

Boundary Element Solution of the Two-Dimensional Groundwater Flow Equation with Stochastic Free Surface Boundary Condition

Sergio E. Serrano* and T. E. Unny

Department of Civil Engineering, University of Waterloo, Waterloo, Ontario, Canada N-2L 3G1

An innovative approach to the approximate solution of stochastic partial differential equations in groundwater flow is presented. The method uses a formulation of the Ito's lemma in Hilbert spaces to derive partial differential equations satisfying the moments of the solution process. Since the moments equations are deterministic, they could be solved by any analytical or numerical method existing in the literature. This permits the analysis and solution of stochastic partial differential equations occurring in two-dimensional or three-dimensional domains of any geometrical shape. The method is tested for the first time in the present paper through a practical application in a sandy phreatic aquifer at the Chalk River Nuclear Laboratories, Ontario, Canada. The equation solved is the two-dimensional Laplace equation with a dynamic, randomly perturbed, free surface boundary condition. The moments equations are derived and solved by using the boundary integral equation method. A comparison is made with a previous analytical solution obtained by applying the randomly forced one-dimensional Boussinesq equation, and some observations on modeling procedures are given.

I. INTRODUCTION

This article attempts to present an innovative approach to the numerical approximation of stochastic partial differential equations (stochastic PDE). The method is the result of new developments in the functional analysis theory of abstract stochastic evolution equations ([1-5] among others) and a formulation of the Ito's lemma in Hilbert spaces, which allows the determination of the moments equations of a stochastic PDE [6]. The solution of these moments equations can be obtained by using any numerical method existing in the literature, since they constitute deterministic equations. The present application employs the boundary integral equation method because of its increasing use in engineering and hydrology. While emphasis is made on groundwater equations, the method could be applied to a variety of physical problems.

Very little work has been done on the theory and applications of stochastic PDE appearing in groundwater flow. One of the earliest investigations was conducted by Sagar [7,8]. The recent work by Serrano et al. [9-12] presented the theory of stochastic PDE from the rigorous point of view of functional analysis and developed some applications for the randomly forced and randomly initiated one-dimensional groundwater flow equation in a phreatic aquifer. These

*Present address: Department of Civil Engineering, University of Kentucky, Lexington, KY 40506.

applications used the concept of the strongly continuous semigroup associated with a particular partial differential operator in a Sobolev space. While the semigroup is relatively simple to derive for a one-dimensional domain, the derivation for a two-dimensional partial differential operator is significantly more difficult, and yet the geometrical domains are limited to square or rectangular shapes because of the integral terms.

The need for a numerical solution procedure of a stochastic PDE is a desirable alternative. Again, very little work has been done in this area. From the abstract-theoretical point of view, Becus [2] and Sun [13] have made important contributions. They noted that the discretization must be done not only in the spatial coordinates, but also in the probabilistic variables. The practical implementation of these schemes may still be a formidable task.

From the applied point of view, most of the work has been directed to the development of numerical techniques for random ordinary differential equations [Refs. 14–18, 18 as an application in hydrology]. One of the first applications of numerical methods to the solution of a one-dimensional stochastic PDE in groundwater flow was presented by Sagar [19]. He used the Galerkin approximation in the usual sense, except that the coefficients in the linear combination were random functions in conjunction with a Taylor series expansion to obtain the first two moments. Tang and Pinder [20] presented a method for the numerical solution of the one-dimensional advective-diffusive PDE with random coefficients. They used the method of decomposition of a stochastic partial differential operator of Adomian [21, 22] and functional-analytic methods to prove that the decomposition sequence converges toward the exact solution.

The present article is evolving from the work by Serrano, et al. [6]. They presented the Ito's lemma as a suitable link between the abstract-theoretical analysis of stochastic PDE and the extensive literature on numerical methods for deterministic PDE. Since the above work was developed on a conceptual basis, there was a concern about the applicability and performance of the method in an actual field situation, and its accuracy with respect to the analytical solution of the equivalent stochastic PDE. Thus in the present article we describe the results from the application of the method to the Twin Lake aquifer at the Chalk River Nuclear Laboratories (CRNL), Ontario, Canada, an evaluation of the method, and some guidelines for applications.

II. THE MATHEMATICAL MODEL

Let us consider the abstract stochastic evolution equation of the form

$$\frac{\partial u}{\partial t} + A(x, \omega)u = g(x, t, \omega), \quad (x, t, \omega) \in G \times [0, T] \times \Omega \quad (1)$$

$$Q(x, t, \omega)u = J(\omega), \quad (x, t, \omega) \in \partial G \times [0, t] \times \Omega \quad (2)$$

$$u(x, 0, \omega) = u_0(x, \omega), \quad x \in G \times \Omega \quad (3)$$

where $u \in H^m(G) \times \Omega$ is the system output; $g \in L_2(\Omega, B, P)$ is a second-order random function; $G \subset \mathbb{R}^n$ is an open domain with boundary ∂G ; t is the

time coordinate; $0 < T < \infty$, Q is a boundary operator; x is the spatial domain; A is a formal partial differential operator given by

$$Au = \sum_{|k|, |l| \leq m} (-1)^{|k|} D^k (p_{ki} D^l u) \tag{4}$$

and is assumed to be elliptic; H^m is the m -th order Sobolev space of second-order random functions [1, 9, 10, 23-27]; Ω is the basic probability sample space of elements ω ; $L_2(\Omega, B, P)$ is the complete probability space of second-order random functions with the probability measure P and B the Borel field or class of ω sets [4]; p_{ki} are real-valued coefficients and D is differentiation in Hilbert space.

According to the way in which randomness enters the equation, we can distinguish five basic problems: (i) the random initial value problem, when u_0 is random; (ii) the random boundary value problem, when J is a random; (iii) the random forcing problem, when g is random; (iv) the random operator problem, when A or Q is random; (v) the random geometry problem. We can also have any combination of the above problems.

The system of Eqs. (1)–(3) is stated in an abstract form because u may represent a variety of physical systems, such as unsteady heat or electric potential as well as groundwater potential in an aquifer or contaminant transport, depending on the particular form of the partial differential operator (4). System (1)–(3) is also representing a general stochastic performance since the forcing function g is a stochastic process accounting for the uncertainties in the source term, the partial differential operator A may be random due to uncertainties in the parameters or errors due to approximations and/or linearizations in the development of the model, and the boundary and initial conditions J and u may be random variables or stochastic processes due to uncertainties and errors in the representation or field measurement of these functions.

Serrano et al. [10] solved the randomly forced and randomly initiated regional groundwater flow equation in a phreatic aquifer (cases i and iii above). Sagar [8] solved the groundwater flow equation for particular forms of random boundary conditions and recharge. The random operator problem in groundwater flow, that is, the PDE with random coefficients (case (iv) above), has received much attention among water resource scientists [e.g., 20, 28-34]. This is because hydraulic conductivity has been assumed as the major source of uncertainty in several cases. Here we are particularly interested in the PDE with random boundary conditions (case ii), when the domain in consideration has a dimension higher than one, the boundaries have an irregular geometry, and an approximate solution of the equation becomes a valuable alternative.

Let us consider the problem of regional groundwater flow in a homogeneous, isotropic, phreatic aquifer [23]

$$\nabla^2 \phi = \frac{S}{K} \frac{\partial \phi}{\partial t} \quad \text{on} \quad G \tag{5}$$

$$\phi|_{\partial G_1} = f_1 \tag{6}$$

$$\phi = \eta, \quad \frac{\partial \eta}{\partial t} = \frac{K}{n_e} \left[\nabla_2 \phi \nabla_2 \eta - \frac{\partial \phi}{\partial z} \right] \quad \text{on} \quad z = \eta \quad (7)$$

$$\phi(t = 0) = \phi_0 \quad \text{on} \quad G \quad (8)$$

where ϕ is the hydraulic potential, S is the specific storage due to the elastic properties of aquifer and water, n_e is the aquifer effective porosity, η is the potential at the free surface, z is the vertical coordinate, ∇_2 is the horizontal gradient operator, ϕ_0 is the initial potential in the aquifer, and f_i is the potential at the boundaries.

Two approaches are followed in solving this boundary value problem. The one most commonly used employs the Dupuit assumptions and derives a different continuity equation, the Boussinesq equation [35]. In the Boussinesq equation the z coordinate no longer exists, ϕ is replaced by h (the water table depth), and the nonlinear boundary condition along the phreatic surface (7) no longer applies. The Boussinesq equation is still a nonlinear equation, but there are several well-known methods of linearization [35].

There is an alternative method based on theoretical and experimental evidence that the elastic storage resulting from compressibility of aquifer and water is much smaller than the specific yield (volume of water released from a vertical column of aquifer of unit horizontal cross section, per unit decline in phreatic surface). This method neglects the elastic storage in the unconfined aquifer [35] and considers the exact statement of the phreatic flow in a homogeneous isotropic domain as: Determine $\phi(x, z, t)$ in the flow domain so that ϕ satisfies

$$\nabla^2 \phi = 0 \quad \text{on} \quad G \quad (9)$$

subject to Eqs. (6) to (8).

The nonlinear boundary condition (7) may be modified to include deep percolation input. It may be linearized by neglecting the quadratic terms.

Based on the above discussion, we shall solve the regional groundwater flow problem in a homogeneous isotropic aquifer governed by Eq. (10) subject to Eqs. (6)–(8), with the linearized boundary condition at the free surface (7) expressed as a random PDE accounting for all uncertainties, and particularly, for all perturbations and fluctuations in the water table.

The study area is located at the northwest end of the Twin Lake (Fig. 1). The Twin Lake area is $3.4 \times 10^4 \text{ m}^2$ and comprises part of the Chalk River Nuclear Laboratories (CRNL), Ontario, Canada, which is located on the south bank of the Ottawa River about 180 km northwest of Ottawa.

The geology of the site consists of a Precambrian massive to gneissic granetiferous monzonite (granite) bedrock unit overlain by compacted sandy glacial till, which in turn is overlain by a range of unconsolidated materials [36, 37]. The deposits overlaying the bedrock are predominantly windblown sands, ranging from very fine to medium in grain size.

The climate at the site is classified as cold snow forest, with a warm summer and no particular dry season [38]. The annual average precipitation is 773 mm and follows a relatively uniform monthly distribution [39]. Of this amount,

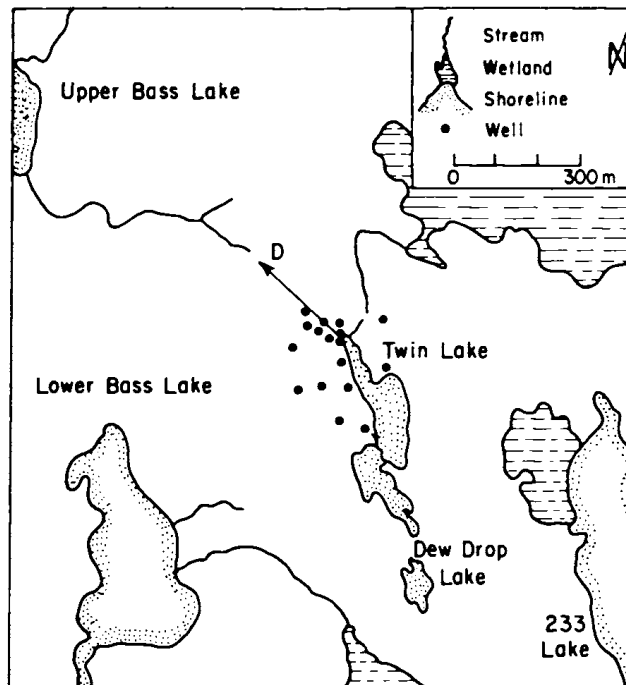


FIG. 1. The Twin Lake area at CRNL.

309 mm have been estimated to contribute to either surface runoff or groundwater recharge in studies at the adjacent Perch Lake watershed. High evaporation rates in June, July, August, and part of September result in moisture deficits and groundwater recharge is almost restricted to spring and late fall. With two exceptions, there is no surface runoff within the study area and most moisture excess produces recharge. There is one intermittent inlet at the north end of the lake. Spring runoff from part of a large wetland northeast of Twin Lake (Fig. 1) follows this channel into the lake. There is no surface outlet from Twin Lake and all of the water leaving the lake recharges the local sand aquifer.

The extreme northwest end of Twin Lake is the area chosen for the modeling study (Figs. 1 and 2). This is a major groundwater discharge area. Groundwater discharge from the lake which follows the general direction of the arrow *D* in Figure 1, enters a stream draining to the southeast corner of Upper Bass Lake.

An extensive number of geological, geophysical, and hydrogeological investigative programs have been conducted in the CRNL area. In particular, Killey and Munch [36] reported borings at over 50 locations around the Twin Lake. Most of the piezometers have been working since 1982 and have multiple piezometer installations. The network was monitored between 1982 and 1984 at monthly intervals.

Figure 2 shows the selected modeling section, the line *X - X*, which is approximately parallel to the main direction of groundwater flow (line *D*,

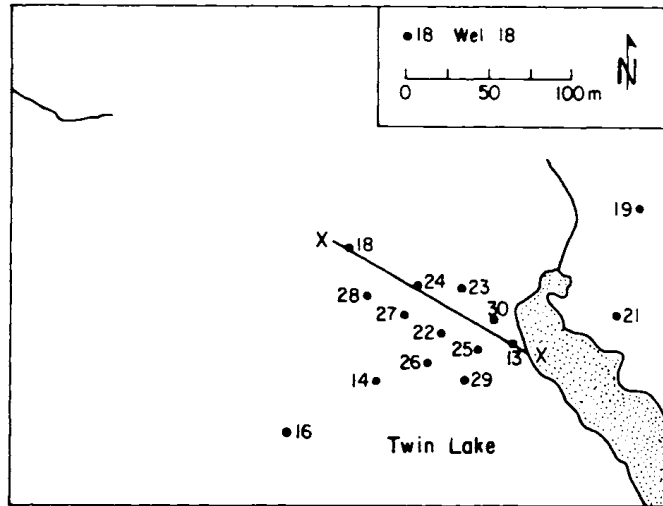


FIG. 2. Problem flow domain X-X.

Fig. 1), as previously determined by the hydrogeological and tracer studies. Line X - X goes through wells TL-18 and TL-13.

Figure 3 shows a cross section of line X - X. Well TL-18 was chosen as the left boundary and origin of the horizontal coordinate. Well TL-13 at $X = L = 116.25$ m was chosen at the right boundary, since no measurements of lake level were available. Wells TL-28, TL-24, TL-22, TL-23, and TL-30 in Figure 2 were projected on line X-X in Figure 3 for an evaluation of the initial conditions and an assessment of the preliminary deterministic model. The simulation period was chosen as coinciding with the measurement period of water levels, that is from July 1, 1982 to May 30, 1984. Bedrock elevation was ob-

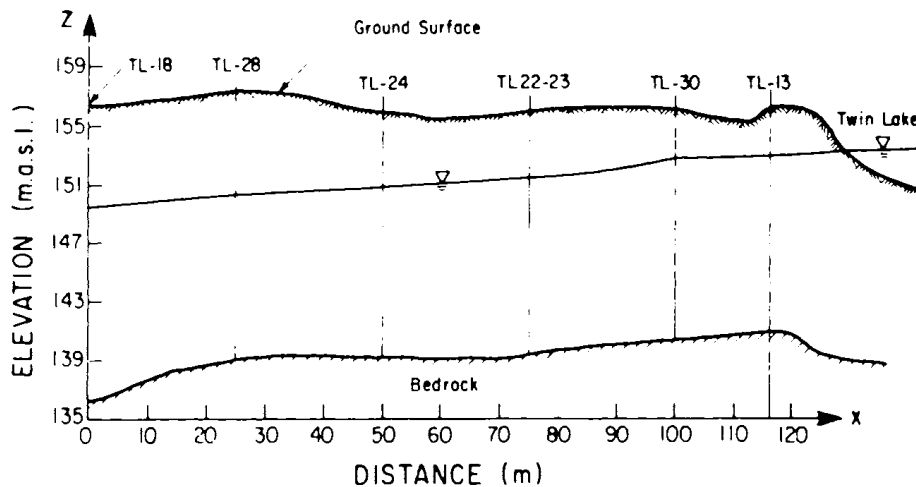


FIG. 3. Cross section of problem flow domain.

tained from the well-drilling data, which was taken as the drill tip elevation upon rejection. Ground surface elevation was also taken from wells data.

A detailed study of the hydrology of the Twin Lake area and aquifer parameters estimation was conducted previously in order to develop an analytical stochastic groundwater flow model. In this study, soon to be published, the one-dimensional groundwater equation with a stochastic forcing function proved to be a satisfactory model for the uncertainty observed in the phreatic surface. Thus one of the aims of this article was to compare the results obtained from the one-dimensional stochastic groundwater flow equation with Dupuit assumptions with the corresponding results obtained by applying the Ito's lemma and the boundary element procedure to the exact statement of flow in a two-dimensional phreatic aquifer. An interesting feature of the last procedure, besides the availability of efficient numerical procedures, is that the irregular form of the bedrock may be considered and that potential values inside the aquifer may be calculated.

The random boundary value problem can now be restated as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{10}$$

$$\phi(0, z, t) = h_1, \quad \phi(L, z, t) = h_2, \quad \frac{\partial \phi}{\partial n}(x, \xi, t) = 0 \tag{11}$$

$$\phi = \eta, \quad \frac{\partial \eta}{\partial t} = -\left(\frac{K}{n_e}\right) \frac{\partial \phi}{\partial z} + \frac{I}{n_e} + \frac{d\beta}{dt} \quad \text{on} \quad z = \eta \tag{12}$$

$$\phi(x, z, 0) = \phi_0(x, z) \tag{13}$$

where $\partial\phi/\partial n$ in Eq. (11) represents the normal derivative of the potential at the bedrock level ξ ; I is the deep percolation for unit horizontal area of the aquifer and has been assumed constant in this case; h_1 and h_2 are the left and right boundary conditions, respectively; and (12) is the linearized free surface boundary condition. The random term $d\beta/dt = w$ is a white noise process in time defined by Ref. 40

$$E\{w(x, t)\} = 0 \tag{14}$$

$$E\{w(x_1, t_1)w(x_2, t_2)\} = q(x_1, x_2)\delta(t_1, t_2) \tag{15}$$

Here q is the white noise variance parameter, and it is a given symmetric positive function, and δ is the delta function. See Unny and Karmeshu [41] for a justification in introducing a Gaussian process as the random component.

The first step in solving (10)–(13) is to transform our boundary value problem in $\phi \in H^1(G)$ into an equivalent one in u , where u belongs to the Sobolev space $H_0^1(G)$ with compact support. In general the PDE in the space $H^m(G)$ must be transformed in its equivalent $m - \text{th}$ -dimensional Sobolev space with compact support $H_0^m(G)$. This is easily done by defining $\phi = u + V$, where $u(x, z, t)$ satisfies the differential equation with homogeneous boundary conditions and $V(x, z)$ satisfies the steady-state problem when t tends to infinity. Hence our original stochastic boundary-value problem (10)–(13) is transformed into the following set of problems:

(i) $V(x, z)$ satisfying the deterministic time-independent set of equations

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad (16)$$

$$V(0, z) = h_1, \quad V(L, z) = h_2, \quad \frac{\partial V}{\partial n}(x, \xi) = 0 \quad (17)$$

$$V = \eta, \quad \frac{\partial V}{\partial z}(x, \eta) = \frac{I}{K}, \quad \text{on } z = \eta \quad (18)$$

where (18) has been deduced from (12) as a steady-state free surface boundary condition.

(ii) $u(x, z, t)$ satisfying the stochastic equation with homogeneous boundary conditions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad (19)$$

$$u(0, z, t) = 0, \quad u(L, z, t) = 0, \quad \frac{\partial u}{\partial n}(x, \xi, t) = 0 \quad (20)$$

$$u(x, z, t) = \eta(x, z, t) - V(x, z), \quad \frac{\partial u}{\partial t} = -\left(\frac{K}{n_e}\right) \frac{\partial u}{\partial z} + \frac{d\beta}{dt} \quad \text{on } z = \eta \quad (21)$$

$$u(x, z, 0) = \phi_0(x, z) - V(x, z) \quad (22)$$

where $u \in H_0^1(G)$.

To Eqs. (19)–(22) we can apply the Ito's lemma in the Sobolev space $H_0^1(G)$.

III. THE ITO'S LEMMA IN HILBERT SPACES

The Ito's lemma in Hilbert spaces may be written as (see Appendix A, B, and C, and Refs. 1 and 27 for proof)

$$\begin{aligned} \Phi(z(t), t) &= \Phi(z(0), 0) + \int_0^t \left(\frac{\partial \Phi}{\partial z}, a \right) ds + \int_0^t \left(\frac{\partial \Phi}{\partial z}, \Gamma d\beta(s) \right) \\ &+ \frac{1}{2} \int_0^t \text{tr} \Gamma^* \frac{\partial^2 \Phi}{\partial z^2} \Gamma q ds + \int_0^t \frac{\partial \Phi}{\partial t} ds \end{aligned} \quad (23)$$

where (\cdot, \cdot) implies inner product regulated by the norm of the Hilbert space in consideration. This equation can be derived from the usual Ito's lemma developed, for example in Jazwinski [40].

Interpreting (1)–(3) in the Ito sense, applying the Ito's lemma with $z(t) = u(t)$, and differentiating we obtain

$$\begin{aligned} \frac{d\Phi(u)}{dt} &= -\left(\frac{\partial \Phi}{\partial u}, Au \right) + \left(\frac{\partial \Phi}{\partial u}, \Gamma w \right) + \left(\frac{\partial \Phi}{\partial u}, g \right) \\ &+ \frac{1}{2} \text{tr} \left[\Gamma^* \frac{\partial^2 \Phi}{\partial u^2} \Gamma q \right] + \frac{\partial \Phi}{\partial t} \end{aligned} \quad (24)$$

By taking expectation it yields

$$\frac{dE\{\Phi(u)\}}{dt} + E\left\{\left(\frac{\partial\Phi}{\partial u}, Au\right)\right\} = E\left\{\left(\frac{\partial\Phi}{\partial u}, g\right)\right\} + \frac{1}{2}trE\left\{\Gamma^* \frac{\partial^2\Phi}{\partial u^2} \Gamma q\right\} + E\left\{\frac{\partial\Phi}{\partial t}\right\} \quad (25)$$

Taking $\Phi(u) = (h, u)$, $h \in V^*$, where h forms a basis in V^* , and $V \subset H_0^1(G)$, Eq. (25) yields an equation for the mean of the solution $M_1 = E\{u(t)\}$:

$$\frac{dM_1}{dt} + AM_1 = g \quad (26)$$

Next set $\Phi(u) = (h_1, u)(h_2, u)$ so that $E\{\Phi(u)\} = (M_2 h_1, h_2)$ for $h_1, h_2 \in V^*$. Then (25) gives an equation for the correlation operator M_2 or the second moment of $u(t)$:

$$\frac{dM_2}{dt} + (A \oplus A)M_2 = (g \oplus g)M_1 + (\Gamma^* \otimes \Gamma q) \quad (27)$$

where \oplus and \otimes denote the direct sum and tensor product of two operators on appropriate tensor product spaces. Basically $A \oplus A$ implies the summation of the operator A on two orthogonal directions to form the complete space.

Similarly, higher order moments may be obtained by defining $M_n(h) = E\{\prod_{i=1}^n (h_i, u)\}$ and using Eq. (25). These moment equations hold in the weak sense. The interesting feature is that Eqs. (26) and (27) are deterministic and may be solved by any analytical or approximate method available in the literature.

Let us now apply the Ito's lemma (25) to Eqs. (19)–(22) representing the transformed random boundary value problem of the Twin Lake in order to obtain:

(i) The first moment equation for $u(x, z, t)$ given by $M_1 = E\{u\}$ as

$$\frac{\partial^2 M_1}{\partial x^2} + \frac{\partial^2 M_1}{\partial z^2} = 0 \quad (28)$$

$$M_1(0, z, t) = 0, \quad M_1(L, z, t) = 0, \quad \frac{\partial M_1}{\partial n}(x, \xi, t) = 0 \quad (29)$$

$$M_1 = E\{\eta(x, z, t) - V(x, \eta)\}, \quad \frac{\partial M_1}{\partial t} = -\left(\frac{K}{n_c}\right) \frac{\partial M_1}{\partial z} \quad \text{on} \quad z = E\{\eta\} \quad (30)$$

$$M_1(x, z, 0) = \phi_0(x, z) - V(x, z) \quad (31)$$

$E\{\phi\}$ can now be calculated from

$$E\{\phi(x, z, t)\} = E\{u(x, z, t)\} + V(x, z) \quad (32)$$

Finally, applying (25) to (19)–(22) [from Eq. (27)],

(ii) The second moment equation for u is given by $M_2 = E\{u^2\}$ as

$$\frac{\partial^2 M_2}{\partial x^2} + \frac{\partial^2 M_2}{\partial z^2} = 0 \quad (33)$$

$$M_2(0, z, t) = 0, \quad M_2(L, z, t) = 0, \quad \frac{\partial M_2}{\partial n}(x, \xi, t) = 0 \quad (34)$$

$$M_2 = E\{\eta(x, z, t) - V(x, \eta)\}^2, \quad \frac{\partial M_2}{\partial t} = -2\left(\frac{K}{n_e}\right) \frac{\partial M_2}{\partial z} + q \quad \text{on} \\ z = E\{\eta\} \quad (35)$$

$$M_2(x, z, 0) = (\phi_0(x, z) - V(x, z))^2 \quad (36)$$

where q is the variance parameter of the white noise process. $E\{\phi^2\}$ can now be calculated from

$$E\{\phi^2\} = E\{u^2\} + 2VE\{u\} - V^2 \quad (37)$$

IV. BOUNDARY ELEMENT APPLICATION AND SOLUTION

The boundary integral equation method (BIEM) can now be used to solve the deterministic boundary value problems described by Eqs. (16)–(18), (28)–(31), and (33)–(36). Basically, BIEM uses the divergence theorem [42], the Green’s second identity and a free-space Green’s function to state

$$\int_{\partial G} \left(U \frac{\partial Q}{\partial n} - Q \frac{\partial U}{\partial n} \right) d(\partial G) = 0 \quad (38)$$

where U is the hydraulic potential ϕ and Q is the two-dimensional free space Green’s function satisfying Laplace equation [43],

$$Q = \ln r \quad (39)$$

where r is the distance between the singular point P and another point P' on the boundary. Replacing (38) in (39) (see Ref. [44] for details on the integration of the singularity) we obtain an expression giving the potential at any point P defined in terms of a boundary integral:

$$2\pi\phi(P) = \int_{\partial G} \left[\phi(Q) \frac{\partial}{\partial n} (\ln r) - \ln r \frac{\partial}{\partial n} \phi(Q) \right] d(\partial G) \quad (40)$$

Since in a well-posed problem ϕ and $\partial\phi/\partial n$ are not known everywhere on ∂G , and instead either ϕ or $\partial\phi/\partial n$ are known at all points of the boundary, the integral equation (40) can be used to find the “missing data,” by moving the point P to the boundary. Hence, (40) becomes

$$\alpha\phi(P) = \int_{\partial G} \left(\frac{\phi}{r} \frac{\partial r}{\partial n} - \ln r \frac{\partial \phi}{\partial n} \right) d(\partial G) \quad (41)$$

where α is the angle in radians between the boundary segments at P , i.e., $\alpha = 2\pi$ where P is a smooth part of the boundary.

Equation (41) can be solved by choosing a finite number of points on the boundary and numerically performing the integration. Thus the procedure is to select a number of nodes $P_j (j = 1, 2, 3, \dots, N)$ on the boundary ∂G and to perform the contour integration using each P_j successively as origin. In the

present application linear elements and linear interpolation functions for the potential and its normal derivative are used for simplicity, although nonlinear elements and higher order interpolation could be used. Integrating over the segment between P_j and P_{j+1} and then summing the contributions from all boundary segments (i.e., using $P_j(j = 1, 2, \dots, N)$ successively) a system of algebraic equations for ϕ_j and $(\partial\phi/\partial n)_j$ is obtained:

$$\sum_{j=1}^N R_{i,j} \phi_j = \sum_{j=1}^N L_{i,j} \left(\frac{\partial\phi}{\partial n} \right)_j \quad j = 1, 2, \dots, N \quad (42)$$

where the coefficients $R_{i,j}$ and $L_{i,j}$ depend only on the geometry of the boundary. For a detailed explanation of the method and the derivation of (42) the reader is referred to Liggett and Liu [44]. Once the known boundary conditions are used in (42), it is possible to solve the system of linear algebraic equations for the unknown ϕ or $\partial\phi/\partial n$ on each boundary node. Then the potential function in the interior of the region can be obtained by performing the integration along the boundary, as indicated by Eq. (40). In order to apply BIEM to the deterministic problem (5)–(8), the relation between ϕ and $\partial\phi/\partial n$ must be known on the free surface.

The random free surface boundary condition (12) may be written in a finite difference form for a time step Δt , which gives the potential at time $k \Delta t$:

$$\begin{aligned} \phi^{k+1} = \phi^k - \frac{K\Delta t}{n_c} \left[\theta \left(\frac{\partial\phi}{\partial n} \right)^{k+1} + (1 - \theta) \left(\frac{\partial\phi}{\partial n} \right)^k \right] \\ + \Delta t \left[\theta \frac{I^{k+1}}{n_c} + (1 - \theta) \frac{I^k}{n_c} \right] + \Delta\beta^{k+1} \end{aligned} \quad (43)$$

where θ is a weighting factor that positions the derivative and recharge between the time levels k and $k + 1$, and $\Delta\beta^{k+1}$ is a sample increment of the Brownian motion process at time level $k + 1$ specified as $N(0, q\Delta t)$. q should also be replaced by $q/\Delta t$ when we are dealing with discrete sample functions [40]. Equation (43) establishes a relation between ϕ and $\partial\phi/\partial n$ at the time level $k + 1$ and it can be used in (42) to generate sample functions of the unknown boundary values. Since ϕ and η are exchangeable, (43) can also be used for computation of sample functions of the free surface elevation η^{k+1} at time $(k + 1)\Delta t$, after the normal derivative of the potential has been calculated.

The first step in the simulations was the computation of the function $M_1 = E\{u\}$. As in the previous studies in the Twin Lake, we used a coefficient of hydraulic conductivity $K = 17.28 \text{ m.d}^{-1}$, the effective porosity of aquifer $n_c = 1.0$, and the left and right boundary conditions at wells *TL-18* and *TL-13* taken from measurements stages during the period of simulation. For all of the computations, the BIEM model was run on a daily basis, and it was assumed that the boundary conditions remained constant during this small time interval, since there were no daily measurements available. Computations of the mean function M_1 were done by successively applying the BIEM procedure to solve Eqs. (28)–(31). The system output at the end of one day was the initial condition of the system at the beginning of the following day and the boundary con-

ditions were adjusted following a simple interpolation between measured dates. This procedure was repeated until the entire period of simulation from July 1, 1982 to May 30, 1984 was completed. Figure 4 illustrates the boundary element grid and the mean potential distribution $E\{\phi\}$ on October 27, 1982.

The differences between mean potential value at the free surface, as simulated by the model, and their corresponding measured values at the same locations and the same dates were interpreted as sample values of the stochastic process perturbing Eq. (12). Studying these differences from the frequency point of view gives valuable information on the statistical properties of the stochastic process. To be consistent with the previous study, it was assumed that the stochastic component can be modeled by a time Gaussian white noise process with the properties given by (14)–(15). This implies that the process is uncorrelated for two successive days, that is that the perturbation is strictly local in time, and that the probability distribution of the process at any given time is normal as a combined effect of the several random perturbations. Based on the above, it was assumed that the variance parameter q in Eq. (15) was constant and equal to $0.034 \text{ m}^2 \cdot \text{d}^{-1}$. This is in contrast with a value of $q = 0.0042 \text{ m}^2 \cdot \text{d}^{-1}$ derived for the previous study using the randomly forced Boussinesq equation. This means that uncertainty generated by using a numerical model is significantly higher.

Some sample functions can now be produced by generating sample functions of the Brownian motion increment process in Eq. (43) and running the BIEM program for the particular dates of interest. For instance, Figure 5 illustrates a sample potential distribution in the aquifer for the same date, October 27, 1982. By comparison with Figure 4, it is possible to note that small random perturbations in the phreatic surface affect the potential distribution in the aquifer. The variation in the shape of the free surface is easier to see in Figure 6.

The final step in the calculations was the computation of the function M_2 from Eqs. (33)–(36), which may be defined in several ways. For instance, let

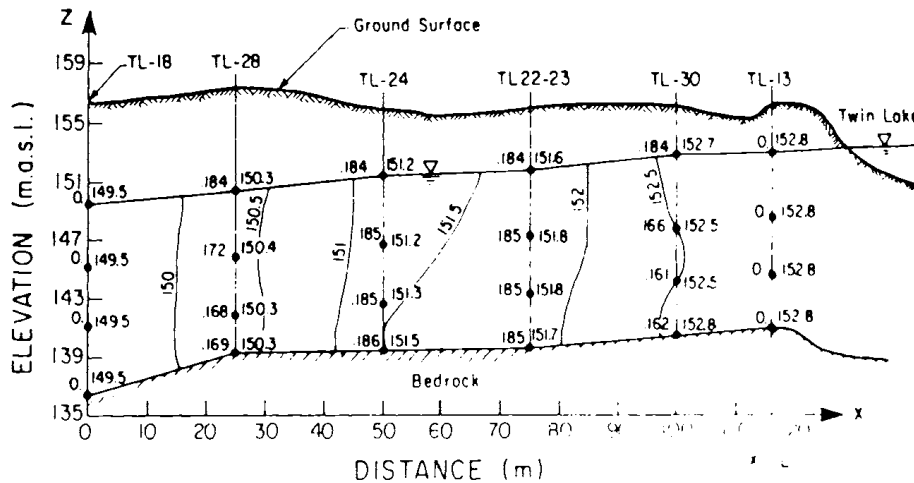


FIG. 4. Boundary element grid, mean potential distribution, and standard deviation on October 27, 1982.

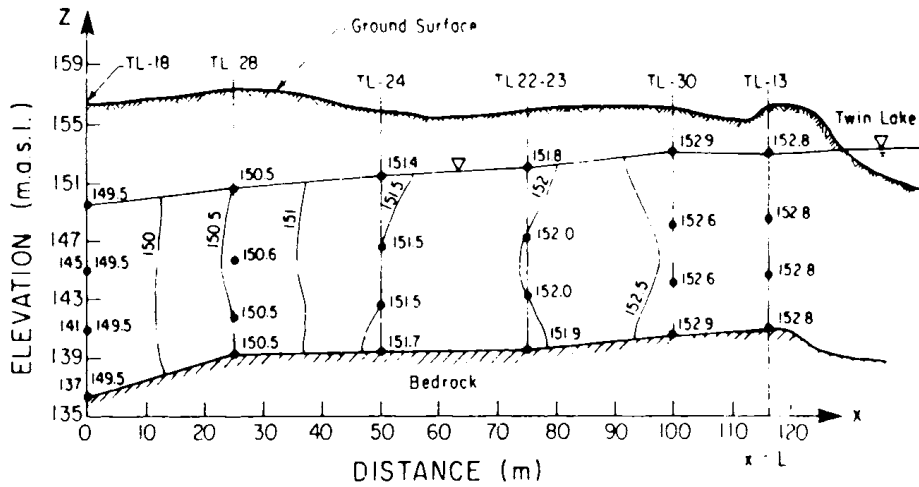


FIG. 5. Boundary element grid and sample potential distribution on October 27, 1982.

us consider M_2 as

$$M_2 = E\{(u - E\{u\})^2\} = E\{u^2\} - E^2\{u\} \quad (44)$$

where coordinates x and z have been neglected because they represent moments at one point. This is equivalent to

$$M_2 = E\{(\phi(t_1) - E\{\phi(t_1)\})^2\} = E\{\phi^2(t_1)\} - E^2\{\phi(t_1)\} \quad (45)$$

For the problem we are analyzing, the value of $M_2 = E\{(\phi(t_1) - E\{\phi(t_1)\})^2\}$ is the variance. At the phreatic surface this variance is equal to the Brownian

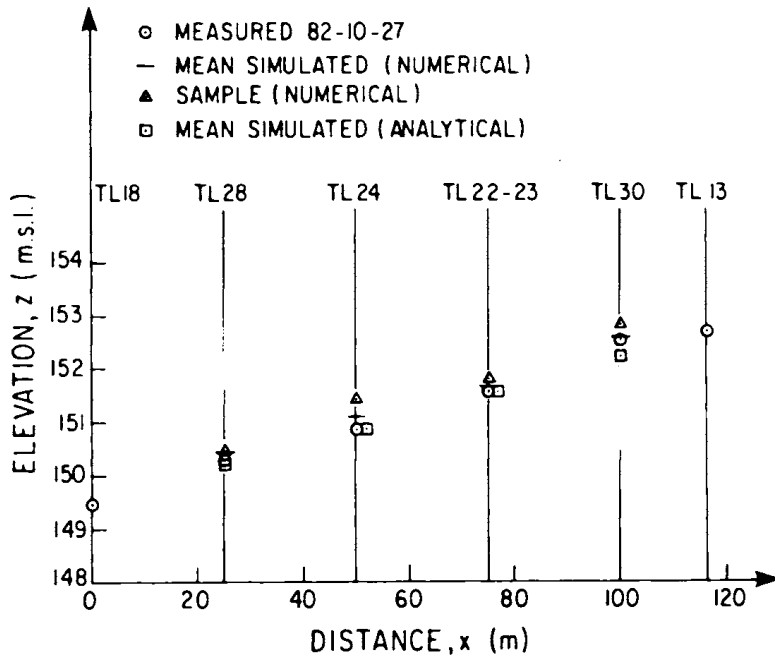


FIG. 6. Free surface potentials on October 27, 1982.

motion variance parameter q at any instant t_1 . Knowing this value at the phreatic surface we can run the program to find the variance at any point inside the aquifer and any instant t_1 . In this case we set Δt equal to zero. This was done for a few dates of interest. Figure 4 shows the potential standard deviation at a few points inside the aquifer. In agreement with the deterministic boundary conditions assumed for this particular problem, the standard deviation is maximum at the center of the aquifer and minimum at the boundaries. Note that this would not be the case if we had introduced stochasticity at the boundaries.

Figures 6 and 7 show the shape of the phreatic surface elevation at the different wells for October 10, 1982 and March 29, 1983, respectively. Of interest is the comparison between the mean free surface elevation as simulated by the boundary element method with the corresponding one simulated by the analytical solution of the randomly forced one-dimensional groundwater flow equation. In general, the mean analytical solution better represents the shape of the free surface, except for well *TL-30*, which is almost always underestimated by the Boussinesq equation because the Dupuit assumption of horizontal flow is not accurate at this point. The mean numerical solution closely simulates the water level elevation at well *TL-30*, perhaps because the two-dimensional approach works better at this point. Figures 6 and 7 also show sample functions for the phreatic surface at those dates, which may serve for qualitative observation of the possible shapes, and the measured values. Tables I and II show the corresponding figures for the same two dates and the standard deviations across the aquifer. Note that corresponding standard deviations for the numerical model are higher.

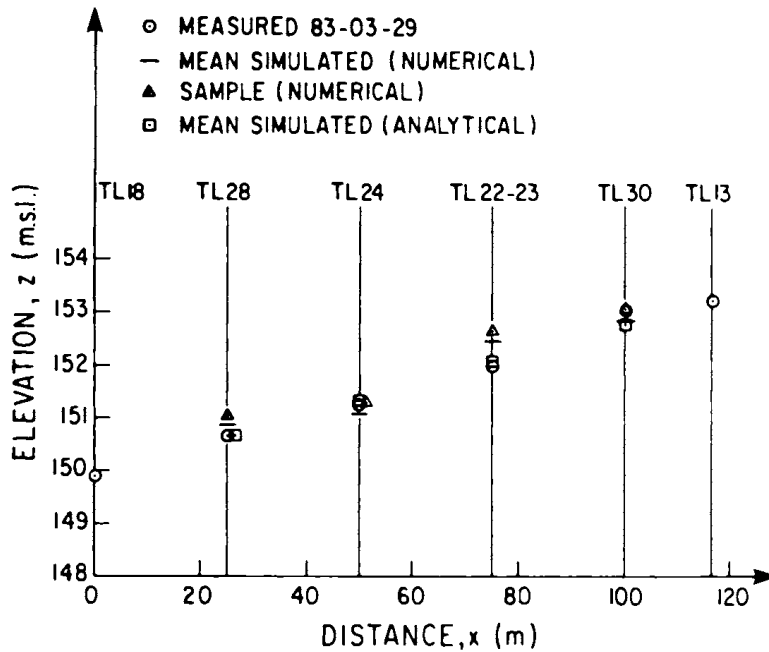


FIG. 7. Free surface potentials on March 29, 1983.

TABLE I. Free surface heads on October 27, 1982 (m).

x	Method	$E\{\phi(x, t)\}$	$\phi(x, t, w)$	σ_ϕ
25.0	Analytical	150.18	150.04	0.037
25.0	Numerical	150.26	150.46	0.168 ^a
50.0	Analytical	150.86	150.64	0.050
50.0	Numerical	151.21	151.43	0.185 ^a
75.0	Analytical	151.54	151.35	0.046
75.0	Numerical	151.56	151.79	0.185 ^a
100.0	Analytical	152.22	152.13	0.027
100.0	Numerical	152.69	152.86	0.161 ^a

^aTaken at half aquifer depth. Boundary conditions, left = 149.501, right = 152.669; Deep Percolation Rate I = 0.0 mm/d.

We may also define the function M_2 as correlation. For instance,

$$M_2 = E\{u(t_1)u(t_2)\} = E\{[\phi(t_1) - V(t_1)][\phi(t_2) - V(t_2)]\} \quad (46)$$

$$M_2 = E\{\phi(t_1)\phi(t_2)\} - V(t_2)E\{\phi(t_1)\} - V(t_1)E\{\phi(t_2)\} + V(t_1)V(t_2). \quad (47)$$

In this case we must perform the calculations stepwise. If we have calculated M_1 previously and V at selected points in the aquifer, then all of the terms in Eq. (47) are known, except $E\{\phi(t_1)\phi(t_2)\}$. We must know this value at the phreatic surface if we want to know M_2 inside the aquifer. We may start an iterative procedure, say setting $t_1 = 0$ and $t_2 = 1$ day. Since the initial condition of the system is deterministic and known, then $E\{\phi(0)\phi(1)\} = \phi_0 E\{\phi(1)\}$. Equation (47) can now be used to calculate M_2 at the free surface and the program could be used with $\Delta t = 1$ day. M_2 is now an output of the system from which $E\{\phi(1)\phi(2)\}$ may be computed from (48) at any point including the free surface. These values of M_2 at the free surface may be used to proceed to the next iteration in a similar manner. We should note that the program should account for the stepwise change in the position of the mean free surface where we wish to find correlations.

TABLE II. Free surface heads on October 27, 1982 (m).

x	Method	$E\{\phi(x, t)\}$	$\phi(x, t, w)$	σ_ϕ
25.0	Analytical	150.62	150.71	0.037
25.0	Numerical	150.86	151.05	0.168 ^a
50.0	Analytical	151.33	151.47	0.050
50.0	Numerical	151.06	151.29	0.185 ^a
75.0	Analytical	152.05	152.17	0.046
75.0	Numerical	152.45	152.65	0.185 ^a
100.0	Analytical	152.76	152.82	0.027
100.0	Numerical	152.85	153.03	0.161 ^a

^aTaken at half aquifer depth. Boundary conditions, left = 149.908, right = 153.228; Deep Percolation Rate I = 0.0 mm/d.

Finally, it is important to remark that for the computation of either sample functions of the potential or the first and second moments it is possible to use the same basic BIEM iteration procedure. We have not emphasized the details on the BIEM iteration, since this subject has already been treated in detail by other authors and there are a number of well-known efficient BIEM algorithms on the market.

V. CONCLUSIONS

The Ito's lemma in Hilbert spaces presents an important alternative to the problem of finding approximate solutions to PDE with stochastic boundary conditions and/or stochastic forcing terms. It presents a link between the abstract-probabilistic analysis of these operator equations and the increasingly used, computer-oriented numerical techniques, by allowing the derivation of PDE satisfying the moments of the output stochastic process. The most important feature of the moments equations derived from the Ito's lemma is that these deterministic equations can be solved by any analytical or numerical method available in the literature. This permits the analysis and solution of stochastic PDE occurring in two-dimensional or three-dimensional domains of any geometrical shape. Although the method as presented here has been illustrated in analysis of regional groundwater flow problems under uncertainty, it could be applied to a number of problems in other areas of physics and engineering where random PDE occur.

The application problem in the Twin Lake aquifer at the CRNL showed that the potential distribution within the aquifer is sensitive to random perturbations in the dynamic free surface boundary condition. Comparison between moments of the output process computed by solving analytically the randomly forced one-dimensional groundwater flow equation, with the corresponding moments computed by numerically solving the two-dimensional moments equations derived from the Ito's lemma showed that the output variances of the numerical solution are significantly higher than the variances of the analytical solution. This implies that additional model uncertainty is generated by using a numerical solution.

Mean values and sample functions of the phreatic surface are better represented by the analytical solution, except at points where the one-dimensionality of the problem is no longer justified. At least for the Twin Lake aquifer, the analytical computation of water table head sample functions and their moments is more efficient than the corresponding computation by the numerical model. However, the one-dimensional solution gives no information about the potential distribution inside the aquifer and its application is quite restricted to very particular cases where the bedrock may be assumed horizontal and the Dupuit assumptions of quasihorizontal groundwater flow and constant hydraulic head with depth are valid. In most other cases where two-dimensional and three-dimensional problems are to be considered because of the irregular shape of the boundaries, the numerical model is a more suitable approach. Furthermore, a two-dimensional numerical model may give valuable information on the poten-

tial distribution inside the aquifer, if information of potential variation with depth is available at the boundaries. Finally, the increasing availability of efficient deterministic numerical methods makes a numerical approach a valuable alternative for the modeler.

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APPENDIX A: A STATEMENT OF THE ITO'S LEMMA IN HILBERT SPACE

In the following a separable Hilbert space is considered. This may be either H^0 defined with the norm $\int_a^b \|f^2(t)\| dt$, or H_0^1 with the norm $\int_a^b \{\|f'(t)\| + \|f^2(t)\|\} dt$. Consider a stochastic process $v(t)$ in H described by:

$$v(t) = v(0) + \int_0^t a(s) ds + \int_0^t B(s) d\xi(s) \tag{A1}$$

where $\xi(s)$ is a Brownian motion process with values in a separable Hilbert space E such that for any $e \in E$, $(\xi(t), e)$ is a real Brownian motion process, that is F^t Martingale measurable on the probability space (Ω, B, P) where F^t is an increasing family of σ - algebras with $B = F^\infty$. Q is the covariance operator on E , then,

$$E\{(\xi(t), e_1)(\xi(t), e_2)\} = (Qe_1, e_2) \min(t_1, t_2) \tag{A2}$$

defines a correlation function.

With $B(t)$ as an adapted process with values in $l(E; H)$ with finite norm, a stochastic integral with values in H can be defined as follows:

$$I = \int_0^t B(s) d\xi(s). \tag{A3}$$

Further, $a(t)$ is also an adapted stochastic process with values in H . $v(0)$ is a random variable in H , that is F^0 measurable.

Let $\Phi(v, t)$ be any Frechet differentiable functional in $H \times [0, T]$ that has continuous first and second derivatives in H and first derivative in t . These derivatives are bounded on bounded sets of H . Then, the following is a statement of the Ito's formula in the Hilbert space H (see Ref. 1 for proof).

$$\begin{aligned} \Phi(v, t) = & \Phi(v(0), 0) + \int_0^t \left(\frac{\partial \Phi}{\partial v}, a \right) ds + \int_0^t \left(\frac{\partial \Phi}{\partial v}, B(s) \right) d\xi(s) \\ & + \frac{1}{2} \int_0^t tr B^*(s) \frac{\partial^2 \Phi}{\partial v^2} B(s) Q ds + \int_0^t \frac{\partial \Phi}{\partial s} ds \end{aligned} \tag{A4}$$

Consider

$$\Phi(v, t) = f(h, v) = (h, v), \tag{A5}$$

where h is an arbitrary but fixed element in H (as opposed to v being random) and $(. . .)$ represents the inner product in H .

By definition $E\{(h, v)\} = (h, M_1)$ for all h in H , where M_1 is the mean value function (first moment) in H .

The Frechet derivative is defined as follows:

$$\frac{\partial \Phi}{\partial v} = h \quad \text{and} \quad \frac{\partial^2 \Phi}{\partial v^2} = 0. \quad (\text{A6})$$

The Ito's formula for the functional in (A5) leads to

$$(h, v) = (h, v(0)) + \int_0^t (h, a) ds + \int_0^t (h, B(s)) d\xi(s). \quad (\text{A7})$$

As this expression is true for all h in H , then by taking the expectation

$$M_1 = \int_c^t a ds \quad \text{or} \quad \frac{dM_1}{dt} = a, \quad (\text{A8})$$

where for convenience $E\{(h, v(0))\}$ is taken as zero.

Let us now consider that

$$\Phi(v, t) = (h_1, v)(h_2, v), \quad (\text{A9})$$

where h_1 and h_2 are different but arbitrary fixed elements in H . Then by definition

$$E\{(h_1, v)(h_2, v)\} = (h_1, M_2 h_2), \quad (\text{A10})$$

where M_2 is the second moment operator.

The Frechet derivatives are obtained as

$$\frac{\partial \Phi}{\partial v} = (h_1, v)h_2 \oplus (h_2, v)h_1, \quad \frac{\partial^2 \Phi}{\partial v^2} = h_1 o h_2 \oplus h_1 o h_2. \quad (\text{A11})$$

The Ito's formula now reads as follows:

$$\begin{aligned} (h_1, v)(h_2, v) &= \int_0^t ((h_1, v)h_2, a) ds \oplus \int_0^t ((h_2, v)h_1, a) ds \\ &+ \frac{1}{2} \int_0^t \text{tr} B^*(s)(h_1 o h_2 \oplus h_2 o h_1)B(s)Q ds. \end{aligned} \quad (\text{A12})$$

Taking the expectation on both sides and for the case when $a(s)$ is a deterministic function of t ,

$$(h_1, M_2 h_2) = \int_0^t (a \oplus a)M_1 dt + \int_0^t B^*(s)B(s)Q ds,$$

for all h_1 and h_2 in H . Hence

$$\frac{dM_2}{dt} = (a \oplus a)M_1 + B^*(s)B(s)Q. \quad (\text{A13})$$

APPENDIX B: DERIVATION OF THE MEAN VALUE EQUATIONS FOR AN ELLIPTIC PARTIAL DIFFERENTIAL EQUATION

Consider the stochastic PDE given by

$$dv = Av dt + B(t) d\xi(t) + g(t) dt \quad (\text{B1})$$

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where $g(t)$ is a deterministic function and A is a spatial elliptic operator. Then the Ito's formula for the functional (h, v) is:

$$(h, v) = (h, v(0)) + \int_0^t (h, Av) ds + \int_0^t (h, g) ds. \quad (B2)$$

Taking expectation in (B2), the moment equation in M_1 becomes

$$M_1 = \int_0^t AM_1 ds + g(t) \quad (B3)$$

Now the Ito's formula for the functional $(h_1, v(0))(h_2, v(0))$ is:

$$\begin{aligned} (h_1, v)(h_2, v) &= (h_1, v(0))(h_2, v(0)) + \int_0^t ((h_1, v)h_2, Av) ds \oplus \int_0^t ((h_2, v)h_1, Av) ds \\ &+ \int_0^t ((h_1, v)h_2, B(s)) d\xi(s) \oplus \int_0^t ((h_2, v)h_1, B(s)) d\xi(s) \\ &+ \int_0^t ((h_1, v)h_2, g(s)) ds \oplus \int_0^t ((h_2, v)h_1, g(s)) ds \\ &+ \frac{1}{2} \int_0^t trB^*(s)h_1oh_2B(s)Q ds \oplus \int_0^t trB^*(s)h_1oh_2B(s)Q ds. \end{aligned} \quad (B5)$$

Evaluating the expectations,

$$E\{((h_1, v)h_2, Av)\} = (h_1, AM_2h_2).$$

Because (B5) is true for all h_1 and h_2 in H , then by taking the expectation we obtain

$$M_2 = \int_0^t AM_2 ds \oplus \int_0^t AM_2 ds + \int_0^t gM_1 ds \oplus \int_0^t gM_1 ds + \int_0^t B^*(s) \oplus B(s)Q ds, \quad (B6)$$

or

$$\frac{dM_2}{dt} = (A \oplus A)M_2 + (g \oplus g)M_1 + B^* \oplus BQ.$$

APPENDIX C: DERIVATION OF FRECHET DIFFERENTIALS

Consider the functional $\Phi = (h_1, v)(h_2, v)$. The derivative in any arbitrary but fixed direction u is given by

$$\left(\frac{\partial \Phi}{\partial v}, u \right) = \frac{\partial}{\partial \epsilon} \{(h_1, v)(h_2, v + \epsilon u)\}_{\epsilon \rightarrow 0} + \frac{\partial}{\partial \epsilon} \{(h_1, v + \epsilon u)(h_2, v)\}_{\epsilon \rightarrow 0}.$$

Since this is true for any u , then

$$\frac{\partial \Phi}{\partial v} = (h_1, v)h_2 \oplus (h_2, v)h_1.$$

Similarly the second derivative is obtained as

$$\left(\frac{\partial^2 \Phi}{\partial v^2}, u \right) = \frac{\partial}{\partial \epsilon} \{(h_1, v + \epsilon u)h_2\}_{\epsilon \rightarrow 0} \oplus \frac{\partial}{\partial \epsilon} \{(h_2, v + \epsilon u)h_1\}_{\epsilon \rightarrow 0}.$$

Since this is true for all u , then

$$\frac{\partial^2 \Phi}{\partial v^2} = h_1 o h_2 \oplus h_2 o h_1 .$$

References

- [1] A. Bensoussan, "Control of Stochastic Partial Differential Equations," in *Distributed Parameter Systems*, W. H. Ray and D. G. Lainiotis, Eds., Marcel Dekker Inc., New York, 1977.
- [2] G. A. Becus, "Variational Formulation of Some Problems for the Random Heat Equation," in *Applied Stochastic Processes*, G. Adomian, Ed., Academic Press, New York, 1980.
- [3] P. L. Chow, "Approximate Solution of Random Evolution Equations," in *Approximate Solution of Random Equations*, A. T. Bharucha-Reid, Ed., Elsevier, North Holland, 1979.
- [4] R. F. Curtain, "Estimation and Stochastic Control for Linear Infinite Dimensional Systems," in *Probabilistic Analysis and Related Topics*, Vol. 1, A. T. Bharucha-Reid, Ed., Academic Press, New York, 1978.
- [5] K. Gopalsamy and A. T. Bharucha-Reid, "On a Class of Parabolic Differential Equations Driven by Stochastic Point Processes," *J. of Appl. Prob.*, **12**, 98–106 (1975).
- [6] S. E. Serrano, T. E. Unny, and W. C. Lennox, "Analysis of Stochastic Groundwater Flow Problems. Part III: Approximate Solution of Stochastic Partial Differential Equations," *J. of Hydrol.*, **82(3–4)**, 285–306 (1985).
- [7] B. Sagar, "Analysis of Dynamic Aquifers with Stochastic Forcing Functions," *Water Resour. Res.*, **14(2)**, 207–216 (1978).
- [8] B. Sagar, "Solution of the Linearized Boussinesq Equation With Stochastic Boundaries and Recharge," *Water Resour. Res.*, **15(3)**, 618–624 (1979).
- [9] S. E. Serrano, T. E. Unny, and W. C. Lennox, "Analysis of Stochastic Groundwater Flow Problems. Part I: Deterministic Partial Differential Equations in Groundwater Flow. A Functional Analytic Approach," *J. of Hydrol.*, **82(3–4)**, 247–263 (1985).
- [10] S. E. Serrano, T. E. Unny, and W. C. Lennox, "Analysis of Stochastic Groundwater Flow Problems. Part II: Stochastic Partial Differential Equations in Groundwater Flow. A Functional Analytic Approach," *J. of Hydrol.*, **82(3–4)**, 265–284 (1985).
- [11] S. E. Serrano, T. E. Unny, and W. C. Lennox, "Analysis of Stochastic Groundwater Flow Problems in Sobolev Space. Approximate Solution of Stochastic Partial Differential Equations," Proceedings Fourth International Symposium on Multivariate Analysis of Hydrologic Processes. Colorado State University, Fort Collins, Colorado, 1985.
- [12] S. E. Serrano, T. E. Unny, and W. C. Lennox, "Stochastic Partial Differential Equations in Hydrology," Proceedings, Symposia on Statistics, University of Western Ontario, London, Ontario, Canada, 1985.
- [13] T. C. Sun, "A Finite Element Method for Random Differential Equations," in *Approximate Solution of Random Equations*, A. T. Bharucha-Reid, Ed., Elsevier, North Holland, 1979.
- [14] W. E. Kohler, and W. E. Boyce, "A Numerical Analysis of Some First Order Initial Value Problems," *SIAM J. Appl. Math.*, **27**, 167–179 (1974).
- [15] G. N. Mil'shtien, "Approximate Integration of Stochastic Differential Equations," *Th. Prob. Appl.*, **19**, 557–562 (1974).
- [16] N. J. Rao, J. D. Borwanker, and D. Ramkrishna, "Numerical Solutions of Ito Integral Equations," *SIAM J. of Control*, **12**, 124–139 (1974).

- [17] M. D. Lax, "Approximate Solutions of Random Differential and Integral Equations," in *Applied Stochastic Processes*, G. Adomian, Ed., Academic Press, New York, 1980.
- [18] T. E. Unny, "Numerical Integration of Stochastic Differential Equations in Catchment Modeling," *Water Resour. Res.*, **20(3)**, 360–368 (1984).
- [19] B. Sagar, "Galerkin Finite Element Procedure for Analyzing Flow Through Random Media," *Water Resour. Res.*, **14(6)**, 1035–1044 (1978).
- [20] D. H. Tang and G. F. Pinder, "Analysis of Mass Transport With Uncertain Physical Parameters," *Water Resour. Res.*, **15(5)**, 1147–1155 (1979).
- [21] G. Adomian, "Random Operator Equations In Mathematical Physics I," *J. Math. Phys.*, **11**, 1069–1084 (1970).
- [22] G. Adomian, "Random Operator Equations In Mathematical Physics II," *J. Math. Phys.*, **12**, 1944–1948 (1971).
- [23] D. H. Griffel, *Applied Functional Analysis*, Ellis Horwood Ltd., 1981.
- [24] V. Hutson and J. S. Pym, *Functional Analysis in Applied Mathematics*, Academic Press, New York, 1980.
- [25] J. T. Oden, *Applied Functional Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1977.
- [26] R. E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Pitman Medical Publishers, London, 1977.
- [27] Y. Sawaragi, T. Soeda, and S. Omatu, in *Modeling, Estimation, and Their Applications for Distributed Parameter Systems. Lecture Notes in Control and Information Sciences*, Vol. 2, A. V. Balakrishnan and M. Thoma, Eds., Springer-Verlag, 1978.
- [28] G. Dagan, "Stochastic Modeling of Groundwater Flow by Unconditional and Conditional Probabilities. 1. Conditional Simulation and The Direct Problem," *Water Resour. Res.*, **18(4)**, 813–833 (1982).
- [29] G. Dagan, "Analysis of Flow Through Heterogeneous Random Aquifers by The Method of Embedding Matrix. 1. Steady Flow," *Water Resour. Res.*, **17(1)**, 107–121 (1981).
- [30] L. W. Gelhar, "Stochastic Analysis of Phreatic Aquifers," *Water Resour. Res.*, **10(3)**, 539–545 (1974).
- [31] L. W. Gelhar, "Effect of Hydraulic Conductivity Variations on Groundwater Flows," Proceedings Inst. IAHR Symp. on Stochastic Hydraulics, 2 Lund, Sweden, 1977.
- [32] L. W. Gelhar and C. L. Axness, "Three-Dimensional Stochastic Analysis of Microdispersion in Aquifers," *Water Resour. Res.*, **19(1)**, 161–180 (1983).
- [33] A. L. Gutjahr and L. W. Gelhar, "Stochastic Models of Subsurface Flow: Infinite Versus Finite Domain and Stationarity," *Water Resour. Res.*, **17(2)**, 337–350 (1981).
- [34] N. T. Kottegoda and G. C. Katuuk, "Effect of Spatial Variation in Hydraulic Conductivity on Groundwater Flow by Alternate Solution Techniques," *J. of Hydrol.*, **65**, 349–362 (1983).
- [35] J. Bear, *Dynamics of Fluids in Porous Media*, American Elsevier, New York, 1972.
- [36] R. W. D. Killey and J. H. Munch, Geology and Hydrogeology of The Twin Lake Tracer Test Site. Report Atomic Energy of Canada Limited AECL 8262, Chalk River, Ontario, Canada, 1985.
- [37] S. B. Lumbars, Mattawa-Deep River (Eastern Half) District of Nipissing and County of Renfrew (Ontario). Ontario Division of Mines Preliminary Map, P. 1197, 1, 63, 360, 1976.
- [38] P. J. Barry, Perch Lake, in *Hydrological Studies on a Small Basin in The Canadian Shield*, P. J. Barry, Ed., Report Atomic Energy of Canada Limited AECL 5041(1):93–130. Chalk River, Ontario, Canada, 1975.
- [39] D. P. Wildsmith, Temporal Distribution of Precipitation at Perch Lake, in *Hydrological and Geochemical Studies in The Perch Lake Basin: A Second Report of*

- Progress*, P. J. Barry, Ed., Report Atomic Energy of Canada Limited AECL 6404:95-120, Chalk River, Ontario, Canada, 1979.
- [40] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*, Academic Press, New York, 1970.
- [41] T. E. Unny and Karmeshu, "Stochastic Nature of Outputs From Conceptual Reservoir Model Cascades," *J. of Hydrol.*, **68**, 161-180 (1984).
- [42] M. R. Spiegel, *Advanced Mathematics for Engineers and Scientists*, Schaum's outline series, McGraw-Hill, New York, 1980.
- [43] M. D. Greenberg, *Applications of Green's Functions in Science and Engineering*, Prentice-Hall, Englewood Cliffs, NJ, 1971.
- [44] J. A. Liggett and P. L. Liu, *The Boundary Integral Equation Method for Porous Media Flow*, Allen and Unwin, London, 1983.