).

2.2. Verification with an exact solution

An exact solution to Eq. (1) may be obtained by the traditional methods of separation of variables and the use of Fourier series (Powers, 1979) as:

$$h(x,y) = W_1(x,y) + W_2(x,y) - \frac{R_g}{4T}(x^2 + y^2) \tag{14}$$

where

$$W_1(x,y) = \sum_{n=1}^{\infty}\{C_n \sinh(\lambda_n x) + D_n \cosh(\lambda_n x)\}\sin(\lambda_n y),$$

$$\lambda_n = \frac{n\pi}{l_y}, \quad n = 0, 1, 2, \ldots$$

$$C_n = \frac{1}{\sinh(\lambda_n l_x)}\left\{\frac{2}{l_y}\int_0^{l_y}\left\{f_2(y) - \frac{R_g}{4T}\left(l_x^2 + y^2\right)\right\}\sin(\lambda_n y)\,dy - D_n\cosh(\lambda_n l_x)\right\} \tag{15}$$

$$D_n = \frac{2}{l_y}\int_0^{l_y}\left(f_1(y) - \frac{R_g y^2}{4T}\right)\sin(\lambda_n y)\,dy$$

and

$$W_2(x,y) = \sum_{i=1}^{\infty}\{G_i \sinh(\gamma_i y) + H_i \cosh(\gamma_i y)\}\sin(\gamma_i x),$$

$$\gamma_i = \frac{i\pi}{l_x}, \quad i = 1, 2, 3, \ldots$$

$$H_i = \frac{2}{l_x}\int_0^{l_x}\left(f_3(x) + \frac{R_g x^2}{4T}\right)\sin(\gamma_i x)\,dx \tag{16}$$

$$G_i = \frac{1}{\sinh(\gamma_i l_y)}\left\{\frac{2}{l_x}\int_0^{l_x}\left[f_4(x) + \frac{R_g}{4T}\left(x^2 + l_y^2\right)\right]\sin(\gamma_i x)\,dx - H_i\cosh(\gamma_i l_y)\right\}$$

To compare a decomposition solution of Eq. (1) with respect to the exact solution, the boundary conditions, Eq. (2), must be known based on water-level measurements at the rivers limiting the aquifer. As an illustration, let us assume arbitrary expressions for the boundary conditions in Eq. (2) representing water level in meters above the sea level:

$$f_1(y) = 100 - 0.2 \times 10^{-2}y, \quad f_2(y) = 103 - 0.1 \times 10^{-2}y \tag{17}$$

$$f_3(x) = 100 + 0.8 \times 10^{-2}x - 0.5 \times 10^{-5}x^2,$$
$$f_4(x) = 99 + 0.883 \times 10^{-2}x - 0.5 \times 10^{-5}x^2$$

where x and y are in meters. A typical recharge rate from rainfall $R_g = 10$ mm/month, aquifer transmissivity of $T = 100\ \mathrm{m}^2$/month, $l_x = 600$ m, and $l_y = 500$ m. Using Eq. (7) through (13), the first four terms of the decomposition solution in the center of the aquifer ($x = l_x/2$ and $y = l_y/2$) are $h_0 = 105.162$, $h_1 = -1.984$, $h_2 = 1.009$, and $h_3 = -0.496$ m. Adding the first four terms, we obtain $h \approx h_0 + h_1 + h_2 + h_3 = 103.691$ m. In comparison, the exact solution at the same location yields $h = 103.527$, and the approximate solution gives an absolute error of 0.165 m. The decomposition solution satisfies the differential Eq. (1) with an error of -0.000083 (month^{-1}) in the center of the aquifer. A bigger picture of the behavior of the solution compares the decomposition and the exact solutions of hydraulic head at various locations. The maximum relative errors of less than 1.2% occur in the center of the aquifer boundaries. The minimum error of about 0.16% occurs in the center of the aquifer. Fig. 1 shows the relative error distribution If we only use the first term in the decomposition series, that is if we set $h \approx h_0$ (Eqs. (7)–(9)), the maximum relative error increases to 1.99%. The simulations illustrate that the rate of convergence of the series solution is high, a typical feature of decomposition. Usually the first few terms are sufficient to assure an accurate solution; in the present case, by using the first term only the error is less than 2%. More importantly, the effort required to produce a decomposition solution is modest, as compared to that required to derive an exact analytical solution (Eqs. (14)–(16)). A simple analytical expression of aquifer head is advantageous in the derivation of hydraulic gradients, seepage velocities, and groundwater fluxes.

2.3. Irregularly-shaped domains

Most analytical solutions of boundary-value initial-value problems are restricted to rectangles, squares, circles and other regular domain shapes. Even though decomposition renders an analytical solution, extension of the above procedure to aquifer domains of irregular geometry is possible if the aquifer boundaries are defined in a functional form. For example, let us assume that in Eq. (1) the upper y river boundary has an irregular shape characterized as $l_y(x) = 500 + 0.4x - 0.001x^2$, after fitting a parabola to a few surveyed points on the river channel. For simplicity, let us assume that

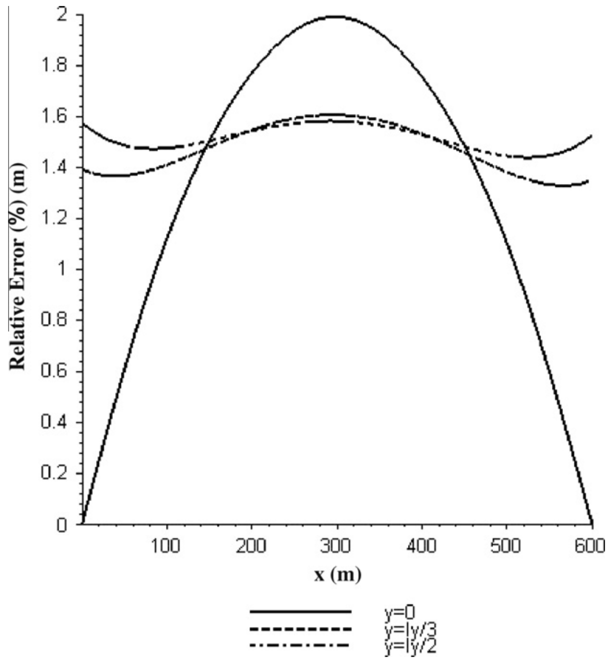

Fig. 1. Comparison between decomposition and exact solution of steady head at various values of y.

the head at the boundaries is described by (18), with $h(x,y) = f_4(x)$, on $y = l_y(x)$. The solution needs to be modified to reflect the fact that the integration must now be done over a variable y domain, which is now reflected in the expansion in y with l_y now being a function of x. Fig. 2 shows a plan view of this aquifer with the head contours calculated by Eqs. (7)–(9), and flow directional vectors calculated with a simple Maple routine. With decomposition, an aquifer of any arbitrary shape in its boundaries can be easily modeled, as long as the boundaries can be represented in analytical form.

2.4. Modeling the effect of wells

Water exploitation from pumping wells affect the spatial distribution of hydraulic heads. Simple decomposition solutions for one-dimensional problems have been obtained by Serrano (1997). In the case of two-dimensional problems, Eq. (1) becomes

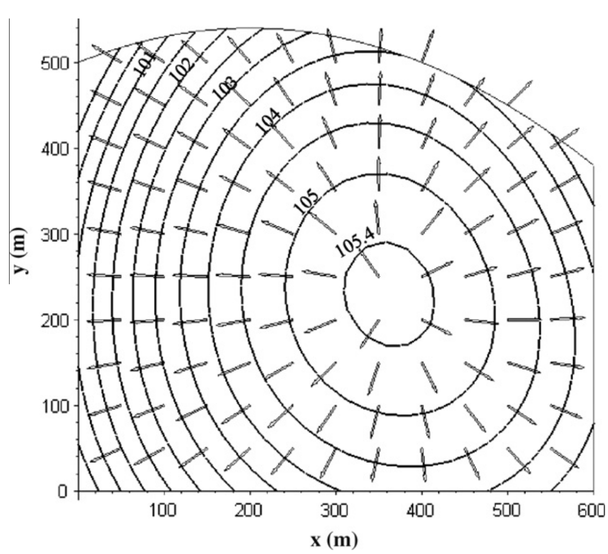

Fig. 2. Head contours (m) and flow direction in an irregularly-shaped aquifer.

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = -\frac{R_g}{T} + W \quad 0 \leqslant x \leqslant l_x, \quad 0 \leqslant y \leqslant l_y \tag{18}$$

where W is the flow rate of pumping wells per unit surface area per unit time [L^{-1}]. Consider a well i located at $x = x_i$, $y = y_i$ pumping at a rate of Q_i (m^3/month), and having a cross-sectional area A_i (m^2). Eq. (18) can now be written as

$$L_x h + L_y h = -\frac{R_g}{T} + \sum_{i=1}^{N} \frac{Q_i}{A_i T} \delta(x - x_i)\delta(y - y_i) \tag{19}$$

where $\delta()$ is the Dirac's delta function, and N is the number of wells pumping the aquifer. The x-partial solution of Eq. (19) is given by

$$h_x = k_1(y) + k_2(y)x - L_x^{-1}\frac{R_g}{T} + L_x^{-1}\sum_{i=1}^{N}\frac{Q_i}{A_i T}\delta(x - x_i)\delta(y - y_i) - L_x^{-1}L_y(h_0 + h_1 + h_2 + \cdots) \tag{20}$$

From the previous development, the pumping contribution from the ith well, h_{ix}, to the first term, h_0, in Eq. (20) satisfies

$$L_x h_{ix} + L_y h_{ix} = \frac{Q_i \delta(x - x_i)\delta(y - y_i)}{A_i T}, \quad h_{ix}(0, y) = h_{ix}(l_x, y) = 0 \tag{21}$$

The solution of Eq. (21) may be written as Myint-U and Debnath (1987)

$$h_{ix} = F_i(x_i, y_i; x, y) + g_{ix}(x_i, y_i; x, y) \tag{22}$$

where F_i is the free-space Green's function for Laplace's operator in Eq. (21),

$$F_i(x_i, y_i; x, y) = \frac{Q_i}{2\pi A_i T}\log\left[(x_i - x)^2 + (y_i - y)^2\right] \tag{23}$$

and g_{ix} satisfies Laplace's equation subject to the homogeneous boundary conditions in Eq. (21). Hence,

$$g_{ix}(x_i, y_i; x, y) = \left(\frac{F_i(x_i, y_i; 0, y) - F_i(x_i, y_i; l_x, y)}{l_x}\right)x - F_i(x_i, y_i; 0, y) \tag{24}$$

Analogous to (20)–(24), a y-partial expansion of Eq. (19) produces a pumping contribution from the ith well, h_{iy}, given by

$$h_{iy} = F_i(x_i, y_i; x, y) + g_{iy}(x_i, y_i; x, y) \tag{25}$$

with g_{iy} given by

$$g_{iy} f(x_i, y_i; x, y) = \left(\frac{F_i(x_i, y_i; x, 0) - F_i(x_i, y_i; x, l_y)}{l_y}\right)y - F_i(x_i, y_i; x, 0) \tag{26}$$

Thus, the first decomposition term in Eq. (19), and the first approximate solution, is given by

$$h_0(x, y) = \left(\frac{h_x(x, y) + h_y(x, y)}{2}\right) + \sum_{i=1}^{N}\left(\frac{h_{ix}(x, y) + h_{iy}(x, y)}{2}\right) \tag{27}$$

where h_x is given by Eq. (7), h_y by Eq. (8), h_{ix} by Eq. (22), and h_{iy} by Eq. (25). If additional accuracy is desired, more decomposition terms may be easily calculated as described by Eqs. (12) and (13) for the case without wells.

These calculations are simple and since the obtained solution is analytical, no specialized groundwater software is required. Plotting head contours or three-dimensional surfaces involve a single command in any computer algebra software, such as Maple. For example, assume that in the aquifer of Fig. 1, in addition to recharge from rainfall there are two wells pumping at a rate of $Q_1 = Q_2 = 500$ m^3/month, with a well-casing area $A_1 = A_2 = 0.1$ m^2, and located at the coordinates $x_1 = 400$, $y_1 = 200$, $x_2 = 100$, and

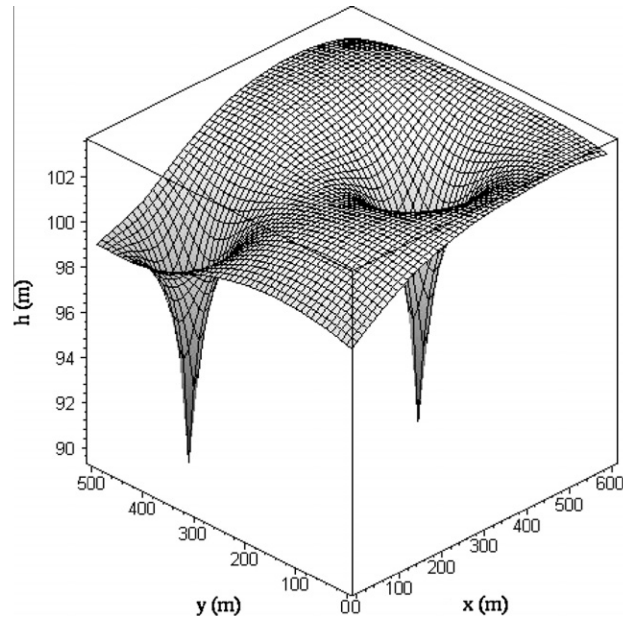

Fig. 3. Groundwater head in an aquifer subject to recharge and pumping wells.

$y_2 = 400$ m. Fig. 3 shows the effect of the pumping wells on the groundwater head surface.

2.5. Modeling heterogeneous aquifers

Most regional aquifers are not homogeneous in the hydraulic conductivity. A well-surveyed aquifer possessing a large number of values of conductivity calls for a numerical model that can account for a detailed variability in the aquifer parameters. However, in many cases the hydrologist has access to only a handful of values; this justifies the utilization of a simpler mathematical model. If the aquifer conductivity field can be described by a functional form fitted to the limited number of measurements, a simple analytical decomposition scheme may be applied. Consider the two-dimensional groundwater flow equation Eq. (1) subject to a spatially-variable conductivity field:

$$\frac{\partial}{\partial x}\left(T(x,y)\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(T(x,y)\frac{\partial h}{\partial y}\right)=-R_g \quad 0\leqslant x\leqslant l_x, \quad 0\leqslant y\leqslant l_y \tag{28}$$

In this section we consider the linear equation resulting from a transmissivity field of spatially-variable hydraulic conductivity, $K(x,y)$ [LT^{-1}], and an average saturated thickness. We later discuss the case of nonlinear groundwater equations arising from a transmissivity that is functionally dependent on a variable saturated thickness. Adopt a set of mixed boundary conditions:

$$h(0,y)=f_1(y), \quad \frac{\partial h}{\partial x}(l_x,y)=0, \quad h(x,0)=f_3(x), \quad h(x,l_y)=f_4(x) \tag{29}$$

Eq. (3) now becomes

$$L_x h + L_y h = -\frac{R_g}{T(x,y)} - \frac{1}{T(x,y)}\frac{\partial T}{\partial x}\frac{\partial h}{\partial x} - \frac{1}{T(x,y)}\frac{\partial T}{\partial y}\frac{\partial h}{\partial y} \tag{30}$$

The x-partial solution Eq. (4) has new terms:

$$h_x = k_1(y)+k_2(y)x - L_x^{-1}\frac{R_g}{T(x,y)} - L_x^{-1}L_y h - L_x^{-1}\frac{1}{T(x,y)}\frac{\partial T}{\partial x}\frac{\partial h}{\partial x} - L_x^{-1}\frac{1}{T(x,y)}\frac{\partial T}{\partial y}\frac{\partial h}{\partial y} \tag{31}$$

Expanding h in the right side as an infinite series $h = h_0 + h_1 + h_2 + \cdots$, Eq. (31) becomes

$$\begin{aligned} h_x &= k_1(y)+k_2(y)x - L_x^{-1}\frac{R_g}{T(x,y)} - L_x^{-1}L_y(h_0+h_1+h_2+\cdots) \\ &\quad - L_x^{-1}\frac{1}{T(x,y)}\frac{\partial T}{\partial x}\frac{\partial}{\partial x}(h_0+h_1+h_2+\cdots) - L_x^{-1}\frac{1}{T(x,y)}\frac{\partial T}{\partial y} \\ &\quad \times \frac{\partial}{\partial y}(h_0+h_1+h_2+\cdots) \end{aligned} \tag{32}$$

As before, a simple choice for a first approximation gives

$$h_x \approx h_0 = k_1(y)+k_2(y)x - L_x^{-1}\frac{R_g}{T(x,y)} \tag{33}$$

where $k_1(y)$ and $k_2(y)$ are such that (33) satisfies the x-boundary conditions in Eq. (29). A y-partial solution of Eq. (30) is given by

$$\begin{aligned} h_y &= k_3(x)+k_4(x)y - L_y^{-1}\frac{R_g}{T(x,y)} - L_y^{-1}L_x(h_0+h_1+h_2+\cdots) \\ &\quad - L_y^{-1}\frac{1}{T(x,y)}\frac{\partial T}{\partial x}\frac{\partial}{\partial x}(h_0+h_1+h_2+\cdots) - L_y^{-1}\frac{1}{T(x,y)}\frac{\partial T}{\partial y} \\ &\quad \times \frac{\partial}{\partial y}(h_0+h_1+h_2+\cdots) \end{aligned} \tag{34}$$

A first approximation to Eq. (34) is

$$h_y \approx h_0 = k_3(x)+k_4(x)y - L_y^{-1}\frac{R_g}{T(x,y)} \tag{35}$$

where $k_3(x)$ and $k_4(x)$ are such that the y boundary conditions in Eq. (2) are satisfied. Substituting Eqs. (33) and (35) into Eq. (9) we obtain a first approximation to Eq. (28). Higher-order terms are obtained by successively combining x-partial solution terms

$$h_i = k_{4i+1}(y)+k_{4i+2}(y)x - L_x^{-1}L_y h_{i-1} - L_x^{-1}\frac{1}{T(x,y)}\frac{\partial T}{\partial x}\frac{\partial h_{i-1}}{\partial x} - L_x^{-1}\frac{1}{T(x,y)}\frac{\partial T}{\partial y}\frac{\partial h_{i-1}}{\partial y} \tag{36}$$

with y-partial solution terms

$$h_i = k_{4i+3}(x)+k_{4i+4}(x)y - L_y^{-1}L_x h_{i-1} - L_y^{-1}\frac{1}{T(x,y)}\frac{\partial T}{\partial x}\frac{\partial h_{i-1}}{\partial x} - L_y^{-1}\frac{1}{T(x,y)}\frac{\partial T}{\partial y}\frac{\partial h_{i-1}}{\partial y} \tag{37}$$

Recent contributions suggest that the choice of the initial term greatly influences the rate of convergence and the complexity in the calculation of individual terms, especially for nonlinear equations (Wazwaz and Gorguis, 2004; Wazwaz, 2000b). Thus, as long as the initial term in a decomposition series, usually the forcing function or the initial condition, is described in analytic form, a partial decomposition procedure may offer a simplified approximate solution to many modeling problems. There is considerable latitude in the choice of a first decomposition term. For instance, in Eqs. (36) or (37) h_1 may be composed of the first three terms in the right side only. From Eqs. (31)–(37), a key element is the choice of analytical form for $K(x,y)$, such that the resulting integrals are calculable. For example, assume that the aquifer hydraulic conductivity gradually decreases with increasing x and y and the resulting transmissivity, fitted to a set of individual points, is given by $T(x,y) = a + bx + cy$, where $a = 500$ m^2/month, $b = -0.2$ m/month, and $c = -0.1$ m/month; $l_x = 840$ m; and the rest of the parameters as before. Thus, from Eq. (33)

$$\begin{aligned} &h_x \approx h_0 = k_1(y)+k_2(y)x - \tfrac{R_g}{b^2}I(x,y) \\ &I(x,y) = T(x,y)\ln(T(x,y)) - T(x,y) \\ &k_1(y) = f_1(y) + \tfrac{R_g}{b^2}I(0,y), \quad k_2(y) = \tfrac{R_g}{b}\ln(T(l_x,y)) \end{aligned} \tag{38}$$

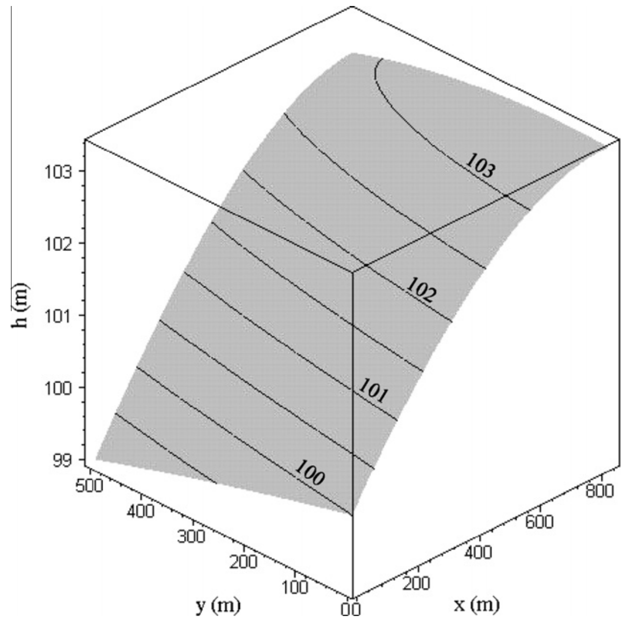

Fig. 4. Groundwater head and contours in a heterogeneous aquifer.

From Eq. (34)

$$h_y \approx h_0 = k_3(x) + k_4(x)y - \tfrac{R_g}{c^2} I(x,y)$$
$$k_3(x) = f_3(x) + \tfrac{R_g}{c^2} I(x,0), \quad k_4(x) = \frac{f_4(x) - k_3(x) + \frac{R_g}{c^2} I(x, l_y)}{l_y} \tag{39}$$

Substituting Eqs. (38) and (39) into Eq. (9) we obtain the first approximation to Eq. (30). From Eqs. (36) and (37) the calculation of higher-order terms suggests a fast convergence in the series. For example the first three decomposition terms at the center of the aquifer ($x = l_x/2$, and $y = l_y/2$) are $h_0 = 104.820$, $h_1 = -1.994$ and $h_2 = 0.668$ meters. Using three terms only, Fig. 4 was produced to depict the water table elevation and contours across the heterogeneous aquifer. As expected, water table elevation tends to be relatively higher in areas, where the transmissivity decreases. Other representations in conductivity spatial variability are possible.

3. Nonlinear flow

One of the most important features of decomposition is its ability to systematically derive solutions to nonlinear equations. Many equations of groundwater flow and groundwater contaminant transport are nonlinear and their linearization may sometimes pose significant errors. In this section we derive approximate decomposition solutions to nonlinear groundwater equations in dimensions greater than one. Consider for example the general nonlinear form of Eqs. 1, 2 in a homogeneous aquifer:

$$\frac{\partial}{\partial x}\left(Kh\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(Kh\frac{\partial h}{\partial y}\right) = -R_g \quad 0 \leqslant x \leqslant l_x,\ 0 \leqslant y \leqslant l_y \tag{40}$$

The x-partial solution of Eq. (40) is given by

$$h \approx h_x = k_1(y) + k_2(y)x - L_x^{-1}\frac{R_g}{Kh} - L_x^{-1}L_y h$$
$$- L_x^{-1}\frac{1}{h}\left(\frac{\partial h}{\partial x}\right)^2 - L_x^{-1}\frac{1}{h}\left(\frac{\partial h}{\partial y}\right)^2 \tag{41}$$

which may be written as

$$h_x = k_1(y) + k_2(y)x - L_x^{-1}L_y h + L_x^{-1}N(h) \tag{42}$$

where k_1 and k_2 are such that the x boundary conditions in Eq. (2) are satisfied, and the nonlinear operator $N(h)$ is given by

$$N(h) = -\frac{1}{h}\left[\frac{R_g}{K} + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right] \tag{43}$$

Eq. (42) may expanded as

$$h_x = h_0 - L_x^{-1}L_y\sum_{i=0}^{\infty} h_i + L_x^{-1}\sum_{i=0}^{\infty} A_i \tag{44}$$

where the A_i are the Adomian polynomials given as a generalized Taylor series expansion about the initial term h_0 :

$$\begin{aligned} A_0 &= Nh_0 \\ A_1 &= h_1\frac{dNh_0}{dh_0} \\ A_2 &= h_2\frac{dNh_0}{dh_0} + \frac{h_0^2}{2!}\frac{d^2Nh_0}{dh_0^2} \\ A_3 &= h_3\frac{dNh_0}{dh_0} + h_1h_2\frac{d^2Nh_0}{dh_0^2} + \frac{h_1^3}{3!}\frac{d^3Nh_0}{dh_0^3} \\ &\vdots \end{aligned} \tag{45}$$

In principle, alternate application of Eqs. (44) and (45) sequentially yields the A_i and the h_i, respectively. However, in practice the number of terms one may calculate depends on the values of the parameters and the complexity in the functional form of $N(h)$. Usually the first few terms in a decomposition series are easily derivable if the initial term, h_0, is simple.

The y-partial solution of Eq. (40) is given by

$$h \approx h_y = k_3(x) + k_4(x)y - L_y^{-1}\frac{R_g}{Kh} - L_y^{-1}L_x h - L_y^{-1}\frac{1}{h}\left(\frac{\partial h}{\partial x}\right)^2$$
$$] \quad - L_y^{-1}\frac{1}{h}\left(\frac{\partial h}{\partial y}\right)^2 \tag{46}$$

which may be written as

$$h_y = k_3(x) + k_4(x)y - L_y^{-1}L_x h + L_y^{-1}N(h) \tag{47}$$

where k_3 and k_4 are such that the y boundary conditions in (2) are satisfied. Eq. (48) may expanded as

$$h_y = h_0 - L_y^{-1}L_x\sum_{i=0}^{\infty} h_i + L_y^{-1}\sum_{i=0}^{\infty} A_i \tag{48}$$

From Eqs. (44) and (2) it is easy to see that a simple choice for the first term, h_0, in the x-partial solution is given by

$$h_0(x,y) = f_1(y) + \frac{f_2(y) - f_1(y)}{l_x}x \tag{49}$$

The second term, h_1, may be obtained from the x-partial solution (44) as

$$h_1 = k_5(y) + k_6(y)x - L_x^{-1}L_y h_0 - L_x^{-1}\frac{1}{h_0}\left[\frac{R_g}{K} + \left(\frac{\partial h_0}{\partial x}\right)^2 + \left(\frac{\partial h_0}{\partial y}\right)^2\right] \tag{50}$$

From Eqs. (48) and (2) a simple first term for the y-partial solution is

$$h_0(x,y) = f_3(x) + \frac{f_4(x) - f_3(x)}{l_y}y \tag{51}$$

and the second term is given from Eq. (48) as

$$h_1 = k_7(x) + k_8(x)y - L_y^{-1}L_x h_0 - L_y^{-1}\frac{1}{h_0}\left[\frac{R_g}{K} + \left(\frac{\partial h_0}{\partial x}\right)^2 + \left(\frac{\partial h_0}{\partial y}\right)^2\right] \tag{52}$$

and combining Eqs. (50) and (52). In general, higher-order terms, h_i, are obtained from

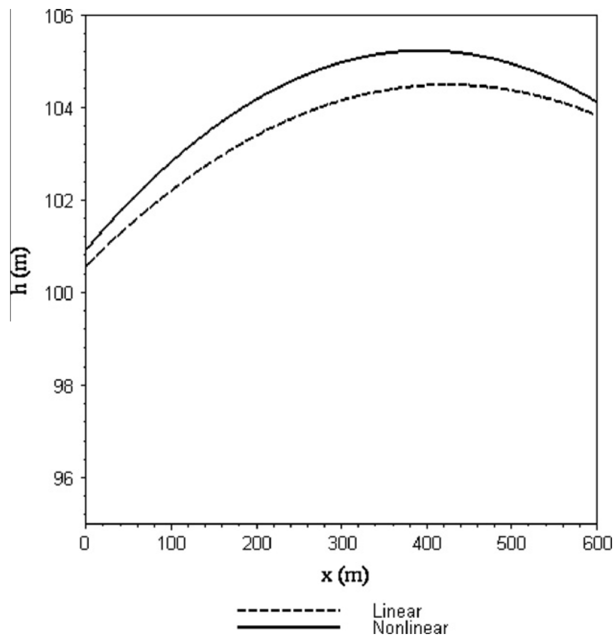

Fig. 5. Comparison between linear and nonlinear head profiles.

$$h_i = \frac{1}{2}\Big[k_{4i+1}(y) + k_{4i+2}(y)x - L_x^{-1}L_y h_{i-1} + L_x^{-1}A_{i-1} + k_{4i+3}(x) + k_{4i+4}(x)y - L_y^{-1}L_x h_{i-1} + L_y^{-1}A_{i-1}\Big] \quad (53)$$

It is instructive to compare linear and nonlinear solutions to the same problem. As an example, assume a hydraulic conductivity $K = 1$ m/month and the rest of the parameters as before. A linear solution is obtained with a transmissivity value $T = K\bar{h}$ with $\bar{h}$ an average saturated thickness estimated as the mean of the heads at the four corners of the aquifer. Fig. 5 shows a comparison between the linear and nonlinear hydraulic head, $h(x,l_y/2)$, through a section in the middle of the aquifer. As expected, the differences between linear and nonlinear solutions tend to be relatively small when an aquifer recharge is small and when hydraulic conductivity values are large. As recharge increases, or conductivity decreases, the errors incurred upon by linearization increase. In other words, linearization may be acceptable in sandy or coarsely-graded aquifers in dry regions. Clay or finely-graded aquifers in wet regions exhibit the highest errors due to linearization. The effect of linearization is to underestimate the magnitude of heads and gradients. This can be seen mathematically, as the linearized model omits several terms in the differential equations.

4. Summary and conclusions

An important limitation of the method of decomposition is that an expansion in a given coordinate explicitly uses the boundary conditions in such axis only, but not necessarily those on the others. In this article several improvements on the method have been proposed, which permit the inclusion of all boundary conditions imposed on a multidimensional boundary-value problem in groundwater; the modeling of irregularly-shaped aquifers; and the consideration of aquifer heterogeneity, multiple sources, and nonlinearity in the equations. The method constitutes a simple modeling procedure for preliminary studies prior to more elaborate numerical analyses. Fast convergence rates, typical of decomposition series, have been shown. Verification was done by comparing decomposition solutions with analytical solutions when available with reasonable agreement. The method can handle aquifer heterogeneity in cases when aquifer heterogeneity in the hydraulic conductivity can be described in an analytical form fitted to a few measured points. Irregularly-shaped aquifer domains can be considered when the aquifer geometrical boundaries are described in an analytical form fitted to a few point on the boundaries. Pumping-well fields, transient flow, and nonlinear equations in several dimensions can be considered. The effect of linearization of the equations is to underestimate the magnitude of the hydraulic heads. Linearization may be a reasonable simplification in coarsely-graded (high conductivity) soils and in dry areas (low recharge). The selection of a simple first term in a decomposition series is important in the calculation difficulty of subsequent ones and the convergence rate.

Appendix A. Convergence of decomposition to exact closed-form solutions

A reviewer of this manuscript suggested a clarification on the convergence of decomposition to an exact linear or nonlinear solution. In this section we present two examples, one linear and one nonlinear, of decomposition series of partial differential equations in groundwater. Consider the problem of a plume migration through an aquifer subject to an irreversible linear first-order kinetic sorption model. The governing equation is given by Serrano (2010)

$$\frac{\partial C}{\partial t} - D\frac{\partial^2 C}{\partial x^2} + u\frac{\partial C}{\partial x} + aC = 0, \quad a = \frac{\rho_b k_1}{n}, \quad -\infty < x < \infty, \quad 0 < t$$
$$C(\pm\infty, t) = 0, \quad C(x,0) = C_i\delta(x) \quad \text{(A1)}$$

where D is the aquifer longitudinal dispersion coefficient (m^2/month) assumed constant; u is the aquifer longitudinal pore velocity (m/month) assumed constant; x is longitudinal distance from the source (m); a is the capacity parameter ($month^{-1}$); ρ_b is the soil dry bulk density (kg/m^3); n is the soil porosity; C_i is the initial contaminant mass per unit cross-sectional area perpendicular to x (kg/m^2); $\delta(x)$ is the Dirac's delta function; and k_1 sub 1is a decay constant (m^3/(kg month)).

Let us define the advective–dispersive differential operator as

$$L_{x,t} = \left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2} + u\frac{\partial}{\partial x}\right) \quad \text{(A2)}$$

Eq. (A2) becomes

$$L_{x,t}C + aC = 0 \quad \text{(A3)}$$

which maybe written as

$$C = -L_{x,t}^{-1}aC \quad \text{(A4)}$$

where the inverse advective–dispersive operator, $L_{x,t}^{-1}$, is given by the convolution integral

$$L_{x,t}^{-1}aC = \int_0^t J_{t-\tau}a\,d\tau \quad \text{(A5)}$$

and the operator $J_t()$ is the strongly continuous semigroup associated with Eq. (A2) and it is given by Serrano (1996)

$$J_t f = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} e^{-\frac{(x-ut-x\prime)^2}{4Dt}} f\,dx\prime \quad \text{(A6)}$$

Now expand c in the right side of Eq. (A4) as the series $C = C_0 + C_1 + C_2 + \cdots$ From Eqs. (A1) and (A6), the first term in the series is given by (see Zauderer, 1983):

$$C_0 = L_{x,t}^{-1}C_i\delta(x) = \frac{C_i e^{-\frac{(x-ut)^2}{4Dt}}}{\sqrt{4\pi Dt}} \quad \text{(A7)}$$

which is the fundamental solution or Green's function of the advective–dispersive equation. The second term in the series is

$$C_1 = -L_{x,t}^{-1} a C_0 = -\frac{atC_i e^{-\frac{(x-ut)^2}{4Dt}}}{\sqrt{4\pi Dt}} \tag{A8}$$

The third term is

$$C_2 = -L_{x,t}^{-1} a C_1 = \frac{a^2 t^2}{2} \frac{C_i e^{-\frac{(x-ut)^2}{4Dt}}}{\sqrt{4\pi Dt}} \tag{A9}$$

In general, the n-th term is given by

$$C_n = -L_{x,t}^{-1} a C_{n-1} = (-1)^n \frac{(at)^n}{n!} \frac{C_i e^{-\frac{(x-ut)^2}{4Dt}}}{\sqrt{4\pi Dt}} \tag{A10}$$

Upon summation, the series converges to

$$C(x,t) = \frac{C_i e^{-\frac{(x-ut)^2}{4Dt} - at}}{\sqrt{4\pi Dt}} \tag{A11}$$

which is the exact closed-form solution to Eq. (A1). Let us now consider the nonlinear advection equation given by Adomian (1991)

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u^2 = 0, \quad u(x,0) = 1/2x, \quad u(0,t) = -1/t \tag{A12}$$

By decomposition, writing $L_t u = -\partial u/\partial x - u^2$, then writing $u = \sum_{n=0}^{\infty} u_n$, and representing u^2 by $u = \sum_{n=0}^{\infty} A_n$ derived for the specific function, where A_n are the Adomian polynomials (Adomian, 1994), we have

$$\begin{aligned} L_t u &= -\frac{\partial}{\partial x} \sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n \\ u &= u_0 - L_t^{-1} \frac{\partial}{\partial x} \sum_{n=0}^{\infty} u_n - L_t^{-1} A_n \\ u_0 &= u(x,0) = \frac{1}{2x} \\ u_1 &= -L_t^{-1} \frac{\partial}{\partial x} u_0 - L_t^{-1} A_0 \\ u_2 &= -L_t^{-1} \frac{\partial}{\partial x} u_1 - L_t^{-1} A_1 \end{aligned} \tag{A13}$$

Substituting the $A_n\{u^2\}$ Adomian (1994) and summing we have

$$u = \frac{1}{2x} + \frac{t}{4x^2} + \frac{t^2}{8x^3} + \frac{t^3}{16x^4} \cdots \tag{A14}$$

which converges to

$$u = \frac{1}{(2x - t)} \tag{A15}$$

The above examples illustrate the principle that decomposition series converge to the exact solution of the differential equation. Rigorous mathematical convergence of decomposition series has already been established in the mathematical community (Abbaoui and Cherruault, 1994; Cherruault, 1989; Cherruault et al., 1992). It is also important to mention the rigorous mathematical framework for the convergence of decomposition series developed by Gabet (1994, 1993, 1992). He connected the method of decomposition to well-known formulations, where classical theorems (e.g., fixed-point theorem, substituted series, etc.) could be used. In many circumstances, including the equations dealt within this paper, a closed-form solution is not possible. As illustrated with the numerical simulations, a fast convergence rate suggests that a few terms in the truncated series constitute a good approximation in hydrologic modeling.

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