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New Approaches to the Propagation of Nonlinear Transients in Porous Media

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Abstract Many problems in regional groundwater flow require the characterization and forecasting of variables, such as hydraulic heads, hydraulic gradients, and pore velocities. These variables describe hydraulic transients propagating in an aquifer, such as a river flood wave induced through an adjacent aquifer. The characterization of aquifer variables is usually accomplished via the solution of a transient differential equation subject to time-dependent boundary conditions. Modeling nonlinear wave propagation in porous media is traditionally approached via numerical solutions of governing differential equations. Temporal or spatial numerical discretization schemes permit a simplification of the equations. However, they may generate instability, and require a numerical linearization of true nonlinear problems. Traditional analytical solutions are continuous in space and time, and render a more stable solution, but they are usually applicable to linear problems and require regular domain shapes. The method of decomposition of Adomian is an approximate analytical series to solve linear or nonlinear differential equations. It has the advantages of both analytical and numerical procedures. An important limitation is that a decomposition expansion in a given coordinate explicitly uses the boundary conditions in such axis only, but not necessarily those on the others. In this article we present improvements of the method consisting of a combination of a partial decomposition expansion in each coordinate in conjunction with successive approximation that permits the consideration of boundary conditions imposed on all of the axes of a transient multidimensional problem; transient modeling of irregularly-shaped aquifer domains; and nonlinear transient analysis of groundwater flow equations. The method yields simple solutions of dependent variables that are continuous in space and time, which easily permit the derivation of heads, gradients, seepage velocities and fluxes, thus minimizing instability. It could be valuable in preliminary analysis prior to more elaborate numerical analysis. Verification was done by comparing decomposition solutions with exact analytical solutions when available, and with controlled experiments, with reasonable agreement. The effect of linearization of mildly nonlinear saturated groundwater equations is to underesti-

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mate the magnitude of the hydraulic heads in some portions of the aquifer. In some problems, such as unsaturated infiltration, linearization yields incorrect results.

Keywords Mathematical models · Nonlinear systems · Adomian's decomposition method · Groundwater · Transient flow

1 Introduction

The problem of stream-aquifer interaction and the exploitation of an aquifer subject to variable pumping require the characterization and forecasting of variables, such as hydraulic heads, hydraulic gradients, and pore velocities. These variables describe hydraulic transients propagating in an aquifer, such as a river flood wave-induced through an adjacent aquifer. The characterization of aquifer variables is usually accomplished via the solution of a transient differential equation subject to time-dependent boundary conditions. Modeling nonlinear wave propagation in porous media is traditionally approached via numerical solutions of governing differential equations. Temporal or spatial numerical discretization schemes permit a simplification of the equations. However, they may generate instability, and require a numerical linearization of true nonlinear problems. Traditional analytical solutions are continuous in space and time, and render a more stable solution, but they are usually applicable to linear problems and require regular domain shapes. The method of decomposition of Adomian (Adomian 1994, 1983; Rach 2008) is an approximate analytical series to solve linear or nonlinear differential equations. It has the advantages of both analytical and numerical procedures. The method of decomposition (Adomian 1994) was introduced as a simple means to solve linear and nonlinear equations in mathematical physics. It consists in deriving an infinite series, much like Fourier series, that in many cases converge to an exact solution. For a simple introduction to the method with applications in surface and subsurface hydrology, engineering analysis, and stochastic methods the reader is referred to Serrano (2011, 2010). For nonlinear equations in particular, decomposition is one of the few *systematic* solution procedures available. With the concepts of partial decomposition and of double decomposition (Adomian 1994), the process of obtaining an approximate solution is simplified. Thus, as long as the initial term in a decomposition series (e.g., the forcing function or the initial condition) is described in analytic form, a partial decomposition procedure may offer a simplified approximate solution to many modeling problems.

A drawback of the method of decomposition limits the use of a set of boundary conditions to a one-dimensional axis only. Thus, a partial decomposition expansion in a given coordinate explicitly uses the boundary conditions in such axis only; applications in two- or three-dimensional domains do not necessarily incorporate the boundary conditions imposed on the other dimensions. Wazwaz (2000a) has shown that a decomposition series can be expressed in terms of the boundary condition not used explicitly and that by equating coefficients of like powers the series can be shown to converge to the exact solution satisfying all of the boundary conditions. For boundary-value problems with simple boundaries (e.g., boundaries at infinity and one-dimensional problems), the procedure is straight forward (Shidfar and Reihani 2010; Serrano 2010). However, this procedure could be elaborate in many cases of practical applications. In this article we present several improvements of the method of decomposition for the practical modeling of groundwater equations: a combination of a partial decomposition expansion in each coordinate in conjunction with successive approximation that permits the consideration of boundary conditions imposed on all of the axes of a multidimensional problem; the analysis of regional flow in irregularly-shaped domains whose geometry can be described functionally; and the solution of nonlinear problems in higher dimensions. Convergence rate is shown to illustrate the fact that in many dissipative systems the first few terms in the series constitute a reasonably accurate approximation. Verification is done when an exact solution is available, and with controlled laboratory measurements in saturated and unsaturated flow problems.

2 Modeling Transient Flow

Consider initially transient regional groundwater flow equation with Dupuit assumptions, with x a horizontal planar coordinate (L), and t the time coordinate (T). A t-partial decomposition solution satisfies the differential equation and its initial condition, but not necessarily the boundary conditions on x. Similarly, an x-partial solution satisfies the differential equation and the x boundary conditions, but not necessarily the initial condition of the system. Although each partial expansion constitutes a general solution to the differential equation (Adomian 1994), a particular solution (i.e., the one needed in actual modeling calculations) requires the evaluation of the constants of integration such that the initial condition and all of the boundary conditions are satisfied. Our objective is to develop a decomposition solution of transient regional groundwater flow that includes all available boundary and initial conditions by combining a decomposition expansion with successive approximation. The governing differential equation is

$$\frac{\partial h}{\partial t} - \frac{T}{S_y}\frac{\partial^2 h}{\partial x^2} = \frac{R_{\mathrm{g}}\,(t)}{S_y}. \qquad 0 \le x \le l_x, \quad 0 < t \tag{1}$$

where h is the hydraulic head (L); R_{g} is mean monthly recharge from rainfall (LT^{-1}); T is the mean aquifer transmissivity (L^{2}T^{-1}); l_x is the aquifer horizontal dimension in the x direction (L); and S_y is the aquifer specific yield. Impose a set of mixed boundary conditions given by

$$h\,(0,t) = V\,(0), \quad \frac{\partial h}{\partial x}\,(l_x, t) = 0, \quad h\,(x,0) = V\,(x) \tag{2}$$

where $V(x)$ is the initial condition representing the head across the aquifer at $t = 0$ (L). Without loss of generality, assume an arbitrary initial condition in (1) to be $V\,(x) = 100 + R_{\mathrm{g}}\,(0)\,l_x x/T - R_{\mathrm{g}}\,(0)\,x^2/2T$, chosen from a corresponding steady-state solution. Assume an arbitrary seasonal recharge from rainfall follows a periodic function of the form

$$R_{\mathrm{g}}\,(t) = a - b\sin\left(\frac{c\pi t}{6}\right) \tag{3}$$

where a (LT^{-1}), b (LT^{-1}), and c (T^{-1}) are constants. This expression represents an aquifer deep recharge from rainfall after infiltration and redistribution in the upper soil layers and could fit a variety of percolation regimes. We seek a solution of the form $h = h_0 + h_1 + h_2 + \ldots$, which may be obtained from partial decomposition solutions to (1). There are two possible decomposition solutions to Equation (1): the t-partial solution, h_t, and the x-partial solution, h_x, (Adomian 1994). The t-partial solution results after defining $L_t = \partial/\partial t$ and $L_x = \partial^2/\partial x^2$, multiplying (1) by L_t^{-1} (i.e., the integral from zero to t) and rearranging:

$$h_t = V\,(x) + L_t^{-1}\frac{R_{\mathrm{g}}}{S_y} + \frac{T}{S_y}L_t^{-1}L_x h_t \tag{4}$$

where h_t is the t-partial solution. Decompose h_t in the right side as $h_t = h_{t0} + h_{t1} + h_{t2} + \ldots$

$$h_t = L_t^{-1} \frac{R_g}{S_y} + \frac{T}{S_y} L_t^{-1} L_x \left(h_{t0} + h_{t1} + h_{t2} + \ldots \right) \tag{5}$$

The first term in the series, h_{t0}, is given by the first term on the right side of (5), which is known:

$$h_{t0} = k_0 (x) + L_t^{-1} \frac{R_g (t)}{S_y} = V (x) + \frac{1}{S_y} \left[at + \frac{6b}{S_y c\pi} \left(1 - \cos \left(\frac{c\pi t}{6} \right) \right) \right] \tag{6}$$

where $k_0 (x)$ satisfies the initial condition, $V(x)$. Equation (6) satisfies the differential Eq. (1) and the initial condition, but not necessarily the x boundary conditions. Now, the x-partial solution results after multiplying Eq. (1) by L_x^{-1} (or the two-fold x indefinite integral), and rearranging:

$$h_x = k_1 (t) + k_2 (t) x - L_x^{-1} \frac{R_g (t)}{T} + \frac{S_y}{T} L_x^{-1} L_t h_x \tag{7}$$

where h_x is the x-partial solution. Decompose h_x in the right side as $h_x = h_{x0} + h_{x1} + h_{x2} + \ldots$ Then,

$$h_x = k_1 (t) + k_2 (t) x - L_x^{-1} \frac{R_g (t)}{T} + \frac{S_y}{T} L_x^{-1} L_t \left(h_{x0} + h_{x1} + h_{x2} + \ldots \right) \tag{8}$$

where $k_1(t)$ and $k_2(t)$ must satisfy the x boundary conditions in (2). If we take h_{x0} as the first three terms in the right side of (8), then

$$h_{x0} \approx h_0 = k_1 (t) + k_2 (t) x - \frac{R_g (t) x^2}{2T} = 100 + \frac{R_g (t) l_x x}{T} - \frac{R_g (t) x^2}{2T} \tag{9}$$

We now have two partial solutions to (1): The t-partial solution Equation (6), and the x-partial solution (9). Since both are approximations to h, a combination of the two yields the first combined decomposition term, h_0:

$$h_0 (x, t) = \frac{1}{2} \left(h_{t0} (x, t) + h_{x0} (x, t) \right) \tag{10}$$

Equation (10) constitutes a first approximate solution to (1), and its boundary and initial conditions (2). Higher-order terms may be obtained. In other words, the ith term in the t-partial expansion is obtained form (5) as

$$h_{ti} = k_{3i} (x) + \frac{T}{S_y} L_x^{-1} L_x h_{i-1}, \quad i > 0 \tag{11}$$

where k_{3i} is such that a homogeneous (i.e., zero) initial condition in (2) is satisfied, and h_{i-1} is the previous combined term in the decomposition series. Then, the ith term in the x-partial expansion is obtained form (8) as

$$h_{xi} = k_{3i+1} (t) + k_{3i+2} (t) x + \frac{S_y}{T} L_x^{-1} L_t h_{i-1}, \quad i > 0 \tag{12}$$

where k_{3i+1} and k_{3i+2} are such that homogeneous (i.e., zero) x-boundary conditions in (2) are satisfied, and h_{i-1} is the previous combined term in the decomposition series. Next, we combine Equations (11) and (12) to derive the next combined term, h_i

$$h_i (x, t) = \frac{1}{2} \left(h_{ti} (x, t) + h_{xi} (x, t) \right) \tag{13}$$

In this manner, by successively adding more terms to the series $h = h_0 + h_1 + h_2 + \ldots$, it is possible to observe a convergence that satisfies the differential equation and all of its boundary conditions. However, many studies indicate that the rate of convergence is so high for dissipative systems that only a few terms yield a good approximation. Recent contributions suggest that the choice of the initial term greatly influences the rate of convergence and the complexity in the calculation of individual terms, especially for nonlinear equations (Wazwaz and Gorguis 2004; Wazwaz 2000). Thus, as long as the initial term in a decomposition series, is described in analytic form, a partial decomposition procedure may offer a simplified approximate solution to many modeling problems. There is considerable latitude in the choice of a first decomposition term. For instance, in (5) h_{t0} may be composed of the first two terms, or just the second term, on the right side of the equality. The convergence of decomposition series has already been established in the mathematical community (Abbaoui and Cherruault 1994; Cherruault 1989; Cherruault et al. 1992). It is also important to mention the rigorous mathematical framework for the convergence of decomposition series developed by Gabet (1994, 1993, 1992)). He connected the method of decomposition to well-known formulations where classical theorems (e.g., fixed-point theorem, substituted series, etc.) could be used.

2.1 Verification with an Exact Solution

To verify the decomposition solution an exact analytical solution was derived (see Appendix A). We arbitrarily adopt the following parameter values: $l_x = 100$ m; $T = 100$ m^2/month; $S_y = 0.1$; $a = 3.14 \times 10^{-3}$ m/month; $b = -3.128 \times 10^{-3}$ m/month; and $c = 1$ month^{-1}. Using (5), (8), and (9–12), we first check for convergence. In this example the most adverse condition occurs at the free boundary, $x = l_x$. The first four terms in the decomposition series at $t = 6$ months are: $h_0(100, 6) = 100.192$ m, $h_1(100, 6) = -0.189$ m, $h_2(100, 6) = 0.091$ m, and $h_3 = -0.007$ m. Using four terms, the decomposition solution yields $h(100, 6) \approx 100.086$ m, compared to an exact value of $h(100, 6) = 100.089$ m, or a 0.004 % in relative error. A maximum relative error of 0.079 % occurs at integer multiples of $t = 12$ month. In this case we obtain a fast convergence and using four decomposition terms we obtain a reasonably accurate solution. The values of the parameters may slow down the rate of convergence. This occurs because aquifer field transmissivity may be two or three orders of magnitude greater than specific yield, and the t-partial solution (4) contains higher-order terms having coefficients T/S_y. In this aquifer, maximum errors occur at the right no-flow boundary, that is at $x = l_x$. Figure 1 shows a comparison between the exact and a two-term decomposition solution of aquifer heads at this location. Water-table profiles at a given time are smooth curves, as can be seen in Fig. 2. We remark that the accuracy of decomposition solutions is reasonable, as compared to the resolution of observed measured variables in groundwater. More importantly, the effort required to produce a decomposition solution is modest, as compared to that of a corresponding exact analytical solution (Appendix A), and especially a numerical solution. A simple analytical expression of aquifer head is advantageous in the derivation of spatially or temporally continuous hydraulic gradients, seepage velocities, and groundwater fluxes. It may be useful in preliminary studies prior to more elaborate numerical modeling.

2.2 Transient Flow in Higher Dimensions

To study a transient problem in higher dimensions, the modeler generates at least one partial decomposition expansion in each dimension, which can be combined to include information

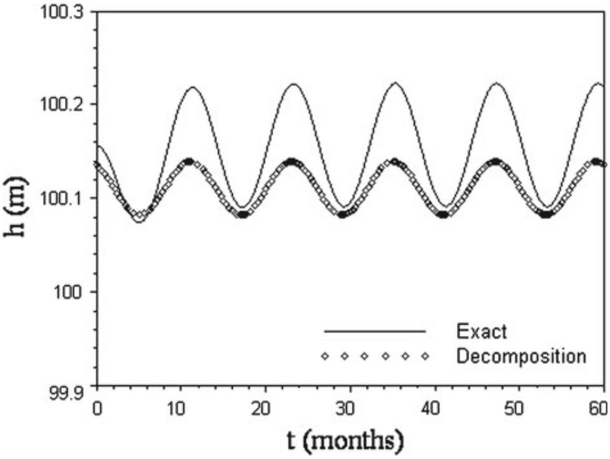

Fig. 1 Comparison between exact and decomposition solutions of transient flow at $x = l_x$

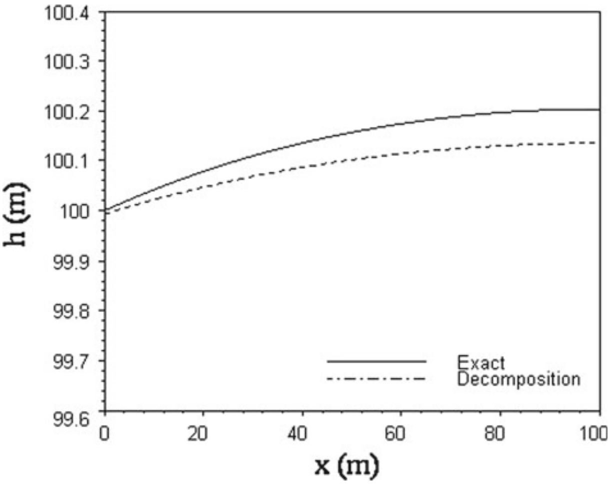

Fig. 2 Exact versus decomposition versions of water-table profile at $t = 12$ months

on all boundary and initial conditions. Consider the transient regional groundwater flow equation with Dupuit assumptions given by

$$\frac{\partial h}{\partial t} - \frac{T}{S_y}\frac{\partial^2 h}{\partial x^2} - \frac{T}{S_y}\frac{\partial^2 h}{\partial y^2} = \frac{R_g}{S_y} \quad 0 \le x \le l_x, \quad 0 \le y \le l_y, \quad 0 < t \tag{14}$$

where and l_x and l_y are the aquifer horizontal dimensions in the x- and y-directions, respectively (L), and the rest of the terms as before. Impose a set of known boundary and initial conditions given as

$$\begin{aligned} h\,(0, y) &= f_1\,(y), \quad h\,(l_x, y) = f_2\,(y), \quad h\,(x, 0) = f_3\,(x), \\ h\left(x, l_y\right) &= f_4\,(x), \quad h\,(x, y, 0) = V\,(x, y), \end{aligned} \tag{15}$$

Equations (14) and (15) describe the elevation of the water table in a long, thin, mildly sloping aquifer, bounded on all sides by streams. The functions f_1, f_2, f_3, and f_4 represent the mean elevation of the water at the boundaries. Assume a transient aquifer recharge given by (3), and an initial condition, $V(x, y)$, given by a steady solution of (14) and (15) as

$$V(x, y) = \frac{1}{2}\left[f_1(y) + \left(\frac{f_2(y) - f_1(y)}{l_x} + \frac{R_g(0)\, l_x}{2T}\right)x - \frac{R_g(0)\, x^2}{2T}\right] + \frac{1}{2}\left[f_3(x) + \left(\frac{f_4(x) - f_3(x)}{l_y} + \frac{R_g(0)\, l_y}{2}\right)y - \frac{R_g(0)\, y^2}{2T}\right] \tag{16}$$

where $R_g(0)$ is the initial recharge rate. The t-partial solution of (14) becomes

$$h_t = h(x, y, 0) + L_t^{-1}\frac{R_g}{S_y} + \frac{T}{S_y}L_t^{-1}L_x(h_{t0} + h_{t1} + \ldots) + \frac{T}{S_y}L_t^{-1}L_y(h_{t0} + h_{t1} + \ldots) \tag{17}$$

from which the first term is

$$h_{t0} = h(x, y, 0) + L_t^{-1}\frac{R_g(t)}{S_y} = h(x, y, 0) + \frac{1}{S_y}\left[at + \frac{6b}{S_y c\pi}\left(1 - \cos\left(\frac{c\pi t}{6}\right)\right)\right] \tag{18}$$

The x-partial solution of (14) is

$$h_x = k_1(y, t) + k_2(y, t)\, x - L_x^{-1}\frac{R_g(t)}{T} + \frac{S_y}{T}L_x^{-1}L_t(h_{x0} + h_{x1} + \ldots) - L_x^{-1}L_y(h_{x0} + h_{x1} + \ldots) \tag{19}$$

where k_1 and k_2 are such that the x boundary conditions in (15) are satisfied. From (19) the first term is given by

$$h_{x0} = f_1(y) + \left(\frac{f_2(y) - f_1(y)}{l_x} + \frac{R_g(t)\, l_x}{2T}\right)x - \frac{R_g(t)\, x^2}{2T} \tag{20}$$

The y-partial solution of (14) is given by

$$h_y = k_3(x, t) + k_4(x, t)\, y - L_y^{-1}\frac{R_g(t)}{T} + \frac{S_y}{T}L_y^{-1}L_t(h_{y0} + h_{y1} + \ldots) - L_y^{-1}L_x(h_{y0} + h_{y1} + \ldots) \tag{21}$$

where $L_y = \partial^2/\partial y^2$, and k_3 and k_4 are such that the y boundary conditions in (15) are satisfied. From (21) the first term is given by

$$h_{y0} = f_3(x) + \left(\frac{f_4(x) - f_3(x)}{l_y} + \frac{R_g(t)\, l_y}{2T}\right)y - \frac{R_g(t)\, y^2}{2T} \tag{22}$$

From (18), (20), and (22) we have three versions of $h_0(x, y, t)$, respectively. Thus, a combination of them will yield the first decomposition term:

$$h_0(x, y, t) + \frac{1}{3}\left(h_{t0}(x, y, t) + h_{x0}(x, y, y) + h_{y0}(x, y, t)\right) \tag{23}$$

Higher-order terms may be derived. The ith term in the t-partial solution, h_{ti}, is obtained from the expansion (17) as

$$h_{ti} = k_{3i+1}(x, y) + \frac{T}{S_y} L_t^{-1} L_x L_{i-1} + \frac{T}{S_y} L_t^{-1} L_y L_{i-1}, \quad i > 0 \tag{24}$$

where h_{i-1} is the $(i-1)$ combined decomposition term, and k_{3i+1} is such that a homogeneous initial condition is satisfied. Similarly, the ith term in the x-partial solution, h_{xi}, is obtained or from the expansion (19) as

$$h_{xi} = k_{3i+2}(y, t) + k_{3i+3}(y, t)\, x + \frac{S_y}{T} L_x^{-1} L_t h_{i-1} - L_x^{-1} L_y h_{i-1}, \quad i > 0 \tag{25}$$

where k_{3i+2} and k_{3i+3} are such that homogeneous x-boundary conditions in (15) are satisfied. Finally, the ith term in the y-partial solution, h_{yi}, is obtained from the expansion (21) as

$$h_{yi} = k_{3i+4}(x, t) + k_{3i+5}(x, t)\, y + \frac{S_y}{T} L_y^{-1} L_t h_{i-1} - L_y^{-1} L_x h_{i-1} \tag{26}$$

where k_{3i+4} and k_{3i+5} are such that homogeneous y-boundary conditions in (15) are satisfied. A combination of the three versions of h_i, Eqs. (24), (25), and (26), respectively, will yield the next term in the decomposition expansion:

$$h_i(x, y, t) + \frac{1}{3}\left(h_{ti}(x, y, t) + h_{xi}(x, y, y) + h_{yi}(x, y, t)\right) \tag{27}$$

As an illustration, let us assume arbitrary expressions for the boundary conditions in (15) representing water level in meters above the sea level:

$$\begin{aligned}
&f_1(y) = 100 - 0.2 \times 10^{-2} y, \quad f_2(y) = 103 - 0.1 \times 10^{-2} y \\
&f_3(x) = 100 + 0.8 \times 10^{-2} x - 0.5 \times 10^{-5} x^2, \\
&\quad f_4(x) = 99 + 0.883 \times 10^{-2} x - 0.5 \times 10^{-5} x^2
\end{aligned}$$

Figure 3 was produced to illustrate the water-table spatial distribution at $t = 8$ months according to a one-term decomposition solution Equations (18), (20), (22), and (23).

2.3 Irregularly-Shaped Domains

Most analytical solutions of boundary-value initial-value problems are restricted to rectangles, squares, circles, and other regular domain shapes. Even though decomposition renders an analytical solution, extension of the above procedure to aquifer domains of irregular geometry is possible if the aquifer boundaries are defined in a functional form. For example, let us assume that in (14) the upper y river boundary has an irregular shape characterized as $l_y(x) = 500 + 0.4x - 0.001x^2$, after fitting a parabola to a few surveyed points on the river channel. For simplicity, let us assume that the head at the boundaries is described by (15), with $h(x, y) = f_4(x)$, on $y = l_y(x)$. The solution needs to be modified to reflect the fact that the integration must now be done over a variable y domain, which is now reflected in (16) and (22) with l_y now being a function of x. Figure 4 shows a plan view of this aquifer with the head contours calculated by (18), (20), (22), and (23), and flow directional vectors calculated after applying Darcy's law and a simple Maple$^{\text{TM}}$ routine. With decomposition, an aquifer of any arbitrary shape in its boundaries can be easily modeled.

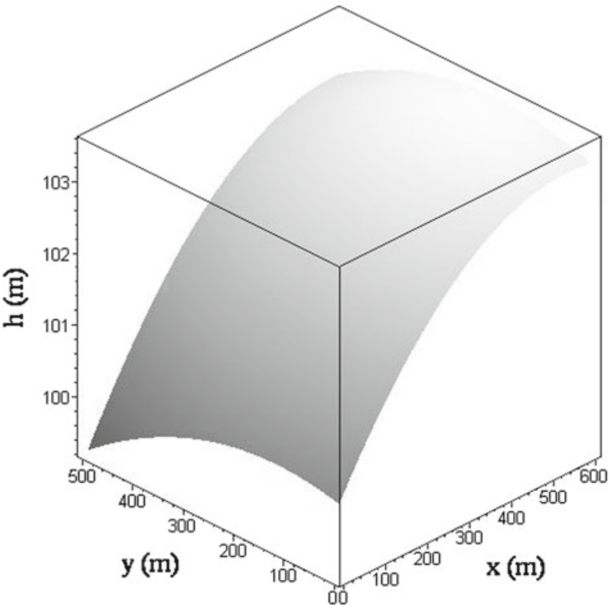

Fig. 3 Transient groundwater flow spatial distribution in at $t = 8$ months

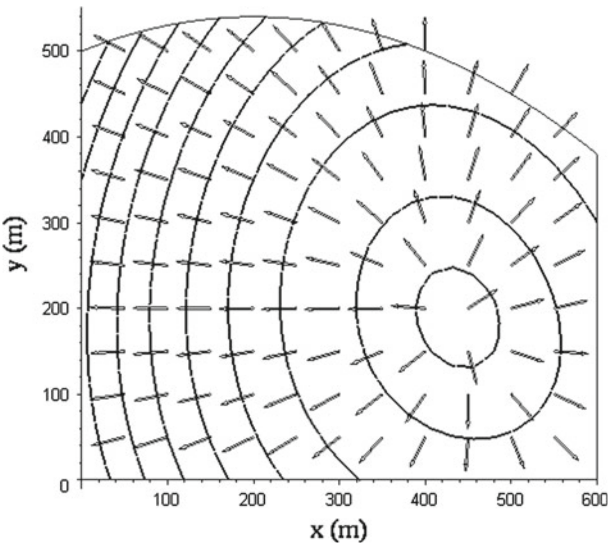

Fig. 4 Transient spatial distribution of hydraulic head at $t = 10$ months and directional velocity vectors in an irregularly-shaped aquifer

2.4 Transient Nonlinear Flow

One of the most important features of decomposition is its ability to systematically derive solutions to nonlinear equations. Many equations of groundwater flow and groundwater contaminant transport are nonlinear and their linearization may sometimes yield significant errors. In this section, we derive approximate decomposition solutions to nonlinear groundwater equations in dimensions greater than one. Consider for example the general nonlinear form of (1) and (2) in a homogeneous aquifer:

$$\frac{\partial h}{\partial t} - \frac{1}{S_y}\frac{\partial}{\partial x}\left(Kh\frac{\partial h}{\partial x}\right) = \frac{R_g(t)}{S_y}, \qquad 0 \le x \le l_x, 0 < t \tag{28}$$

where K is the aquifer hydraulic conductivity (LT^{-1}). The t-partial solution of (28), h_t, becomes

$$h_t = V(x) + L_t^{-1}\frac{R_g}{S_y} + \frac{K}{S_y}L_t^{-1}N(h_t) \tag{29}$$

where the nonlinear operator $N(h_t)$ is given by

$$N(h_t) = \left[h_t L_x h_t + \left(\frac{\partial h_t}{\partial x}\right)^2\right] \tag{30}$$

(29) may be expanded as

$$h_t = V(x) + L_t^{-1}\frac{R_g}{S_y} + \frac{K}{S_y}L_t^{-1}\sum_{i=0}^{\infty} A_i \tag{31}$$

where the A_i are the Adomian (Adomian 1994) polynomials given as a generalized Taylor series expansion about an initial term h_0:

$$\begin{aligned}
A_0 &= Nh_0\\
A_1 &= h_1\frac{\mathrm{d}Nh_0}{\mathrm{d}h_0}\\
A_2 &= h_2\frac{\mathrm{d}Nh_0}{\mathrm{d}h_0} + \frac{h_0^2}{2!}\frac{\mathrm{d}^2Nh_0}{\mathrm{d}h_0^2}\\
A_3 &= h_3\frac{\mathrm{d}Nh_0}{\mathrm{d}h_0} + h_1h_2\frac{\mathrm{d}^2Nh_0}{\mathrm{d}h_0^2} + \frac{h_1^3}{3!}\frac{\mathrm{d}^3Nh_0}{\mathrm{d}h_0^3}
\end{aligned} \tag{32}$$

Equation (32) is not the most efficient way to calculate the A_i polynomials. In recent years, several researcher have developed efficient algorithms for the derivation of the Adomian polynomials. Rach (2008) presented a generalization and new foundations for the efficient calculation of the polynomials, along with a revision of existing literature. New simpler algorithms with promising perspectives for highly nonlinear equations have been developed by Duan and Rach (2011a,b).

In principle, alternate application of (31) and (32) sequentially yields the A_i and the h_i, respectively. However, in practice the number of terms one may calculate depends on the values of the parameters and the complexity in the functional form of $N(h)$. Usually the first few terms in a decomposition series are easily derivable if the initial term, h_0, is simple. From (31) and (2) it is easy to see that a simple choice for the first term, h_{t0}, in the t-partial solution is given by $h_{t0} = V(x)$. The second term, h_{t1}, may be obtained from the t-partial expansion (31) and the first polynomial in (32) applied to h_{t0}:

$$h_{t1} = k_3(x) + L_t^{-1}\frac{R_g}{S_y} + \frac{K}{S_y}L_t^{-1}\left[h_{t0}L_x h_{t0} + \left(\frac{\partial h_{t0}}{\partial x}\right)^2\right] \tag{33}$$

where $k_3(x)$ satisfies a homogeneous initial condition. The x-partial solution of (28) is given by

$$h_x = -\frac{R_g(t)}{K} L_x^{-1} N(h_x) \tag{34}$$

where

$$N(h_x) = -\frac{1}{h}\left[1 + \frac{S_y}{R_g} L_t h_x - \frac{K}{R_g}\left(\frac{\partial h_x}{\partial x}\right)^2\right] \tag{35}$$

where k_1 and k_2 are such that the x boundary conditions in (2) are satisfied. Equation (34) may be expanded as

$$h_x = k_1(t) + k_2(t)\,x - \frac{R_g(t)}{K} L_x^{-1} \sum_{i=0}^{\infty} A_i \tag{36}$$

From (35) and (2), a simple first term for the x-partial solution is $h_{x0} = V(x)$ and the second term is given by (35) and the first polynomial in (32) applied to h_{x0}:

$$h_{x1} = k_4(t) + k_5(t)\,x - \frac{R_g(t)}{K} L_x^{-1} - \frac{1}{h_{x0}}\left[1 + \frac{S_y}{R_g} L_t h_{x0} - \frac{K}{R_g}\left(\frac{\partial h_{x0}}{\partial x}\right)^2\right] \tag{37}$$

where $k_4(t)$ and $k_5(t)$ must satisfy homogeneous boundary conditions in (2). Combining the two-term t-partial solution with the two-term x-partial solution as in (10), we obtain a simple solution to (28).

Higher-order terms in the t-partial solution, h_{ti}, may be derived from (31) and (32) as

$$h_{ti} = k_{3i+1}(x) + \frac{K}{S_y} L_t^{-1} A_{i-1}, \quad i > 0 \tag{38}$$

where $k_{3i+1}(x)$ satisfies a homogeneous initial condition in (2) and A_{i-1} is the $(i-1)$ polynomial in (32) applied on h_{ti-1}. Similarly, higher-order terms in the x-partial solution, h_{xi}, may be derived from (36) and (32) as

$$h_{xi} = k_{3i+2}(t) + k_{3i+3}(t)\,x + \frac{R_g(t)}{K} L_x^{-1} A_{i-1}, \quad i > 0 \tag{39}$$

where $k_{3i+2}(t)$ and $k_{3i+3}(t)$ satisfy homogeneous x boundary conditions in (2), and A_{i-1} is the $(i-1)$ polynomial in (32) applied on $h_{xi-1}(x)$. Finally the the next combined term, h_i, is given by (13).

It is instructive to compare linear and nonlinear solutions to the same problem. As an example, assume a hydraulic conductivity $K = 1$ m/month and the rest of the parameters as before. A linear solution is obtained with a transmissivity value $T = K\bar{h}$ with $\bar{h}$ an average saturated thickness estimated as the mean of the heads at the four corners of the aquifer. Figure 5 shows a comparison between the linear exact (Appendix A) and nonlinear hydraulic head at $t = 10$ months. The linear solution underestimates the magnitude of hydraulic head at some portions of the aquifer. It is important to remark that in this particular example the values of some of the parameters make some of the decomposition terms diverge. In particular, the second term in the x-partial expansion (37) has values of S_y/R_g that violate the Lipschitz condition. As in any other modeling scheme, the hydrologist must calculate these terms in dimensionless domains that satisfy stability requirements (Oden and Demkowicz 2010).

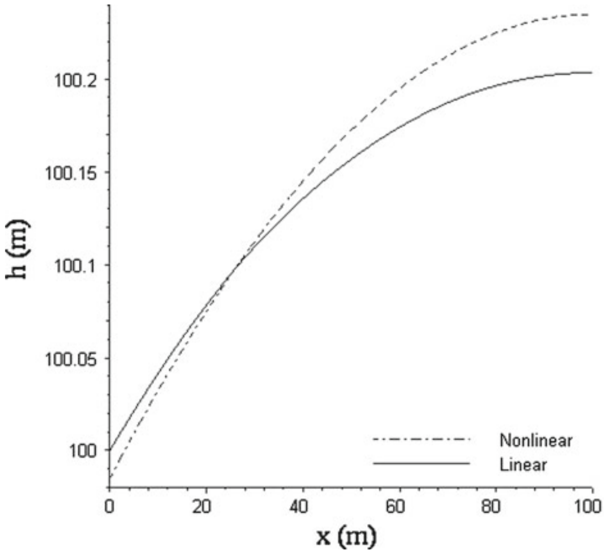

Fig. 5 Comparison between linear and nonlinear spatial distribution of head profiles at $t = 10$ months

2.5 Verification with the Horizontal Infiltration Equation Under a Sharp Wetting Front

In this exercise we compare a decomposition solution to the horizontal infiltration equation with laboratory experimentation and with various classical numerical solutions. We attempt a combination of decomposition with successive approximation that yields a simple approximate solution to a highly nonlinear equation, when traditional numerical solutions present numerous accuracy, complexity, and instability problems. Consider the horizontal infiltration equation in a semi-infinite homogeneous soil with a constant boundary condition maintained on one end is

$$\begin{aligned}&\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial x}\left(D(\theta)\frac{\partial \theta}{\partial x}\right) = 0, \quad 0 < x < \infty, \quad 0 < t \\ &\theta(0,t) = \theta_b, \quad \theta(\infty,t) = \theta_i, \quad \theta(x,0) = \theta_i\end{aligned} \tag{40}$$

where θ is soil volumetric water content; x = horizontal distance (L); t is time (T); θ_b is the water content at the left boundary; θ_i, the initial water content; and $D(\theta)$ is the soil-water diffusivity ($\mathrm{M}^2\mathrm{T}^{-1}$). For the experiment reported in Serrano (1997), $\theta_b = 0.458$, $\theta_i = 0.086$, the soil-water diffusivity (m^2/h) is given by

$$D(\theta) = c_1 e^{\lambda\,\theta^{\alpha}} - 1 \tag{41}$$

where $c_1 = 1\ \mathrm{m}^2/\mathrm{h}$; $\lambda = 500$; and $\alpha = 11$. The t-partial decomposition expansion of (40) is

$$\theta = L_t^{-1}\frac{\partial}{\partial x}\left(D(\theta)\frac{\partial \theta}{\partial x}\right) = L_t^{-1}\frac{\partial}{\partial x}\left(\sum_{j=1}^{\infty} E_j \frac{\partial \theta}{\partial x}\right) \tag{42}$$

where the E_j polynomials are recursively calculated based on successive components of the series $\theta \approx \sum_{j=1}^{N} \theta_j$. From (32), (41), and (42),

$$\theta_0 = \theta_b, \quad E_0 = D(\theta_0) = D(\theta_b) = c_1 e^{\lambda\theta_b^{\alpha}} - 1 \tag{43}$$

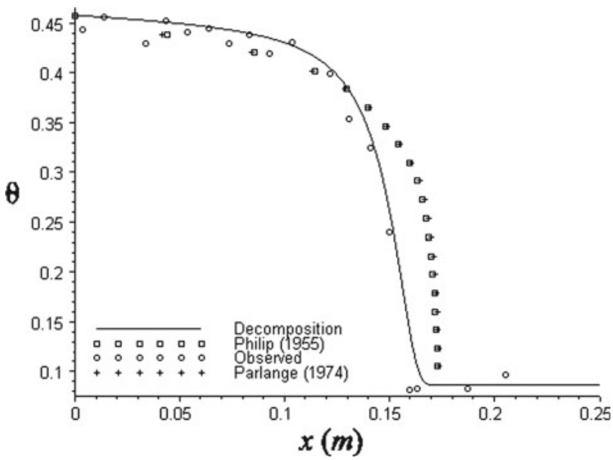

Fig. 6 Water content versus distance at $t = 1$ h

If we use the approximation $D(\theta) \approx d(\theta_0) = E_0$, then (40) reduces to the classical heat flow equation with constant coefficient and whose solution is (Myint-U and Debnath 1987)

$$\theta_1 = \theta_i + (\theta_b - \theta_i)\,\mathrm{erfc}\left(\frac{x}{\sqrt{4E_0 t}}\right) \tag{44}$$

where erfc () denotes the "error function complement." Now from (32) and (44) calculate E_1 and obtain an improved diffusivity:

$$\mathrm{D}(\theta) \approx E_0 + E_1 = D(\theta_0) + \theta_1 \frac{\mathrm{d}D(\theta_0)}{\mathrm{d}\theta_0} = D(\theta_0) + \alpha\lambda\theta_1\theta_0^{\alpha}\left(D(\theta_0)\right) + 1 \tag{45}$$

The improved solution to (40) becomes

$$\theta_2 \approx \theta_i + (\theta_b - \theta_i)\,\mathrm{erfc}\left(\frac{x}{\sqrt{4(E_0 + E_1)\,t}}\right) \tag{46}$$

This process may be continued. However, we hope the series converges fast and only a few terms are needed. Thus, if θ_2 is a good approximation, we may use it to obtain an improved, final, version of the diffusivity:

$$D(\theta) \approx (E_0 + E_1 + E_2)\big|_{\theta_2} = D(\theta_2) + 2\theta_2 \frac{\mathrm{d}D(\theta_2)}{\mathrm{d}\theta_2} = D(\theta_2) + 2\alpha\lambda\theta_2^{\alpha+1}\left(D(\theta_2) + 1\right) \tag{47}$$

which we use in (40) to obtain a final solution

$$\theta \approx \theta_i + (\theta_b - \theta_i)\,\mathrm{erfc}\left(\frac{x}{\sqrt{4(E_0 + E_1 + E_2)\,t}}\right) \tag{48}$$

Figure 6 shows profiles of the water content versus distance profiles at $t = 1$ h, according to four sources: Eq. (48), the classical numerical solution of Philip (1955), the numerical solution of Parlange (1971), and experimental observations. Figure 7 shows the same situation at at $t = 2$ h. Equation (48) appears to be in good agreement with the other solutions and with the observed data. In fact, the decomposition solution (48) appears to better predict the

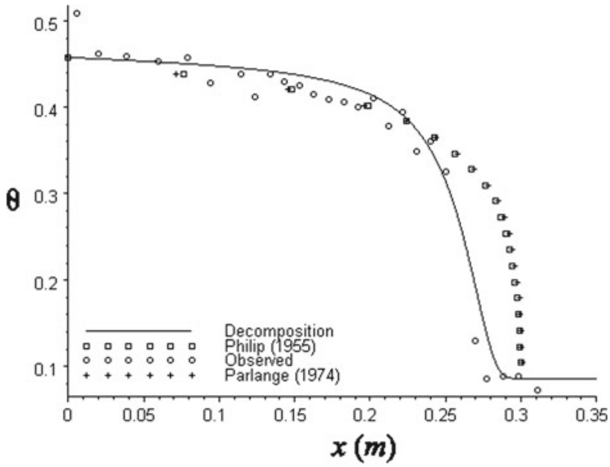

Fig. 7 Water content versus distance at $t = 3$ h

position of the wetting front and the shape of the tail after that than Philip (1955) or Parlange (1971) solutions. It is important to remark that in this problem the linearized solution (not shown) yields incorrect results since it does not exhibit a wetting front. The decomposition solution is simpler, it provides a continuous spatio-temporal description, and it does not exhibit the stability and discretization restrictions of numerical solutions.

3 Summary and Conclusions

An important limitation of the method of decomposition is that an expansion in a given coordinate explicitly uses the boundary conditions in such axis only, but not necessarily those on the others. In this article several improvements on the method have been proposed, which permit the inclusion of all boundary conditions imposed on a multidimensional transient boundary-value problem in groundwater; the modeling of irregularly-shaped aquifers, and nonlinearity in the equations. The method constitutes a simple modeling procedure for preliminary studies prior to more elaborate numerical analyses. Fast convergence rates, typical of decomposition series, have been shown. Verification was done by comparing decomposition solutions with analytical solutions when available, and with controlled experiments, with reasonable agreement. The method can handle aquifer heterogeneity in cases when aquifer heterogeneity in the hydraulic conductivity can be described in an analytical form fitted to a few measured points. Irregularly-shaped aquifer domains can be considered when the aquifer geometrical boundaries are described in an analytical form fitted to a few points on the boundaries. Pumping-well fields, transient flow, and nonlinear equations in several dimensions can be considered. The effect of linearization of mildly nonlinear saturated groundwater equations is to underestimate the magnitude of the hydraulic heads in some portions of the aquifer. In some problems, such as unsaturated infiltration, linearization yields incorrect results.

Appendix A: Exact Solution of Linear Transient Equation

An exact analytical solution to (40) and (41) can be derived by using the concept of the semigroup operator (Serrano and Unny 1987):

$$h\left(x,t\right) = V\left(x\right) + J_t\left(h\left(x,0\right) - V\left(x\right)\right) + \int\limits_0^t J_{t-\tau} \frac{\left(R_{\mathrm{g}}\left(\tau\right) - R_{\mathrm{g}}\left(0\right)\right)}{S_y} \mathrm{d}\tau \tag{B1}$$

In Equation (B1) $V(x)$ is the steady-state part of the solution given as

$$V\left(x\right) = 100 + \frac{R_{\mathrm{g}}\left(0\right) l_x x}{T} - \frac{R_{\mathrm{g}}\left(0\right) x^2}{2T} \tag{B2}$$

The second term in the right side of (B2) represents the semigroup operator J_t acting on the initial condition minus the steady-state:

$$J_t\left(h\left(x,0\right) - V\left(x\right)\right) = \sum_{n=1}^{\infty} b_n\left(h\left(x,0\right) - V\left(x\right)\right) \varphi_n\left(x\right) M_n\left(t\right) \tag{B3}$$

where b_n is the Fourier coefficient given by

$$b_n\left(h\left(x,0\right) - V\left(x\right)\right) = \frac{2}{l_x} \int\limits_0^{l_x} \left(h\left(x,0\right) - V\left(x\right)\right) \sin\left(\xi_n x\right) \mathrm{d}x \tag{B4}$$

ξ_n are the eigenvalues given by

$$\xi_n = \left(\frac{2n-1}{2l_x}\right)\pi, \quad n = 1, 2, 3, \ldots, \tag{B5}$$

$\varphi_n\left(x\right)$ are the basis functions given by

$$\varphi_n\left(x\right) = \sin\left(\xi_n x\right) \tag{B6}$$

and

$$M_n\left(t\right) = e^{-\frac{\xi_n^2 T t}{S_y}} \tag{B7}$$

Now the third term in (B1) represents a convolution integral of the semigroup operator acting on the transient recharge minus the steady part of the recharge:

$$\int\limits_0^t J_{t-\tau}\left(\frac{R_{\mathrm{g}}\left(\tau\right) - R_{\mathrm{g}}\left(0\right)}{S_y}\right)\mathrm{d}\tau = \sum_{n=1}^{\infty} b_n\left(1\right) \varphi_n\left(x\right) \int\limits_0^t \left(\frac{R_{\mathrm{g}}\left(t-\tau\right) - R_{\mathrm{g}}\left(0\right)}{S_y}\right) M_n\left(\tau\right) \mathrm{d}\tau \tag{B8}$$

References

Abbaoui, K., Cherruault, Y.: Convergence of Adomian's method applied to differential equations. Comp. Math. Appl. **28**(5), 103–109 (1994)

Adomian, G.: Stochastic systems. Academic Press, New York (1983)

Adomian, G.: Solving Frontier Problems in Physics: the Decomposition Method. Kluwer, Dordrecht (1994)

Cherruault, Y.: Convergence of Adomian's method. Kybernetes **18**(2), 31–38 (1989)

Cherruault, Y., Saccomardi, G., Some, B.: New results for convergence of Adomian's method applied to integral equations. Math. Comput. Model. **16**(2), 85–93 (1992)

Duan, J.-S., Rach, R.: New higher-order numerical one-step methods based on the Adomian and the modified decomposition methods. Appl. Math. Comput. **218**, 2810–2828 (2011)

Duan, J.-S., Rach, R.: A new modification of the Adomian decomposition method for solving boundary value problems for higher-order nonlinear differential equations. Appl. Math. Comput. **218**, 4090–4118 (2011)

Gabet, L.: Equisse d'une Théorie Décompositionnelle et Application aux Equations aux Dérivées Partialles. Dissertation, Ecole Centrale de Paris, Paris (1992)

Gabet, L.: The decomposition method and linear partial differential equations. Math. Comput. Model. **17**(6), 11–22 (1993)

Gabet, L.: The decomposition method and distributions. Comput. Math. Appl. **27**(3), 41–49 (1994)

Myint-U, T., Debnath, L.: Partial Differential Equations for Scientists and Engineers. North Holland, Amsterdam (1987)

Oden, J.T., Demkowicz, L.: Applied Functional Analysis. Chapman & Hall, London; CRC Press, Boca Raton (2010)

Parlange, J.Y.: Theory of water movement in soils: I. One-dimensional absorption. Soil Sci. **111**(2), 134–137 (1971)

Philip, J.R.: Numerical solution of equations of the diffusion type with diffusivity concentration-dependent. Trans. Faraday Soc. **51**(7), 391 (1955)

Rach, R.: A new definition of the adomian polynomials. Kybernetes **37**(7), 910–955 (2008)

Serrano, S.E.: The Theis Solution in Heterogeneous Aquifers. Gr. Water **35**(3), 463–467 (1997)

Serrano, S.E.: Hydrology for Engineers, Geologists, and Environmental Professionals. An Integrated Treatment of Surface, Subsurface, and Contaminant hydrology. HydroScience Inc., Ambler, PA (2010)

Serrano, S.E.: Engineering Uncertainty and Risk Analysis. A Balanced Approach to Probability, Statistics, Stochastic Models, and Stochastic Differential Equations. 2nd edn. HydroScience Inc., Ambler, PA (2011)

Serrano, S.E., Unny, T.E.: Semigroup solutions of the unsteady groundwater flow equation with stochastic parameters. Stoch. Hydrol. Hydraul. **1**(4), 281–296 (1987)

Shidfar, A., Reihani, P.: A series solution of nonlinear injection–extraction well equation. Commun. Nonlinear Sci. Numer. Simulat. **15**, 2278–2283 (2010)

Wazwaz, A.-M.: A new algorithm for calculating adomian polynomials for nonlinear operators. Appl. Math. Comput. **111**, 53–69 (2000)

Wazwaz, A.-M.: A note on using adomian decomposition method for boundary value problems. Found. Phys. Lett. **13**(5), 493–498 (2000a)

Wazwaz, A.M., Gorguis, A.: Exact solutions for heat like and wave like equations with variable voefficients. Appl. Math. Comp. **149**, 15–29 (2004)