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ANALYSIS OF STOCHASTIC GROUNDWATER FLOW PROBLEMS. PART III: APPROXIMATE SOLUTION OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

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Following the theory presented in Part I and Part II of these series of articles, functional analysis theory and a formulation of the Ito's lemma in Hilbert spaces are outlined as a practical alternative to the problem of finding the equations satisfying the moments of a stochastic partial differential equation of the type appearing in groundwater flow. By combining the moments equations derived from Ito's lemma and the strongly continuous semigroup associated with a particular partial differential operator in a Sobolev space, very simple solutions of the moments equations can be obtained. The most important feature of the moments equations derived from Ito's lemma is that these deterministic equations can be solved by any analytical or numerical method available in the literature. This permits the analysis and solution of stochastic partial differential equations occurring in two-dimensional or three-dimensional domains of any geometrical shape. The method provides a rigorous bridge between the abstract-theoretical analysis of stochastic partial differential equations and computer-oriented numerical techniques. An illustrative example showed the potential applications of the method in regional groundwater flow analysis subject to general white noise disturbances. The example used the present method in combination with the boundary integral equation method to solve the problem of regional groundwater flow in a two-dimensional domain subject to a random phreatic surface. Given the stochastic properties of the boundary condition, the first two moments of the potential as well as sample functions were found.

1. INTRODUCTION

Following the theory and applications of stochastic PDE presented in Part I and Part II, our intention here is the development of a solution method for stochastic PDE appearing in two-dimensional or three-dimensional domains. Our aim is the construction of a functional-analytic link between the abstract-probabilistic theory of stochastic PDE and the increasingly used numerical techniques.

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Let us recall the abstract stochastic evolution equation (see also the list of symbols):

$$\frac{\partial u}{\partial t}(x, t) + \mathbf{A}(x)u(x, t) = g(x, t) + w(x, t) \tag{1}$$

$$u|_{\partial G} = 0 \tag{2}$$

$$u(x, 0) = u_0(x) \tag{3}$$

where $u(x, t) \in H_0^m(G) \times \Omega$; H_0^m is the mth order Sobolev space of second-order random functions; $G \subset \mathbb{R}^n$ is an open domain with boundary ∂G; Ω is the basic probability sample space; $\mathbf{A}$ is a partial differential operator of the form given in Part I.

The state of the system at time t is the function $u(x, t)$, which represents hydraulic potential or a similar other groundwater flow variable. g and u_0 are given deterministic inputs whereas $w(x, t)$ is a white noise process in time and it depends smoothly on the space variable x:

$$\mathbf{E}\{w(x, t)\} = 0 \tag{4}$$

$$\mathbf{E}\{w(x_1, t_1)w(x_2, t_2)\} = q(x_1, x_2)\delta(t_1 - t_2) \tag{5}$$

Here q is a given symmetric positive function and δ is the delta function. In the case when the operator $\mathbf{A}$ is independent of time t, the solution of the evolutional system (1)—(3) is given by:

$$u(x, t) = \mathbf{J}_t u_0 + \int_0^t \mathbf{J}_{t-s}g(s)\mathrm{d}s + \int_0^t \mathbf{J}_{t-s}\mathrm{d}\beta(s) \tag{6}$$

where $\mathbf{J}_t$ is a strongly continuous semigroup in the Hilbert space associated with the evolution operator $\mathbf{A}$ (see Parts I and II).

The expected value of the process $u(t)$ is then:

$$\mathbf{E}\{u(x, t)\} = \mathbf{J}_t u_0 + \int_0^t \mathbf{J}_{t-s}g(s)\mathrm{d}s \tag{7}$$

Using the properties of semigroups (see Part I; Curtain and Pritchard, 1978), the correlation function of the process $u(x, t)$ is:

$$\begin{aligned}\mathbf{E}\{u(x, t_1)u(x, t_2)\} = {} & \mathbf{J}_{t_1+t_2}u_0 + \mathbf{J}_{t_1}u_0\int_0^{t_2}\mathbf{J}_{t_2-\zeta}g(\zeta)\mathrm{d}\zeta + \mathbf{J}_{t_2}u_0\int_0^{t_1}\mathbf{J}_{t_1-s}g(s)\mathrm{d}s \\ & + \int_0^{t_1}\int_0^{t_2}\mathbf{J}_{t_1-s}g(s)\mathbf{J}_{t_2-\zeta}g(\zeta)\mathrm{d}s\mathrm{d}\zeta + q_1\int_0^{t_1}\mathbf{J}_{t_1+t_2-2s}\mathrm{d}s\end{aligned} \tag{8}$$

It was accepted then that there is enough information in the first two moments of the solution of u and that because of analytical difficulties the scope of the solution is limited to that of determining the first two moments.

Equations (7) and (8) were applied to the case of one-dimensional regional groundwater flow, with an initial water table composed of a sinusoidal curve, as conceived by Toth, 1963a.

It is possible to apply the above formulation to two-dimensional problems, although the complexity of the eqn. (8) is significantly increased in terms of the semigroup operator **J**. The solution can be arrived at in a straightforward manner, because it is possible to derive the analytic semigroup of a two-dimensional groundwater flow equation. However, the integral terms would seriously limit the shape of the domain to square or rectangular forms. Since groundwater-flow problems appearing in nature are often not so simple the need for an approximate solution method for a random PDE in a domain of arbitrarily specified geometric shape arises. This method should provide a rigorous mathematical link between the probabilistic analysis of the equation and the existing deterministic numerical methods of solution of partial differential equations (finite differences, finite elements, boundary elements, etc.).

An excellent account of the available approximate methods of solutions for random differential equations is presented by Lax (1980). They can be described as perturbation techniques (Chow, 1972, 1975); Hierarchy techniques (Richardson, 1964; Beran, 1968); stochastic Green's function approach (Adomian, 1970, 1971, 1976, 1983); reduction to deterministic PDE (Srinivasan and Vasudevan, 1971; Soong, 1973; Soong and Chuang, 1973; Kohler and Boyce, 1974; Boyce, 1979); numerical methods (Kohler and Boyce, 1974; Mil'shtein, 1974; Rao et al., 1974). Successive approximation and stochastic approximation techniques (Tsokos and Padgett, 1974; Becus, 1979); and the method of moments, which is a version of Galerkin method for approximation of operator equations and should not be confused with the moments of a random process (Lax and Boyce, 1976; Lax, 1976, 1977, 1979).

Most of the above methods have been applied to random ordinary differential equations and indeed little work has been done on the development of workable numerical methods to approximate solutions of random PDE. Some important work has been done by Sun (1979a,b). One of the first applications of finite element analysis to problems in groundwater flow through random media was done by Sagar (1978b). He used the Galerkin formulation in the usual sense, except that the coefficients in the linear combination were random functions in conjunction with a Taylor series expansion to obtain the first two moments.

A theoretical justification of these discretization procedures was presented by Becus (1980). He defined an equivalent variational formulation for problem (1)–(3) and proved the existence and uniqueness of a solution by a spatial discretization. He noted that the discretization must be done not only in the spatial coordinates, but also in the probabilistic variables $\omega \in (\Omega, B, P)$, since it involves an internal approximation of V. This fact was also noted by Sun (1979a) in his work on finite element solution of random equations. Becus (1980) also concluded that the practical implementation of these approximation schemes may still be a formidable task.

The method presented in this paper is an alternative way to obtain the first two moments of the solution by an application of Ito's lemma. The method has proved to be very useful in the development of moment equations of random ordinary differential equations (Jazwinski, 1970). See for example Unny (1984) for an interesting application of Ito's lemma to random differential equations in catchment modeling.

2. THE ITO'S LEMMA IN HILBERT SPACES

Let us first define the Ito's lemma in Hilbert spaces. Let $a(t)$ be a stochastic process with values in H satisfying:

$$\int_0^T \|a(t)\|_H \,\mathrm{d}t < +\infty$$

for all T, and $z(t)$ a continuous stochastic process in H of the form:

$$z(t) = z(0) + \int_0^t a(s)\mathrm{d}s + \int_0^t \Gamma(s)\mathrm{d}\beta(s) \tag{9}$$

where $z(0)$ is a random variable in H, $\mathrm{d}\beta(s)$ is Brownian motion increment and $\Gamma(s)$ is a stochastic process in H such that:

$$\mathbf{E}\left\{\int_0^T \|\Gamma\|^2 \,\mathrm{d}t\right\} < +\infty, \qquad \text{for } t \quad \textit{finite}$$

Now let $\phi(z, t)$ be a functional on $H \times [0, T]$ which is twice continuously differentiable in H and once continuously differentiable in t. Additionally, $\frac{\partial \phi}{\partial z}$ and $\frac{\partial^2 \phi}{\partial z^2}$ are assumed to be bounded on bounded sets of H.

Hence, the Ito's lemma in Hilbert spaces may be written as (Bensoussan and Iria-Laboria, 1977; Sawaragi et al., 1978):

$$\begin{aligned}\phi[z(t), t] = {} & \phi[z(0), 0] + \int_0^t \left(\frac{\partial \phi}{\partial z}, a\right)\mathrm{d}s + \int_0^t \left[\frac{\partial \phi}{\partial z}, \Gamma \mathrm{d}\beta(s)\right] \\ & + \tfrac{1}{2}\int_0^t tr\Gamma^* \frac{\partial^2 \phi}{\partial z^2}\Gamma q \mathrm{d}s + \int_0^t \frac{\partial \phi}{\partial t}\mathrm{d}s\end{aligned} \tag{10}$$

This equation can be derived from the usual Ito's lemma developed, for example in Jazwinski (1970).

Interpreting eqns. (1)—(3) in the Ito sense, applying the Ito's lemma with $z(t) = u(t)$, and differentiating we obtain:

$$\begin{aligned}\frac{\partial \phi(u)}{\partial t} = {} & -\left(\frac{\partial \phi}{\partial u}, \mathrm{A}u\right) + \left(\frac{\partial \phi}{\partial u}, \Gamma w\right) + \left(\frac{\partial \phi}{\partial u}, g\right) \\ & + \tfrac{1}{2} tr\left[\Gamma^* \frac{\partial^2 \phi}{\partial u^2}\Gamma q\right] + \frac{\partial \phi}{\partial t}\end{aligned} \tag{11}$$

By taking expectation it yields:

$$\frac{\partial \mathbf{E}\{\phi(u)\}}{\partial t} + \mathbf{E}\left\{\left(\frac{\partial \phi}{\partial u}, \mathbf{A}u\right)\right\} = \mathbf{E}\left\{\left(\frac{\partial \phi}{\partial u}, g\right)\right\} + \tfrac{1}{2} tr\, \mathbf{E}\left\{\Gamma^* \frac{\partial^2 \phi}{\partial u^2} \Gamma q\right\} + \mathbf{E}\left\{\frac{\partial \phi}{\partial t}\right\} \tag{12}$$

Taking $\phi(u) = (h, u)$, $h \in V^*$, where h forms a basis in V^*, eqn. (12) yields an equation for the mean of the solution $M_1 = \mathbf{E}\{u(t)\}$:

$$\frac{\mathrm{d}M_1}{\mathrm{d}t} + \mathbf{A}M_1 = g \tag{13}$$

Next set $\phi(u) = (h_1, u)(h_2, u)$ so that $\mathbf{E}\{\phi(u)\} = (M_2 h_1, h_2)$ for $h_1, h_2 \in V^*$. Then eqn. (12) gives an equation for the correlation operator M_2 or the second moment of $u(t)$ (Chow, 1979):

$$\frac{\mathrm{d}M_2}{\mathrm{d}t} + (\mathbf{A} \oplus \mathbf{A})M_2 = gM_1 + (\Gamma^* \otimes \Gamma q) \tag{14}$$

where $\oplus$ and $\otimes$ denote the direct sum and tensor product of two operators on appropriate tensor product spaces. Basically $\mathbf{A} \oplus \mathbf{A}$ implies the summation of the operator $\mathbf{A}$ on two orthogonal directions to form the complete space.

Similarly, higher-order moments may be obtained by defining $M_n(h) = \mathbf{E}\{\prod_{j=1}^{n} (h_j, u)\}$ and using eqn. (12). These moment equations hold in the weak sense. The interesting feature is that eqns. (13) and (14) are deterministic and may be solved by any analytical or approximate method available in the literature.

Consider as an example the homogenized one-dimensional groundwater flow equation in the Sobolev space $H_0^1(0, L)$ solved in Part I:

$$\frac{\partial y}{\partial t} - \frac{T}{S}\frac{\partial^2 y}{\partial x^2} = w_1(t) \tag{15}$$

$$y(0, t) = 0 \tag{16}$$

$$\frac{\partial y}{\partial x}(L, t) = 0 \tag{17}$$

$$y(x, 0) = h_0(x) - V(x) + \beta_2(x) = y_0(x) + \beta_2(x) \tag{18}$$

For the case of random initial conditions only, we can now apply the Ito's lemma in the Sobolev space H_0^1 and from eqn. (13) we obtain the equation for the first moment:

$$\frac{\partial M_1}{\partial t} - \frac{T}{S}\frac{\partial^2 M_1}{\partial x^2} = 0 \tag{19}$$

with the homogeneous boundary conditions:

$$M_1(0, t) = 0 \tag{20}$$

$$\frac{\partial M_1}{\partial x}(L, t) = 0 \tag{21}$$

and the initial condition:

$$M_1(x, 0) = M_1(0) = \mathbf{E}\{y_0(x) + \beta_2(x)\} = y_0(x) \tag{22}$$

The solution of eqns. (19)—(22) is given by:

$$M_1 = \mathbf{E}\{y(t)\} = \mathbf{J}_t M_1(0) \tag{23}$$

where $\mathbf{J}_t$ is the semigroup associated with the partial differential operator in eqn. (19) given by:

$$\mathbf{J}_t y = \frac{2}{L}\sum_{n=1}^{\infty} \exp\left(\frac{-\lambda_n^2 Tt}{S}\right) \sin(\lambda_n x) \int_0^L y(s)\sin(\lambda_n s)\mathrm{d}s \tag{24}$$

Thus the complete solution for the first moment is:

$$M_1 = \frac{2}{L}\sum_{n=1}^{\infty} \exp\left(\frac{-\lambda_n^2 Tt}{S}\right) \sin(\lambda_n x) \int_0^L y_0(s)\sin(\lambda_n s)\mathrm{d}s \tag{25}$$

Replacing $y(x, t)$ by $h(x, t) - V(x)$, eqn. (25) becomes the mean value in the potential head h.

Now applying eqn. (14), we can derive an equation for the second moment:

$$\frac{\partial M_2}{\partial t} - 2\,\frac{T}{S}\,\frac{\partial^2 M_2}{\partial x^2} = 0 \tag{26}$$

$$M_2(0, t) = 0 \tag{27}$$

$$\frac{\partial M_2}{\partial x}(L, t) = 0 \tag{28}$$

$$M_2(x, 0) = M_2(0) = \mathbf{E}\{[y_0 - \beta_2(x)]^2\} = y_0^2(x) + qx \tag{29}$$

The solution of eqns. (26)—(29) is given by:

$$M_2 = \mathbf{E}\{[y(t)]^2\} = \mathbf{J}_{2t} M_2(0) \tag{30}$$

where $\mathbf{J}_{2t}$ represents the square of the semigroup $\mathbf{J}_t$, that is:

$$M_2 = \frac{4}{L^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \exp\left(\frac{-\lambda_n^2 Tt}{S}\right) \exp\left(\frac{-\lambda_m^2 Tt}{S}\right) \sin(\lambda_n x)\sin(\lambda_m x) \cdot \int_0^L\int_0^L [y_0^2(x) + q_2 x]\sin(\lambda_n x)\sin(\lambda_m x)\mathrm{d}x\mathrm{d}x \tag{31}$$

Noting that:

$$M_2 = \mathbf{E}\{[h - V(x)]^2\} = \mathbf{E}\{h^2\} - 2\mathbf{E}\{hV\} + \mathbf{E}\{V^2\} \tag{32}$$

and that $y_0^2 = [h_0 - V(x)]^2$, it is easy to see that eqn. (31) is exactly the correlation function derived in Part II when $x_1 = x_2$. It is interesting to observe that by applying the Ito's lemma the derivation of the correlation function is considerably simplified.

Now for the case of random forcing term in eqns. (15)—(18) only, we can also use the Ito's lemma to derive the moment equations. Now applying eqn. (13), we obtain an equation for the first moment:

$$\frac{\partial M_1}{\partial t} - \frac{T}{S}\frac{\partial^2 M_1}{\partial x^2} = 0 \tag{33}$$

$$M_1(0, t) = 0 \tag{34}$$

$$\frac{\partial M_1}{\partial x}(L, t) = M_1(0) = \mathbf{E}\{y_0(x)\} = y_0(x) \tag{35}$$

The solution of eqns. (33)—(35) is then:

$$M_1 = \mathbf{E}\{y(t)\} = \mathbf{J}_t M_1(0) \tag{36}$$

or:

$$M_1 = \frac{2}{L}\sum_{n=1}^{\infty} \exp\left(\frac{-\lambda_n^2 Tt}{S}\right) \sin(\lambda_n x) \int_0^{\infty} y_0(x)\sin(\lambda_n x)\mathrm{d}x \tag{37}$$

Solving as before in terms of the hydraulic potential $h(x, t)$ we obtain an expression for $\mathbf{E}\{h(x, t)\}$.

Now applying eqn. (14) to eqns. (15)—(18) and random forcing term only, we obtain an equation for the second moment:

$$\frac{\partial M_2}{\partial t} - 2\frac{T}{S}\frac{\partial^2 M_2}{\partial x^2} = q_1 \tag{38}$$

$$M_2(0, t) = 0 \tag{39}$$

$$\frac{\partial M_2}{\partial x}(L, t) = 0 \tag{40}$$

$$M_2(x, 0) = M_2(0) = \mathbf{E}\{y_0^2(x)\} = y_0^2(x) \tag{41}$$

The solution of eqns. (38)—(41) is:

$$M_2 = \mathbf{E}\{y^2(t)\} = \mathbf{J}_{2t} M_2(0) + \int_0^t \mathbf{J}_{2(t-s)} q_1 \,\mathrm{d}s \tag{42}$$

$\mathbf{J}_{2t}$ is again the square of the semigroup $\mathbf{J}_t$.

$$M_2 = \frac{4}{L^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \exp\left(\frac{-\lambda_n^2 Tt}{S}\right) \exp\left(\frac{-\lambda_m^2 Tt}{S}\right) \sin(\lambda_n x)\sin(\lambda_m x)$$
$$\cdot \int_0^L\int_0^L y_0 y_0 \sin(\lambda_n x)\sin(\lambda_m x)\mathrm{d}x\,\mathrm{d}x$$

$$+\frac{4q_1}{L^2}\int_0^t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \exp\left[\frac{-\lambda_n^2 T(t-s)}{S}\right] \exp\left[\frac{-\lambda_m^2 T(t-s)}{S}\right]$$

$$\cdot \sin(\lambda_n x)\sin(\lambda_m x)\int_0^L\int_0^L \sin(\lambda_n x)\sin(\lambda_m x)\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}s \tag{43}$$

Transforming this expression in terms of $\mathrm{E}\{h^2(x, t)\}$ as before and solving, we obtain the equation for the correlation function, which is exactly the same one derived in Part II when t_1 is equal to t_2.

Clearly the use of Ito's lemma is an efficient and practical procedure to derive the moment equations of a stochastic PDE. We shall study its applications to problems with more complex geometrical domains.

3. APPLICATION TO RANDOM GROUNDWATER FLOW PROBLEMS IN COMPLEX GEOMETRIES

Let us consider the interesting problem of two-dimensional regional groundwater flow as conceived by Toth (1962, 1963a). Discussions after the publication of Toth's papers (Davis, 1963; Toth, 1963b), and their extensions to multi-layered watersheds (Freeze, 1969), led to the work by Vandenberg (1980). In his article Vandenberg modified the steady state Laplace equation with sinusoidal water table used by Toth, to a more realistic one in which the governing equation is the unsteady groundwater flow equation subject to an oscillating, time-dependent, water table. He concluded that, depending on the shape of the watershed and on the period of oscillation, the deviation from the steady state was considerable. This is specially true for short fluctuation periods in the water table. This suggests that daily perturbations in the water-table level may have an important effect on the potential distribution within the aquifer.

There is also some controversy about the use of the elasticity and compression storage S in an unconfined aquifer. Vandenberg himself (1980, appendix) recognized the difficulties and argued that at least the elastic storage is the essential parameter determining the response of groundwater potential and flow velocities in the subsurface to local perturbations, even though the cause of the perturbations is associated with water-table fluctuations responding to recharge or exploitation.

However, there is sufficient theoretical and experimental evidence that the elastic storage (Bear, 1972, p. 8) resulting from compressibility of aquifer and water is much smaller than the specific yield (volume of water released from a vertical column of aquifer of unit horizontal cross-section, per unit decline in phreatic surface).

Hence, the problem of regional groundwater flow in a homogeneous, isotropic, phreatic aquifer is described by (Bear, 1972):

$$\nabla^2\phi = \frac{S}{K}\frac{\partial\phi}{\partial t} \qquad \text{on } G \tag{44}$$

$$\phi|_{\partial G_1} = f_1 \tag{45}$$

$$\phi = \eta \qquad \text{on } z = \eta \tag{46}$$

$$\frac{\partial \eta}{\partial t} = \frac{K}{n_e}\left(\nabla_2 \phi \nabla_2 \eta - \frac{\partial \phi}{\partial z}\right) \qquad \text{on } z = \eta \tag{47}$$

$$\phi(t = 0) = \phi_0 \qquad \text{on } G \tag{48}$$

where ϕ is the hydraulic potential, S is the specific storage due to the elastic properties of aquifer and water, n_e is the aquifer effective porosity, η is the potential at the free surface, z is the vertical coordinate, ∇_2 is the horizontal gradient operator, ϕ_0 is the initial potential in the aquifer, f_1 is the potential at the boundaries, and the rest of the terms are as in Part II.

Two approaches are followed in solving this boundary value problem. The most commonly used one employs the Dupuit assumptions and derives a different continuity equation, the Boussinesq equation (Bear, 1972, Chapter 8). In the Boussinesq equation the z coordinate no longer exists, ϕ is replaced by h (the water-table head) and the nonlinear boundary condition along the phreatic surface (eqn. 47) no longer applies. The Boussinesq equation is still a nonlinear equation, but there are several well-known methods of linearization (see Bear, 1972, pp. 408—430). This was essentially the method used in Parts I and II.

An alternative method, which is explored in this section, is to neglect the elastic storage in the unconfined aquifer (see Bear, 1979, p. 331) and to consider the exact statement of the phreatic flow in a homogeneous isotropic domain as: Determine $\phi(x, z, t)$ in the flow domain so that ϕ satisfies:

$$\nabla^2 \phi = 0 \qquad \text{on } G \tag{49}$$

subject to eqns. (45)—(48) (see Bear, 1972, p. 423 for additional discussion).

The nonlinear boundary condition (47) may be modified to include deep percolation input. It may be linearized by neglecting the quadratic terms (Bear, 1972, p. 259).

Based on the above discussion, we shall solve the regional groundwater flow problem in a homogeneous isotropic aquifer governed by eqn. (49) subject to eqns. (45)—(48), with the linearized boundary condition at the free surface (eqn. 47) expressed as a random PDE accounting for all uncertainties, and particularly, for all perturbations and fluctuations in the water table.

To be precise, let us consider the half two-dimensional section of a symmetric watershed represented elsewhere in Fig. 2. We follow some of the assumptions made by Toth: (1) that at some depth a subhorizontal formation of relatively low permeability exists; and (2) that there is no appreciable flow component in the y direction. We use a similar geometrical shape in flow domain, except that no assumption is made to approximate it to a rectangular shape. We should reiterate that in this case the flow domain could be represented by any geometrical shape resulting from the previous geological investigations on the shape and underground properties of the aquifer or aquifers in the watershed. In that sense, the present approach has all the advantages of existing numerical techniques. We represent the left, right and bottom boundary conditions as straight lines for comparison with Toth's results and for simplicity in the illustration of the method.

The random boundary value problem can now be re-stated as:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{50}$$

$$\frac{\partial \phi}{\partial x}(0, z, t) = 0 \tag{51}$$

$$\frac{\partial \phi}{\partial x}(L, z, t) = 0 \tag{52}$$

$$\frac{\partial \phi}{\partial z}(x, 0, t) = 0 \tag{53}$$

$$\phi = \eta \qquad \text{on } z = \eta \tag{54}$$

$$\frac{\partial \eta}{\partial t} = -\left(\frac{K}{n_e}\right)\frac{\partial \phi}{\partial z} + \frac{I}{n_e} + \frac{\mathrm{d}\beta}{\mathrm{d}t} \qquad \text{on } z = \eta \tag{55}$$

$$\phi(x, z, 0) = \phi_0(x, z) \tag{56}$$

where I is the deep percolation for unit horizontal area of the aquifer and has been assumed constant in this case, and eqn. (55) is the linearized free-surface boundary condition. The random term $\frac{\mathrm{d}\beta}{\mathrm{d}t}$ is a white noise process in time defined by eqns. (4) and (5). See Unny and Karmeshu (1984) for a justification in introducing a Gaussian process as the random component.

The first step in solving eqns. (50)—(56) is to transform our boundary value problem in $\phi \epsilon H^1(G)$ into an equivalent one in u, where u belongs to the Sobolev space $H_0^1(G)$ with compact support. This easily done by defining $\phi = u + V$, where $u(x, z, t)$ satisfies the differential equation with homogeneous boundary conditions and $V(x, z)$ satisfies the steady state problem when t tends to infinity. Hence our original stochastic boundary value problem (50)—(56) is transformed into the following set of problems.

(1) $V(x, z)$ satisfying the deterministic time-independent set of equations:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{57}$$

$$\frac{\partial V}{\partial x}(0, z) = 0 \tag{58}$$

$$\frac{\partial V}{\partial x}(L, z) = 0 \tag{59}$$

$$\frac{\partial V}{\partial z}(x, 0) = 0 \tag{60}$$

$$V = \eta \qquad \text{on } z = \eta \tag{61}$$

$$\frac{\partial V}{\partial z}(x, \eta) = \frac{I}{K} \tag{62}$$

where eqn. (62) has been deduced from eqn. (55) as a steady-state free-surface boundary condition.

(2) $u(x, z, t)$ satisfying the stochastic equation with homogeneous boundary conditions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{63}$$

$$\frac{\partial u}{\partial x}(0, z, t) = 0 \tag{64}$$

$$\frac{\partial u}{\partial x}(L, z, t) = 0 \tag{65}$$

$$\frac{\partial u}{\partial z}(x, 0, t) = 0 \tag{66}$$

$$u(x, z, t) = \eta(x, z, t) - V(x, z) \qquad \text{on } z = \eta \tag{67}$$

$$\frac{\partial u}{\partial t} = -\left(\frac{K}{n_e}\right)\frac{\partial u}{\partial z} + \frac{d\beta}{dt} \qquad \text{on } z = \eta \tag{68}$$

$$u(x, z, 0) = \phi_0(x, z) - V(x, z) \tag{69}$$

Now we can apply the Ito's lemma in the Sobolev space H_0^1 to problem (63)–(69) to obtain (from eqn. 13):

(3) The first moment equation for $u(x, z, t)$ in (2) given by $M_1 = \mathbf{E}\{u\}$ as:

$$\frac{\partial^2 M_1}{\partial x^2} + \frac{\partial^2 M_1}{\partial z^2} = 0 \tag{70}$$

$$\frac{\partial M_1}{\partial x}(0, z, t) = 0 \tag{71}$$

$$\frac{\partial M_1}{\partial x}(L, z, t) = 0 \tag{72}$$

$$\frac{\partial M_1}{\partial z}(x, 0, t) = 0 \tag{73}$$

$$M_1 = \mathbf{E}\{\eta(x, z, t) - V(x, \eta)\} \qquad \text{on } z = \mathbf{E}\{\eta\} \tag{74}$$

$$\frac{\partial M_1}{\partial t} = -\left(\frac{K}{n_e}\right)\frac{\partial M_1}{\partial z} \qquad \text{on } z = \mathbf{E}\{\eta\} \tag{75}$$

$$M_1(x, z, 0) = \phi_0(x, z) - V(x, z) \tag{76}$$

$\mathbf{E}\{\phi\}$ can now be calculated from:

$$\mathbf{E}\{\phi(x, z, t)\} = \mathbf{E}\{u(x, z, t)\} + V(x, z) \tag{77}$$

Finally, applying eqn. (14) to (63)—(69) (see Serrano, 1985 for details),

(4) The second moment equation for u in (2) is given by $M_2 = \mathbf{E}\{u^2\}$ as:

$$\frac{\partial^2 M_2}{\partial x^2} + \frac{\partial^2 M_2}{\partial z^2} = 0 \tag{78}$$

$$\frac{\partial M_2}{\partial x}(0, z, t) = 0 \tag{79}$$

$$\frac{\partial M_2}{\partial x}(L, z, t) = 0 \tag{80}$$

$$\frac{\partial M_2}{\partial z}(x, 0, t) = 0 \tag{81}$$

$$M_2 = \mathbf{E}\{[\eta(x, z, t) - V(x, \eta)]^2\} \qquad \text{on } z = \mathbf{E}\{\eta\} \tag{82}$$

$$\frac{\partial M_2}{\partial t} = -2\left(\frac{K}{n_e}\right)\frac{\partial M_2}{\partial z} + q \qquad \text{on } z = \mathbf{E}\{\eta\} \tag{83}$$

$$M_2(x, z, 0) = [\phi_0(x, z) - V(x, z)]^2 \tag{84}$$

where q is the variance parameter of the white noise process. $\mathbf{E}\{\phi^2\}$ can now be calculated from:

$$\mathbf{E}\{\phi^2\} = \mathbf{E}\{u^2\} + 2V\mathbf{E}\{u\} - V^2 \tag{85}$$

Hence, solution in terms of moments of the random boundary value problem (50)—(56) may be obtained by solving the three deterministic boundary value problems (57)—(62), (70)—(76) and (78)—(84). Any numerical procedure available in the literature can be used.

In recent years the boundary integral equation method (BIEM) has become an important numerical method (Brebbia, 1980; Crouch and Starfield, 1983). BIEM has the advantages of finite elements while considerably reducing the disadvantages (Liggett and Liu, 1983). Because of its simplicity and innovative approach, BIEM is going to be used in the solution of the moment equations and generation of sample functions of our regional groundwater flow problem.

Basically, BIEM uses the divergence theorem (Spiegel, 1980), the Green's second identity and a free-space Green's function to state:

$$\int_{\partial G}\left(U\frac{\partial \mathbf{Q}}{\partial n} - \mathbf{Q}\frac{\partial U}{\partial n}\right) \mathrm{d}(\partial G) = 0 \tag{86}$$

where U is the hydraulic potential ϕ and $\mathbf{Q}$ is the two-dimensional free-space Green's function satisfying the LaPlace equation (Greenberg, 1971):

$$\mathbf{Q} = \ln r \tag{87}$$

where r is the distance between the singular point P and another point P' on the boundary. Replacing eqn. (87) in (86) (see Liggett and Liu, 1983, for details on the integration of the singularity) we obtain an expression giving the potential at any point P defined in terms of a boundary integral:

$$2\pi\phi(P) = \int_{\partial G} \left[\phi(\mathbf{Q}) \frac{\partial}{\partial n} (\ln r) - \ln r \frac{\partial}{\partial n} \phi(\mathbf{Q}) \right] \mathrm{d}(\partial G) \tag{88}$$

Since in a well-posed problem ϕ and $\frac{\partial \phi}{\partial n}$ are not known everywhere on ∂G, and instead either ϕ or $\frac{\partial \phi}{\partial n}$ are known at all points of the boundary, the integral eqn. (88) can be used to find the "missing data", by moving the point P to the boundary. Hence, eqn. (88) becomes:

$$\alpha\phi(P) = \int_{\partial G} \left(\frac{\phi}{r} \frac{\partial r}{\partial n} - \ln r \frac{\partial \phi}{\partial n} \right) \mathrm{d}(\partial G) \tag{89}$$

where α is the angle in radians between the boundary segments at P, i.e. $\alpha = 2\pi$ where P is a smooth part of the boundary.

Equation (89) can be solved by choosing a finite number of points on the boundary and numerically performing the integration. Thus the procedure is to select a number of nodes P_j ($j = 1,2,3, \ldots, N$) on the boundary ∂G and to perform the contour integration using each P_j successively as origin. In the present example linear elements and linear interpolation functions for the potential and its normal derivative are used for simplicity, although nonlinear elements and higher-order interpolation could be used. Integrating over the segment between P_j and P_{j+1} and then summing the contributions from all boundary segments [i.e., using P_j ($j = 1,2, \ldots, N$) successively] a system of algebraic equations for ϕ_j and $\left(\frac{\partial \phi}{\partial n}\right)_j$ is obtained:

$$\sum_{j=1}^{N} R_{i,j}\phi_j = \sum_{j=1}^{N} L_{i,j} \left(\frac{\partial \phi}{\partial n}\right)_j \qquad j = 1,2, \ldots, N \tag{90}$$

where the coefficients $\mathbf{R}_{i,j}$ and $L_{i,j}$ depend only on the geometry of the boundary. For a detailed explanation of the method and the derivation of eqn. (90) the reader is referred to Liggett and Liu (1983). Once the known boundary conditions are used in eqn. (90), it is possible to solve the system of linear algebraic equations for the unknown ϕ or $\frac{\partial \phi}{\partial n}$ on each boundary node. Then the potential function in the interior of the region can be obtained by performing the integration along the boundary, as indicated by eqn. (88). In order to apply BIEM to our example problem (50)–(56), the relation between ϕ and $\frac{\partial \phi}{\partial n}$ must be known on the free surface. For two-dimensional problems, the linearized free-surface boundary conditions are (Bear, 1972):

$$\phi = \eta(x, z, t) \qquad \text{on } z = \eta \tag{91}$$

$$\frac{\partial \eta}{\partial t} = -\frac{K}{n_e}\frac{\partial \phi}{\partial n} + \frac{I}{n_e} + \frac{d\beta}{dt} \qquad \text{on } z = \eta \tag{92}$$

This free-surface boundary condition may be written in a finite difference form for a time step Δt, which gives the potential at time $k\Delta t$:

$$\phi^{k+1} = \phi^k - \frac{K\Delta t}{n_e}\left[\theta \left(\frac{\partial \phi}{\partial n}\right)^{k+1} + (1-\theta)\left(\frac{\partial \phi}{\partial n}\right)^{k}\right] + \Delta t\left[\theta \frac{I^{k+1}}{n_e} + (1-\theta)\frac{I^k}{n_e}\right] + \Delta\beta^{k+1} \tag{93}$$

where θ is a weighting factor that positions the derivative and recharge between the time levels k and $k + 1$, and $\Delta\beta^{k+1}$ is a sample increment of the Brownian motion process at time level $k + 1$ specified as $N(0, q\Delta t)$. Equation (93) establishes a relation between ϕ and $\partial\phi/\partial n$ at the time level $k + 1$ and it can be used in eqn. (90) to generate sample functions of the unknown boundary values. Since ϕ and η are exchangeable, eqn. (93) can also be used for computation of sample functions of the free-surface elevation η^{k+1} at time $(k + 1)\Delta t$, after the normal derivative of the potential has been calculated.

As an example, in Fig. 1 L was assumed to be equal to 1000 m and C equal to 500 m. The effective porosity of aquifer $n_e = 1.0$, hydraulic conductivity $K = 10^{-3}$ cm s^{-1} as before. The basis of the program used was the FORTRAN listing of the program GM8 appearing in the appendix of the

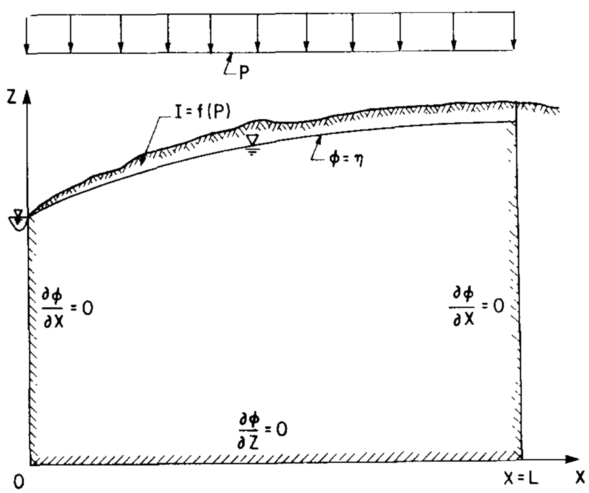

Fig. 1. Definition sketch.

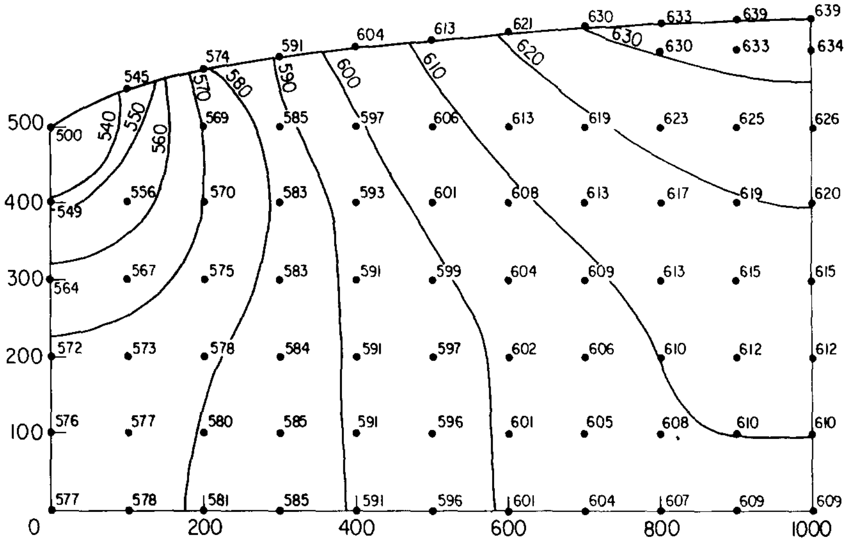

Fig. 2. Initial potential distribution (m).

book by Liggett and Liu (1983). This program was modified and considerably extended to be capable of solving for M_1, M_2 and generate random unsteady free-surface profiles, through eqn. (93), update the new domain geometry and boundary coordinates in the free surface, and successively generate new sample profiles and sample internal potential distributions.

This first author was originally involved with problems related to the hydrology of desert areas. This interest, combined with the recent human disasters in regions of the third world due in part to severe drought, has motivated the writer to use the example as a model for prolonged recession. Therefore, deep percolation was assumed equal to zero in all the trial runs. Figure 2 shows the initial potential distribution across the aquifer. It was obtained by oversaturating the aquifer with $I = 10\,\mathrm{mm\,h^{-1}}$ during several months (with an arbitrary initial condition) and then allowing it to recede for about the same period of time; random components were not acting at this point.

Simulations were started thereafter. The noise variance parameter was assumed equal to 1.0, which is somewhat large, but serves for illustration purposes, given the scale of the aquifer. Figure 4 shows the sample potential function after 300 rainless days (or 10 months), which is an average dry period in several semi-desert areas of the torrid zone. Note the way in which the boundary was divided in nodes and that the point ($x = 0, z = 500$) always has a potential of 500 m. This is due to the assumption that the river, or the natural surface drainage, maintains a relatively constant level. It is interesting to note that for this example the water-table level may decrease up to 18 m.

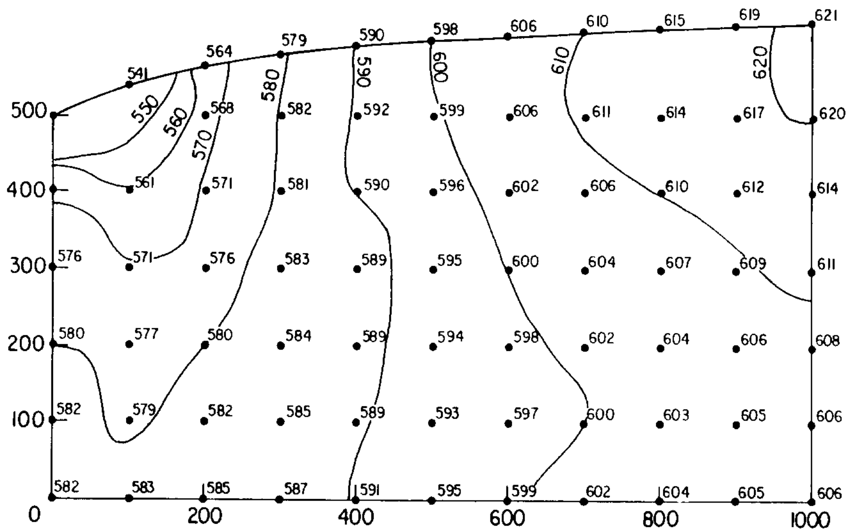

Fig. 3. Mean potential after 300 rainless days (m).

The homogenizing function $V(x, z)$ satisfying eqns. (57)–(62) is simply $V(x, z) = 500$ everywhere in the domain (note that this would not be the case when deep percolation occurs). The program solved for M_1 the problem (70)–(76), and the mean potential distribution after 300 rainless days was obtained at several points in the aquifer. Figure 3 shows the results.

Finally the program solved for M_2 in the problem (78)–(84) and the time standard deviation function was computed for the same internal points.

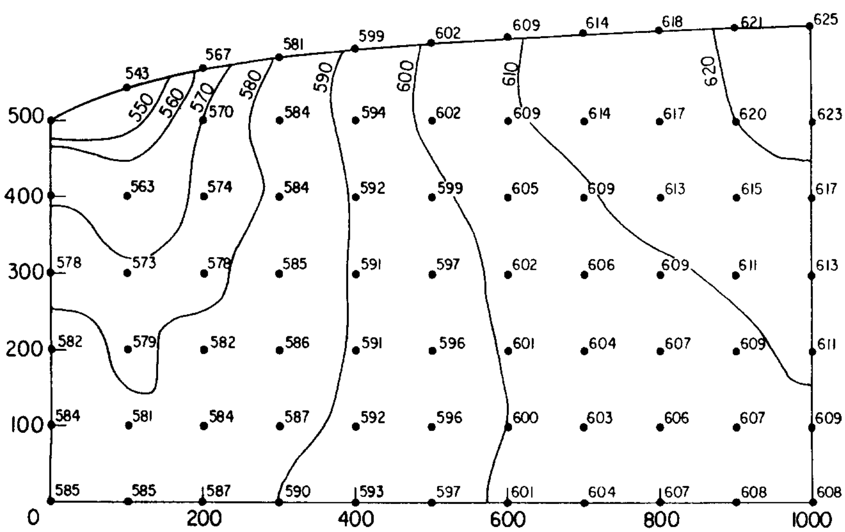

Fig. 4. Sample potential function after 300 rainless days (m).

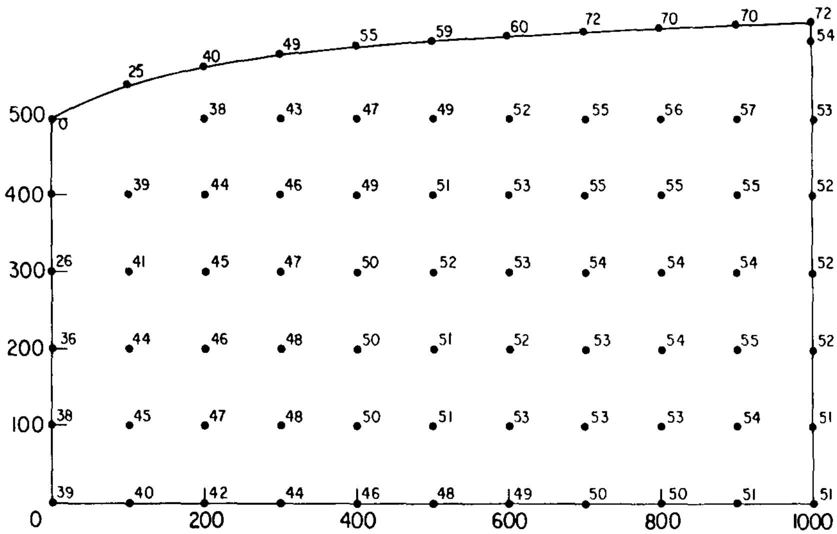

Fig. 5. Standard deviation (m) in aquifer after 300 rainless days.

Basically, the program runs similarly, except that instead of potential values at the boundaries and at the internal points it computes second-moment values $\mathrm{E}\{\phi^2(x_i,z_i,t_1)\}$, given the parameter q and the time level (t_1). Figure 5 shows the spatial distribution of the time standard deviation $[\mathrm{E}\{\phi^2(x_i,z_i, 300)\} - \mathrm{E}^2\{\phi(x, z, 300)\}]^{1/2}$ after 300 rainless days. The lowest values of the standard deviation are at the point (0, 500) and its surroundings, where the variation in potential is minimum, and the highest values in the standard deviation are on the free surface at the points where the maximum variation in potential with time occurred.

The sensitivity of the standard deviation to the noise variance parameter was not studied, but it is obvious from the equations that at greater values of q, greater values of second moment will occur. It is also suspected that the second moment, and thus the dispersion, increases with time.

We can conclude that generally the potential distribution within a deep unconfined regional aquifer is sensitive to random-time perturbations in the free surface. The magnitude of the internal variations depends on the variance parameter q of the perturbing white noise. A random-unsteady free-surface boundary condition seems to be a more appropriate way of treating regional unconfined groundwater flow to account for unknown connections with other watersheds, approximations, inaccurate data and all unexplained processes perturbing the theoretical equations. The choice between two-dimensional unconfined flow as above, or one-dimensional flow with Dupuit approximations depends on the aquifer depth and geometry, in agreement with results found elsewhere.

The model developed could easily be adapted for application purposes by including a functional relationship between the free surface and the ground surface level. Implementations of this type, as well as additional calibration goes beyond the scope of the present work.

CONCLUSIONS AND RECOMMENDATIONS

The Ito's lemma in Hilbert spaces presents an important and practical alternative to the problem of finding the equations satisfying the moments of a stochastic PDE. By combining the moments equations derived from the Ito's lemma and the semigroup associated with a particular partial differential operator in stochastic PDE, very simple solutions of the moment equations can be obtained. The method proved most useful for semigroups of partial differential operators in a one-dimensional domain. For higher dimensions, although the solutions are conceptually possible, the method is not as efficient due to the complexity in the analytical expression of a semigroup in more than one dimension.

However, the most important feature of the moment equations derived from the Ito's lemma is that these deterministic equations can be solved by any analytical or numerical method available in the literature. This permits the analysis and solution of stochastic PDE occurring in two-dimensional or three-dimensional domains of any geometrical shape. The illustrative example presented in Section 3 showed the potential applications of the method in regional groundwater flow analysis subject to general white noise disturbances. The method provides the needed rigorous link between the abstract-theoretical analysis of random PDE and the increasingly used numerical techniques in engineering applications.

The example application also showed that the potential distribution within a deep regional unconfined aquifer is sensitive to random-time perturbations in the free surface. The magnitude of the internal variations depend on the white noise variance parameter q.

A random-unsteady free-surface boundary condition seems to be a more appropriate way of treating regional unconfined groundwater flow to account for unexplained connections with other watersheds, inaccurate data, approximations, and all uncertainties and deviations from the theoretical equations.

The choice between one-dimensional flow with Dupuit assumptions and two-dimensional flow with an unsteady free-surface boundary condition depends on the depth and geometry of the aquifer. One-dimensional Dupuit flow works reasonably well in areas of the aquifer far from the recharge and the discharge points, in agreement with previous research.

The model presented in Section 3 could easily be adapted for application purposes by including a functional relationship between the free-surface elevation and the ground surface elevation.

LIST OF SYMBOLS

$\mathbf{A}$	A partial differential operator, a probability space
$\mathbf{A}^*$	Dual operator of $\mathbf{A}$
$b_i(t)$	Unidimensional Brownian motion processes
B	A Borel field
C	Constant piezometric head at $x = 0$
d^n	nth ordinary derivative
$\mathrm{E}\{\cdot\}$	Expectation operator
f	A functional
f_1	Dirichlet boundary condition
g	Forcing function, generally
∂G	The boundary of the G domain
$g(x, t, \omega)$	Random forcing function, generally
G	Spatial domain in $\mathbb{R}^n$
h	Piezometric head
h_0	Initial piezometric head
H	Separable Hilbert space
H_0^1	First-order Sobolev space of first differentiable random functions with compact support
$H^1(G)$	First-order Sobolev space of first differentiable random functions $f \in L_2(G)$ such that all the first partial derivatives belong to $L_2(G)$
H_0^m	mth order Sobolev space with compact support
I	Deep percolation
$\mathbf{J}_t$	A strongly continuous semigroup
$\mathrm{J}(\omega)$	Random boundary condition
l	Linear space
ln	Natural logarithm
L	Aquifer length, linear space
L_1	Space of integrable distributions
$L_2(\Omega, P; H)$	Space of square-integrable random functions from $\Omega \times (\cdot)$ into H
L_2	Space of square-integrable distributions
$L_{i,j}$	A matrix
M_1	First moment of u
M_2	Second moment of u
n	Counter, normal direction, an index
n_e	Aquifer effective porosity
N	A constant, number of half-cycles for the sinusoidal representation of the water table, number of boundary nodes
$P(\cdot)$	A probability measure
P_j	Geometric points
$\mathbf{q}$	Covariance operator
$\mathbf{Q}$	A boundary operator, a twice-differentiable function
$R(k)$	Standard normal deviates
$R_{i,j}$	A matrix
S	Specific storage
t	Time coordinate
T	Time domain, aquifer transmissivity
u	A dummy variable, a subindex
u_0	Initial condition of the system
$u(x, t, \omega)$	State of system at time t
$u_0(x, \omega)$	Random initial condition

U	A twice-differentiable function
V	A real separable Hilbert space
V'	The dual of V
$V(x)$	Steady state function, a Sobolev space
w	White noise process
$w_1(t)$	White noise process in time
$w_2(x)$	White noise process in space
x	Spatial coordinate
y	A random process, spatial coordinate
z	Spatial coordinate
$z(t)$	A continuous stochastic process
$\beta(t)$	Brownian (or Wiener) process in time
$\Gamma(t)$	A function, a matrix coefficient
Δ	Increment or interval
λ_i	Inverse of Brownian motion variance parameter
λ_n	Eigenvalues
η	Phreatic surface function
ζ	Dummy variable
ϕ	Fluid potential
π	3.1415. . .
Π	Product
Σ	Summation
ω	Sample elements of probability space Ω
Ω	Probability space
∇	Gradient
$\mathbb{R}$	One-dimensional real space
$\mathbb{R}^n$	nth dimensional real space
∂	Partial derivative
∞	Infinity
$\oplus$	Direct sum of two operators
$\otimes$	Tensor product of two operators

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