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## PREDICTING GROUNDWATER FLOW IN A PHREATIC AQUIFER

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### ABSTRACT

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An innovative approach to the solution of groundwater forecasting problems using mathematical models is presented. The approach specifically accounts for the errors generated in the development and solution of the groundwater flow equations, and the uncertainty generated by the use of data subject to environmental fluctuations and measurement errors. Since both the input data and the model are subject to considerable uncertainty, the groundwater heads simulated by a mathematical model should be presented in a statistical sense. Validation of a mathematical model should be assessed by the ability of the model to reproduce the statistical properties of the field measurements. The article describes in detail the development, solution and validation of two mathematical models describing groundwater potential at the Twin Lake aquifer, Chalk River, Ontario, Canada. The first one results from the application and solution of the stochastic Boussinesq equation with Dupuit assumptions. The second one results from the boundary elements solution of the two-dimensional Laplace equation with stochastic free surface boundary condition.

### 1. INTRODUCTION

One of the difficult problems in groundwater hydrology is the forecasting of the regional groundwater flow conditions subject to measurable hydrologic, hydrogeologic and meteorologic variables. Many practical problems in hydrology require a thorough understanding of the spatial distribution of the groundwater heads and an ability to predict their time evolution according to specified conditions. For example, in an irrigation district it is necessary to calculate an optimal estimate of the groundwater elevation due to an available weather forecast in order to determine an adequate irrigation depth for maximum crop yield. In other situations, optimal groundwater management decisions are taken after a detailed study of the effect of a groundwater development project on the overall regional groundwater flow regime. Mining activities demand an investigation of the groundwater drainage requirements for the mine and a posterior evaluation of the extent of potential mineral pollution of surrounding streams and aquifers. Thus the modern hydrologist is also required to assess the environmental impact of development projects. This leads

us to one of the most difficult problems in modern hydrology, namely the investigation of the effect of industrial, urban and nuclear waste management activities on the water quality of the regional aquifer system and the surrounding streams. It is clear that in aquifers with relatively high hydraulic conductivity, the dominant physical process is mechanical dispersion, which is ultimately controlled by the spatial and temporal distribution of the hydraulic gradients.

From above, it is apparent the importance of an understanding and an ability to predict the regional groundwater flow regime. This preoccupation for perfecting a tool to forecast the groundwater potential in an aquifer system has resulted in a variety of mathematical models. Early contributions by researchers such as Toth (1963), Venetis (1969) and Venetis (1971) concentrated on the development of analytical solutions of the arising partial differential equations describing groundwater flow in one-dimensional and two-dimensional domains. In practice, it has been found that the validation of these mathematical models faces a number of obstacles. Often, it has been noted that the simulated groundwater potentials do not approach the corresponding measured groundwater potentials. The differences between the simulated and the corresponding measured potentials sometimes follow a consistent trend and sometimes follow a semiperiodic pattern in which the differences are positive for a period of time and negative for another. The inability of a model to exactly reproduce the true potentials is due to the assumptions imposed on the derivations of the equations, which represent our best judgement and knowledge of the physical system, and the number of approximations for a solution of the equations to be possible. A common assumption refers to essentially horizontal flows or flows averaged over the vertical plane. One of the most outstanding assumptions is the one which reduces the partial differential equation with a three-dimensional hydraulic conductivity tensor to an essentially isotropic heterogeneous system. The equations themselves are the result of the continuum approximation to molecular porous media flow in an effort to use the mathematics of averaged continuous functions. With the proliferation of high-speed computers, it is now feasible to obtain numerical approximations of the partial differential equations (e.g. Freeze and Whitherspoon, 1966; Freeze, 1971). Hence, the solution of unsteady flow problems in a three-dimensional heterogeneous aquifer is relatively simple to obtain by the use of finite element techniques. However, this flexibility and speed gained by using these techniques is compensated by the additional convergence and roundoff errors generated by numerical approximations.

If the best possible mathematical model is a desirable feature in the solution of groundwater forecasting problems, we should realize that these models are strongly dependent on the quality of aquifer parameters and hydrologic information available. This is the measurement problem, which inevitably involves as many errors and uncertainty as the modeling problem. The validation of mathematical models is done by comparing in a number of ways the groundwater potential distribution simulated by the mathematical model with respect

to a set of isolated field measurements. In traditional hydrologic practice, the limited field measurements have been considered as "the true values", and the simulated values have been considered as "the incorrect values". From the above discussion, the reader will understand that the measured values are also subject to a high degree of uncertainty and that it may not be appropriate to validate and adjust a mathematical model by measuring its accuracy with respect to equally uncertain data.

In the present article, the authors propose an unconventional procedure to the groundwater forecasting problem, which specifically accounts for the uncertainty in the modeling and measurement phases. This procedure is based on the assumption that the measured values are subject to the same degree of environmental fluctuation and uncertainty as the simulated values. Under these considerations, a more adequate description of the groundwater potential should be given in a statistical sense. This implies that the statistical properties of the groundwater potential are function of the statistical properties of the aquifer parameters, the boundary conditions and the recharge. This will give the hydrologist a more realistic tool to estimate the high variability and uncertainty associated with the groundwater flow regime. For instance, calculation of the variance of the groundwater potential at different points in space and time will provide the hydrologist with a measure of the uncertainty and fluctuation of the head estimation. This will also provide her/him with a quantitative evaluation of the uncertainty inherent to the system at different points in time and space, and at the same time with the degree of reliability of the model as a predicting tool at different points in space and time.

Let us postulate the fundamental assumptions behind this type of analysis.

(1) Mathematical models of regional groundwater flow are subject to a varied degree of uncertainty due to the errors generated by the conceptual assumptions in the development of the differential equations and errors produced by the simplifications and approximations in the solution process.

(2) Input data to mathematical models, such as aquifer parameters, boundary conditions and recharge, are subject to environmental fluctuations and measurement errors. Strictly speaking the data functions are not deterministic functions but stochastic functions. The best way to describe this information is by the statistical measures of the ensemble family of functions.

(3) Groundwater potentials calculated by a mathematical model fed by stochastic data are themselves stochastic. Thus the simulated groundwater potentials are best described by the statistical measures of the resulting stochastic process.

(4) Since both the simulated groundwater potential and the measured data are stochastic processes, the validation of mathematical models should be done by comparing the statistical properties of the simulated potentials with respect to the statistical properties of the measured potentials. If the two sets of statistical measures are correspondingly close one another, then the model is considered an acceptable predictive tool. Hence the ultimate modeling objective will be to show that both sets of information, that is the simulated poten-

tials and the measured potentials, were generated by the same class of stochastic process.

The above conception presents a whole new concept in the mathematics and in the process of groundwater forecasting. If the objective of forecasting is to be able to predict the groundwater potential distribution in space and in time subject to a known set of aquifer parameters, boundary conditions and recharge, the ultimate objective of the proposed analysis is to predict the stochastic nature of the groundwater potential subject to the known stochastic nature of the aquifer parameters, the boundary conditions and the recharge.

The authors were in particular interested in the development of a mathematical model capable of predicting the stochastic nature of the groundwater heads in a phreatic aquifer at the Chalk River Nuclear Laboratories (CRNL), Ontario, Canada. The area of study is the Twin Lake at the CRNL. The area had previously been the object of groundwater measurements and tracer tests (Killey and Munch, 1985; Molyaner, 1985) in an attempt to correlate measurable aquifer properties with parameters in groundwater equations. The Twin Lake is the groundwater outlet of one of the main low-level nuclear waste disposal sites of the CRNL and for many years the scientific staff at the environmental Research Branch have concentrated on the evaluation of the magnitude and extent of the impact of the radionuclide migration on the aquifers and surface water systems, and on the biological life of the area. The aquifer selected for the study receives recharge from the Twin Lake (Fig. 1) and

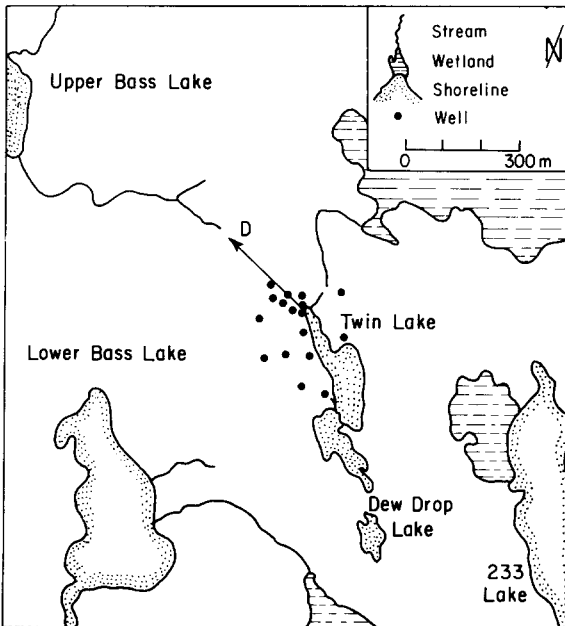


Fig. 1. The Twin Lake area at CRNL.

a concern has existed on predicting the groundwater flow regime subject to a known set of measurements of precipitation and groundwater heads at a few piezometers. Calculation of the groundwater heads will make possible the estimation of the groundwater velocity distribution in space and time and a computation of the rate of migration of groundwater pollutants could be accomplished. Thus the uncertainty inherent to the calculations in groundwater transport could by far be explained by the stochastic nature of the groundwater flow model. Recognizing the high variability and uncertainty of both mathematical models and input data, it has been decided that a groundwater head forecast which specifically accounted for the stochastic nature of the system and the environmental fluctuations would represent a more realistic estimation.

Let us now state the objectives of the present study:

(1) To develop a mathematical model capable of predicting the stochastic nature of the groundwater head distribution in the Twin Lake aquifer in space and time subject to known measured values of precipitation and a few piezometric heads. This model should serve as input for estimation of the random nature of contaminant migration in the aquifer.

(2) To validate the mathematical model by investigating the convergence of the stochastic properties of the simulated values towards the corresponding stochastic properties of measured groundwater heads.

(3) To present this innovative groundwater modeling methodology to the international hydrologic community, and discuss the procedures and implications of the approach.

In compliance with the above objectives, the authors develop one analytical solution of the arising one-dimensional stochastic partial differential equation, and the results are compared with a numerical solution of the two-dimensional groundwater flow equation applied to the same aquifer. These two models are the result of recent mathematical developments in functional analytic theory which originated a new method to analyze and solve stochastic partial differential equations in hydrology (Serrano and Unny, 1986, 1987a; Serrano et al., 1985a, b, c).

In section 2 of this paper the geologic and hydrologic properties of the study area are briefly described. In section 3 the mathematical models describing the random groundwater table and the stochastic potential distribution are developed. Little emphasis is made on the theoretical aspects and mathematical meanders, and the functional analysis jargon is reduced to a minimum. Expressions for the mean groundwater level, the correlation function, the covariance, the standard deviation and sample functions are derived. These are fundamental tools for the hydrologist to draw conclusions about the degree of uncertainty of different regions in the model and the degree of reliability of the groundwater forecast. In section 4 the computational algorithm is described and some results on the mean groundwater level, its standard deviation and some sample functions are presented for points in space located near piezometers and for times coinciding with groundwater level measurements.

## 2. THE TWIN LAKE AREA

The study area is located at the northwest end of the Twin Lake. The Twin Lake area is  $3.4 \times 10^4 \text{ m}^2$  and comprises part of the CRNL property, which is located on the South bank of the Ottawa River about 180 km northwest of Ottawa.

The geology of the site consists of a Precambrian massive to gneissic granetiferous monzonite (granite) bedrock unit overlain by compacted sandy glacial till, which in turn is overlain by a range of unconsolidated materials (Lumbars, 1976; Killey and Munch, 1985). The deposits overlaying the bedrock are predominantly sand, ranging from very fine to medium in grain size. Other minor deposits of silt and clayey silt are also observed. Recent organic deposits can be found at the surface in some locations, while erosion has exposed the bedrock and till in others. In general, however, bedrock exposure at the site is very limited. Within the Twin Lake study area (Fig. 1), surficial materials are almost entirely wind-blown sands. Small areas of bedrock outcrop are found in the southeast side of Twin Lake. Relief is dominated by two dune ridges at about 170 m asl. Dry closed depressions are common (Killey and Munch, 1985).

The climate at the site is classified as cold snow forest, with a warm summer and no particular dry season (Barry, 1975). Soils are typically frozen from late November until late April. The annual average precipitation is 773 mm and follows a relatively uniform monthly distribution (Wildsmith, 1979). Of this amount 309 mm has been estimated to contribute to either surface runoff or groundwater recharge in studies at the adjacent Perch Lake watershed. High evaporation rates in June, July, August and part of September result in moisture deficits and groundwater recharge is almost restricted to spring and late fall. With two exceptions, there is no surface runoff within the study area and most moisture excess produces recharge. There is one intermittent inlet at the north end of the lake. Spring runoff from part of a large wetland northeast of Twin Lake (Fig. 1) follows this channel into the lake. Beavers are numerous and average low water levels of about 1 m in the lake may correspond to beaver dams blocking this seasonal inflow. There is no surface outlet from Twin Lake and all of the water leaving the lake recharges the local sand aquifer.

The Twin Lake is naturally recharged by the aquifers located on its eastern side. These aquifers have been the disposal site of low-level nuclear waste for more than thirty years. Although radioactivity above background levels has not been detected in the lake, there is a concern about the future contamination of the lake and its subsequent spread towards the surrounding aquifers.

The extreme northwest end of Twin Lake is the area chosen for the modeling study (Figs. 1 and 2). This is a major groundwater discharge area, and the objective of the modeling exercise is to predict the random nature of the groundwater discharge regime. This information will be the input for groundwater transport estimations when the Twin Lake is contaminated. Groundwater discharge from the lake, which follows the general direction of the arrow D in Fig. 1, enters a stream draining to the southeast corner of Upper Bass Lake.

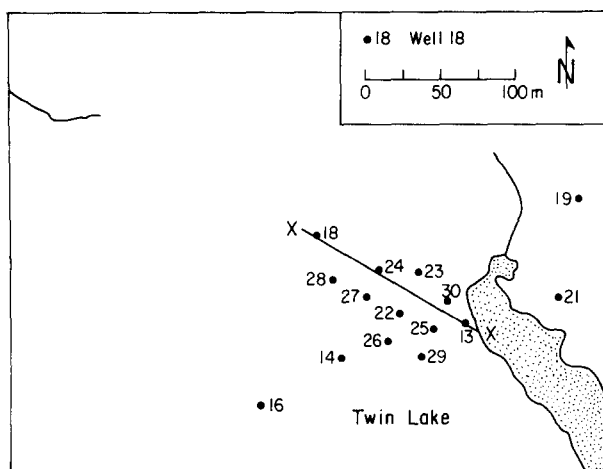


Fig. 2. Problem flow domain X-X.

An extensive number of geological, geophysical and hydrogeological investigative programs have been conducted in the CRNL area. In particular Killey and Munch (1985) reported borings at over 50 locations around the Twin Lake. Most of the piezometers have been working since 1982 and have multiple piezometer installations. The network was monitored between 1982 and 1984 at monthly intervals. They also employed several techniques to estimate hydraulic conductivity of various geologic materials found at site. These methods included permeameter tests and single-well response tests as well as estimates based on the grain size distribution of various split spoon samples taken during the borings. A ground probing radar survey of the site did not show any continuity in the hydraulic permeability zonations. The Twin Lake aquifer, therefore, was classified as homogeneous. The study concluded that the hydraulic conductivity values  $K \pm \sigma$ , where  $\sigma$  is the standard deviation, are the following: for the aeolian and granetiferous sands  $(1.16 \pm 0.47) \times 10^{-2} \text{ cm s}^{-1}$ ; for the medium fluvial sands  $(1.56 \pm 0.21) \times 10^{-2} \text{ cm s}^{-1}$ . No strong vertical or lateral gradients were found and mainly the horizontal gradients and flow is controlled by the bedrock slope.

Following the Killey and Munch study, two natural-gradient radioactive tracer tests, real analogs of groundwater contamination by a nonreactive radionuclide, were conducted at the Twin Lake area in 1982 and 1983 in order to identify the field scale heterogeneities and to evaluate the hydraulic and dispersive properties of the Twin Lake aquifer. Information related to the tracer tests results can be found in Moltyaner and Killey (1983), Killey and Moltyaner (1984), Moltyaner and Paniconi (1984), and Moltyaner (1985). The estimate of hydraulic conductivity determined from the tracer tests  $K \pm \sigma$  is  $(2.0 \pm 0.33) \times 10^{-2} \text{ cm s}^{-1}$  or an average of  $17.28 \pm 2.85 \text{ m d}^{-1}$ . Although the temporal and spatial variations of tracer concentration are measured at bore-

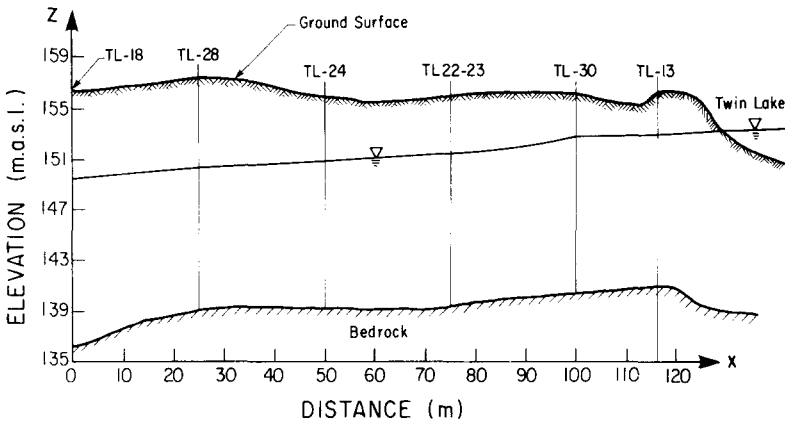


Fig. 3. Cross section of problem flow domain.

holes, the concentration history at borehole locations reflect the tracer behaviour at the interwell zones. This feature makes parameters measured from tracer tests more accurate than those obtained from well logging and coring. For this reason we have adopted the above value as the one describing the hydraulic conductivity magnitude of the Twin Lake aquifer. Figure 2 shows the selected modeling section, the line X-X, which is approximately parallel to the main direction of groundwater flow (line D, Fig. 1), as previously determined by the hydrogeological and tracer studies. Line X-X goes through wells TL-18 and TL-13.

Figure 3 shows a cross section of line X-X. Well TL-18 was chosen as the left boundary and origin of the horizontal coordinate. Well TL-13 at  $X = L = 116.25$  m was chosen at the right boundary, since no measurements of lake level were available. Our task will be to predict the random behaviour of the potential distribution in the problem flow domain limited by wells TL-18 and TL-13 subject to the known groundwater levels at these wells, and an estimation of the recharge. Wells TL-28, TL-24, TL-22, TL-23 and TL-30 in Fig. 2 were projected on line X-X in Fig. 3 for an evaluation of the initial conditions. The simulation period was chosen as coinciding with the measurement period of water levels, that is from July 1, 1982 to May 30, 1984. Bedrock elevation was obtained from the well-drilling data, which was taken as the drill tip elevation upon rejection. For the modeling purposes the bedrock elevation was assumed to be represented by a straight line of constant slope, that is  $y_r = 137.2 + 0.0292x$ . Ground surface elevation was also taken from well data.

The average aquifer transmissivity was computed as the mean hydraulic conductivity value, as before, times the mean flow depth obtained from well data and groundwater level measurements. This would produce a minimum transmissivity value of  $T_{\min} = (189.73 \pm 31.29) \text{ m}^2 \text{ d}^{-1}$  and a maximum of  $T_{\max} + (247.10 \pm 40.75) \text{ m}^2 \text{ d}^{-1}$ . It will be seen later that the small range of fluctuation



in the water table and the high transmissivity values make a mean value of  $218.4 \text{ m}^2 \text{ d}^{-1}$  as a suitable one for modeling.

For the specific yield parameter  $S$ , use was made of the extensive grain size measurements reported in Killey and Munch (1985) and a theoretical relationship between grain size and  $S$  (Bear, 1979). For an average grain size of  $10^{-2}$  mm,  $S$  is estimated to be equal to 0.15.

### 3. DEVELOPMENT OF THE MATHEMATICAL MODELS

In the present section we will describe two distinct mathematical models to represent the random behaviour of the potential in the Twin Lake aquifer. Regardless of the modeling technique, the hydrologist always finds that the physical equations may only explain a part of the groundwater potential. Hence each model should possess a random component to describe the unexplained part. This component will depend on the particular features of the model and on the fluctuating nature of the groundwater potential. The first model developed here is the one-dimensional Boussinesq equation, resulting from applying the Dupuit assumptions to the two-dimensional groundwater flow equation in an unconfined aquifer. The random component in this model is represented by a stochastic process in the form of a source term, the properties of which are calculated based on the frequency behaviour of the fluctuating and unpredictable groundwater table. This random term will account for the variations in groundwater head which the physical equations cannot explain. An analytical solution of this model will be presented. The second model results from an exact mathematical statement of the two-dimensional groundwater flow in an unconfined aquifer, that is the Laplace equation subject to a time-dependent free-surface boundary condition. The random component in this model is a stochastic process perturbing the dynamic free-surface boundary condition, following the reasoning that the most uncertain term in the modeling process is constituted by the recharge estimation. A numerical solution of this equation will be presented by using the boundary integral equation method.

The modeling procedure goes along similar lines to deterministic modeling: (1) Defining the problem (forecasting of groundwater potential); (2) Geometrical domain definition (left: well TL-18, right: TL-13, bottom: bedrock, top: free surface); (3) Development of governing differential equation and boundary and initial conditions (either model 1 or model 2 below); and (4) solution of boundary value problem. The main difference between deterministic and stochastic modeling is that in stochastic modeling the hydrologist goes one step further and presents the results in a more realistic statistical fashion, and that the boundary value problem is a random boundary value problem which requires a different mathematical treatment. In what follows, the authors will attempt to present the basic mathematical equations and the procedure that a hydrologist would follow in making the assumptions and adjustments when confronted with the usual limitations of data availability and accuracy. The stochastic

component in the model will account for the errors generated by this procedure. Evaluation of the specific statistical properties of the random component is presented in section 4.

*Model 1: stochastic Boussinesq equation and analytical solution*

From the analysis of information given in section 2, and particularly the observation that the saturated thickness, the horizontal and vertical hydraulic gradients and bedrock slope are small, it is possible to assume that the random water table elevation between wells TL-18 and TL-13 (Fig. 3) may be described by the one-dimensional groundwater flow equation in a phreatic aquifer (Bear, 1979), which is randomly perturbed by a stochastic process:

$$\frac{\partial y}{\partial t}(x, t) - \frac{\partial}{\partial x} \left[ Kh \frac{\partial y}{\partial x}(x, t) \right] = \frac{I}{S} + \frac{\xi(x, t)}{S} \quad (1)$$

subject to:

$$y(0, t) = y_1(t) \quad (2)$$

$$y(L, t) = y_2(t) \quad (3)$$

$$y(x, 0) = y_0(x) \quad (4)$$

where  $y(x, t)$  is the elevation above the sea level of the water table (m);  $t$  is the time coordinate (d);  $S$  is the aquifer specific yield;  $K$  is the hydraulic conductivity ( $\text{m d}^{-1}$ );  $I$  is the input function representing deep percolation to the aquifer and assumed uniformly distributed along  $x$  ( $\text{m d}^{-1}$ );  $h$  is the water table depth with respect to the bedrock (m);  $\xi(x, t)$  is a time-space stochastic process accounting for the model uncertainty. Dupuit assumptions of essentially horizontal flow and constant hydraulic gradient over the whole depth of the aquifer are implicit in the derivation of eqn. (1).

Now assuming that the bedrock elevation is represented by a line of constant slope  $\alpha$ :

$$y = h + \alpha x \quad (5)$$

eqn. (1) becomes:

$$\frac{\partial h}{\partial t} - \frac{1}{S} \frac{\partial}{\partial x} \left( Kh \frac{\partial h}{\partial x} \right) - \frac{\alpha}{S} \frac{\partial}{\partial x} (Kh) = \frac{I}{S} + \frac{\xi}{S} \quad (6)$$

At this point we should mention that other researchers have attempted the solution of eqn. (6) for the steady-state case (e.g. Dagan, 1979, 1981, 1982; Gelhar, 1974, 1977; Gutjahr and Gelhar, 1981; Kottegoda and Katuuk, 1983) after assuming that  $K$  could be represented as a stochastic process in space. The authors believe that for the present case the information required on  $K$  with respect to space would make those approaches unfeasible. On the other hand, the fact that the Twin Lake aquifer is constituted by homogeneous sands

makes us conclude that the continuum assumption of a constant bulk value for  $K$  is more reasonable and that the time components are relatively more important in our system. Recently, Serrano and Unny (1987b) obtained a solution for the more complex, and perhaps more general, case in which the time stochasticity of  $K$  is the dominant random component.

In order to linearize eqn. (6) (Bear, 1979), let us assume that the aquifer is homogeneous, that is  $K = \text{constant}$  and that  $h$  in the second term may be described as  $h = \bar{h} + h'$ , with  $\bar{h} \gg h'$ , so that  $Kh \approx \bar{T} + T'$  with  $\bar{T} \gg T'$ , the average transmissivity. Making  $\bar{T} = T$  for convenience in notation, eqn. (6) becomes:

$$\frac{\partial h}{\partial t} - \frac{T}{S} \frac{\partial^2 h}{\partial x^2} - \frac{\alpha K}{S} \frac{\partial h}{\partial x} = \frac{I}{S} + \frac{\xi}{S} \quad (7)$$

In order to obtain an analytical solution, we must have an analytical expression for the time-dependent boundary conditions, eqns. (2) and (3). However, as is usual in hydrologic practice, water table measurements are only available at unequal intervals of about one month. We could then consider the functions  $y_1(t)$  and  $y_2(t)$  as stochastic functions composed of a periodic deterministic part accounting for the seasonality plus a random term. The periodic function would be quite empirical in nature, since with the available sample size for the Twin Lake of sixteen measurements it would be difficult to identify a continuous function. It is also suspected that the variance introduced in the model would be so large that it would not justify an additional analytical solution for points in between the boundary conditions.

The above discussion suggests that a daily simulation with daily updating of boundary conditions estimated from interpolation between measurements is a simpler approach and perhaps a more accurate one than the monthly simulation with stochastic boundary conditions. Furthermore, for a simulation on a monthly basis an analytical expression for the forcing function  $I$  would be required. Sagar (1978) suggested the possible form for such a function as a sequence of pulse excitations  $\rho(t_i, \omega)$  acting at random instants of time  $t = t_1, t_2, \dots$  with random intensity, that is:

$$\rho(t, \omega) = \sum_i \rho\{t_i(\omega)\delta[t - t_i(\omega)]\} \quad (8)$$

where  $\delta(\ )$  is the Dirac delta function. Again for the two years of simulation the evaluation of precipitation parameters for eqn. (8) and the assessment of the effect of soil, evaporation and transpiration would be difficult.

Hence a preliminary, and thus the simplest, stochastic model one could adopt in view of the reality of the problem and the availability of data, is one in which the source term and the boundary conditions are deterministically estimated and one that has a single stochastic process perturbing the equation as an error and/or uncertainty term. This assumption is reinforced by the fact that the average water table fluctuation in a month is about 0.20 m, which in turn represents an average daily water table fluctuation of about 0.0067 m.

Assuming constant boundary conditions, such a small variation in the magnitude of the boundary conditions would not affect to a large extent the computation of water table depths in the flow domain at the end of one day. Additional total daily precipitation values could be used and assumed constant through the day. Hence eqn. (7) becomes:

$$\frac{\partial h}{\partial t} - \frac{T}{S} \frac{\partial^2 h}{\partial x^2} - \frac{\alpha K}{S} \frac{\partial h}{\partial x} = \frac{I}{S} + \frac{d\beta(t)}{dt} \quad (9)$$

$$h(0, t) = h_1 \quad (10)$$

$$h(L, t) = h_2 \quad (11)$$

$$h(x, 0) = h_0(x) \quad (12)$$

where  $h_1$  and  $h_2$  are the left and right boundary conditions, assumed constant, and  $d\beta(t)/dt = \xi/S = w$  is a white Gaussian noise process in time and smooth in space.  $\beta(t)$  is of course the Brownian (or Wiener) process and  $d\beta(t)/dt = w$  is the formal derivative of the Brownian motion, that is white Gaussian noise (Jazwinski, 1970). This process accounts for the errors generated in the development of the model, such as linearization, daily simulation with constant infiltration and constant boundary conditions. The stochastic process should also account for the errors in the estimation of the parameter values, deep percolation and environmental fluctuations not considered or unknown.

The water table fluctuation due to variations in infiltration rates is an extremely complex process. From a mathematical viewpoint, the magnitude of infiltration is a function of many variables. To predict the water table fluctuations accurately one would in the first place have to understand and quantify the nature of physical processes involved in infiltration; in other words, the functions that define infiltration would need to be completely specified.

Obviously, complete definitions of these functions and exact measurement of associated parameters are formidable tasks. We are forced to simplify the approach by constructing models based upon field observations of relations between physical quantities characterizing processes. The best way to construct such a model is to postulate the underlying generating mechanisms that may have produced the observed quantities. This can be achieved by considering the generating process to be the sum of a deterministic function  $I(t)$  and a stochastic component  $\xi(t)$ , the stochastic component being a Gaussian white noise process  $d\beta(t)/dt$ . The justification for this approach stems from the fact that owing to the characteristics of the aquifer (clean sandy material of significant horizontal extent), the infiltration rates are uniformly distributed all over the aquifer surface. The generating process can be thought of as the parallel water table fluctuations with random independent increments in time.

The properties of  $w$  are:

$$E[w(t)] = 0 \quad (13)$$

$$E[w(t_1)w(t_2)] = q\delta(t_2 - t_1) \quad (14)$$

where  $E[ \ ]$  represents the expectation operator, and  $q = \sigma^2/S$  is the variance parameter ( $m^2 d^{-1}$ ).

For a detailed description on the use of functional analysis and the mathematical procedure to obtain a solution of the random boundary value problem given by eqns. (9)–(12) the reader is referred to Appendix A. The general solution is given by:

$$\begin{aligned}
 h(x, t) = & V(x) + e^{-\alpha Kx/2T} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \sin\left(\frac{n\pi x}{L}\right) \\
 & \times \frac{2}{L} \int_0^L [h_0(x) - V(x)] e^{\alpha Kx/2T} \sin\left(\frac{n\pi x}{L}\right) dx \\
 & + e^{-\alpha Kx/2T} \int_0^t \sum_{n=1}^{\infty} e^{-\lambda_n^2(t-s)} db_n(s) \sin\left(\frac{n\pi x}{L}\right)
 \end{aligned} \tag{15}$$

where  $V(x)$  is given by:

$$V(x) = h_1 + \frac{\left(h_2 - h_1 + \frac{IL}{\alpha K}\right)}{\left(e^{-\alpha KL/T} - 1\right)} \left(e^{-\alpha Kx/T} - 1\right) - \frac{I}{\alpha K} x \tag{16}$$

$\lambda_n$  are the eigenvalues given by:

$$\lambda_n^2 = \frac{n^2 \pi^2 T}{L^2 S} + \frac{\alpha^2 K^2}{4TS} \tag{17}$$

and  $db_n$  is a unidimensional scalar Brownian motion process with incremental variance parameter given by (Appendix A):

$$q dt \sum_{n=1}^{\infty} 1/\lambda_n$$

Sample functions of the groundwater table may be obtained by generating sample realizations of the Brownian motion process. This may serve for comparative purposes, for a visualization of the evolution of the groundwater table with time, and for testing and obtaining the response of our system subject to continuous variable conditions. At this point we are in a position to calculate statistical measures of the water table. We are very interested in calculating the mean water table with respect to space and time. This is given by (see Appendix A):

$$\begin{aligned}
 E[h(x, t)] = & V(x) + e^{-\alpha Kx/2T} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \sin\left(\frac{n\pi x}{L}\right) \\
 & \times \frac{2}{L} \int_0^L [h_0(x) - V(x)] e^{\alpha Kx/2T} \sin\left(\frac{n\pi x}{L}\right) dx
 \end{aligned} \tag{18}$$

where  $E [ \ ]$  is the expectation operator. We are also very interested in calculating the standard deviation  $\sigma_h$  of the water table with space and time. This is given by (see Appendix A):

$$\sigma_h = E(\{h(x, t) - E[h(x, t)]\}^2) = \frac{4q}{L^2} e^{-\alpha Kx/T} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)}{\lambda_m^2 + \lambda_n^2} \times \frac{[1 - e^{\alpha KL/2T} \cos(m\pi)] [1 - e^{\alpha KL/2T} \cos(n\pi)]}{\left(1 + \frac{\alpha^2 K^2 L^2}{4T^2 m^2 \pi^2}\right) \left(1 + \frac{\alpha^2 K^2 L^2}{4T^2 n^2 \pi^2}\right)} (1 - e^{-t(\lambda_m^2 + \lambda_n^2)}) \quad (19)$$

Similarly, other statistical measures like correlation and covariance could be computed depending on our particular interests (see Appendix A). These statistical measures may help to obtain additional information of the behaviour of  $h(x, t)$ .

### *Model 2: Laplace equation with stochastic free-surface boundary condition and boundary element solution*

An alternative model for the present problem could be obtained by considering the exact formulation for the problem of two-dimensional groundwater flow in a phreatic aquifer, when the Dupuit assumptions are disregarded. The governing differential equation is the Laplace equation subject to a dynamic free-surface boundary condition (Bear, 1979). The random component in this model is a stochastic process perturbing the partial differential equation describing the free surface boundary condition:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (20)$$

$$\phi(0, z, t) = h_1, \quad \phi(L, z, t) = h_2, \quad \frac{\partial \phi}{\partial n}(x, \xi, t) = 0 \quad (21)$$

$$\phi = \eta, \quad \frac{\partial \eta}{\partial t} = -\left(\frac{K}{n_e}\right) \frac{\partial \phi}{\partial z} + \frac{I}{n_e} + \frac{d\beta}{dt}, \quad \text{on } z = \eta \quad (22)$$

$$\phi(x, z, 0) = \phi_0(x, z) \quad (23)$$

where  $\phi$  is the hydraulic potential;  $n_e$  is the aquifer effective porosity;  $\eta$  is the potential at the free surface;  $\phi_0$  is the initial potential in the aquifer;  $\partial\phi/\partial n$  in eqn. (21) represents the normal derivative of the potential at the bedrock level  $\xi$  (Fig. 3);  $I$  is the deep percolation for unit horizontal area of the aquifer and has been assumed constant in this case;  $h_1$  and  $h_2$  are the left and right boundary conditions; and (22) is the linearized free surface boundary condition. The random term  $d\beta/dt = w$  is a white noise process in time defined by eqns. (13)–(14).

The solution of eqns. (20)–(23) will produce the statistical properties of the potential  $\phi$  at discrete points inside the aquifer and at different times. The detailed procedure is described in the recent article by Serrano and Unny (1986). The method used a formulation of the Ito's lemma in Hilbert spaces to derive partial differential equations satisfying the moments of  $\phi$ . Since the moments equations are deterministic, they could be solved by any analytical or numerical procedure available in the literature. The method used in solving the moments equations was the boundary integral equation, which allows for more flexibility, accuracy and convergence speed in this particular case with respect to the finite elements method.

In section 4 of this article, we will present a comparison between the analytical solution and the numerical solution of the first two moments (mean and standard deviation) of the potential at the Twin Lake aquifer for the period of simulation.

#### 4. CALCULATIONS AND RESULTS

We will begin our calculations of the statistical properties of the potential in the Twin Lake aquifer by using Model 1 above. The first step was the computation of the mean water table depth  $E[h(x, t)]$  at selected points across the flow domain. These points were the  $x$  values obtained by projecting wells TL-28, TL-24, TL22–23 and TL-30 on line X–X in Fig. 2. The equation used was eqn. (18) with  $V(x)$  given by eqn. (16). As stated before, the assumed bedrock slope was  $0.0292 \text{ m m}^{-1}$ ; the coefficient of hydraulic conductivity  $K = 17.28 \text{ m d}^{-1}$ ; the aquifer transmissivity  $T = 218.4 \text{ m}^2 \text{ d}^{-1}$  (see section 2). The integral involving  $h_0(x)$  and  $V(x)$  in the second term of eqn. (18) was solved by using a simple numerical algorithm.

The model was run on a daily basis. This implied solving eqns. (16) and (18) with the boundary conditions  $h_1$  and  $h_2$  given by measurements in wells TL-18 and TL-13, respectively, on the first day of record, that is on June 22, 1982. Corresponding measurements on wells TL-28, TL-24, an average between TL-22 and TL-23, and TL-30 were used as initial conditions. The model output produced the expected water table depth  $E[h(x, t)]$  with  $x$  at the end of day June 22, 1982. These values were used as initial conditions for the computation the following day. The new boundary conditions were computed after a simple linear interpolation between the monthly values measured at wells TL-18 and TL-13 and were updated every day. Hence eqns. (16) and (18) were successively solved each day with the output of the system at a day becoming the initial conditions of the system for the comparisons on the following day.

The daily deep percolation to the aquifer was estimated based on the total daily precipitation record measured at the CRNL during the same simulation period, which covered the time from June 22, 1982 to May 31, 1984. Following the observation that overland runoff is almost nonexistent (see section 2), the difference between total daily precipitation and total daily potential evapotranspiration was interpreted as deep percolation to the aquifer. Given the

mean depth of the water table of about 5 m and the sandy texture of soil, evaporation from the groundwater reservoir and the effect of a capillary fringe were neglected. It was also assumed in the above computation that the mean soil-water content was virtually unchanged below the root zone. The daily potential evapotranspiration was estimated based on monthly lake evaporation values computed at nearby Perch Lake. Of lake evaporation 70% of the values was assumed to be equal to evapotranspiration rates at the Twin Lake. This figure was taken from long-term experience in the area.

In applying deep percolation values to the groundwater model, due consideration was given to seasonal restrictions. A simple model releasing accumulated melted snow depth when soil temperatures were frankly above 0°C was used for this purpose.

Generally the convergence of the series in eqn. (18) was rather fast since only a few terms were necessary to compute in order to approximate the transient term up to an accuracy of 0.0001 m.

The next task is the estimation of the parameters of the random process perturbing eqn. (18). This is easily done by subtracting eqn. (18) from eqn. (15). This will yield sample realizations of the purely random component in our model. If we assume that the unknown stochastic process perturbing the model is the same one perturbing the sample realizations of the water table measured in the field, the above subtraction is simply the difference between the mean calculated heads and the measured heads at the same points in space and time. Since the simulated values give the mean groundwater depths at every point and every time, and since the measured groundwater levels at the boreholes represent sample values from an unknown probabilistic population, the difference  $d\beta(x, t)$  between the two corresponding values represents a sample point from the perturbing stochastic process, that is from the last term in the right hand side of eqn. (18). As stated before, this deviation is interpreted as a purely random effect originated from the combined action of errors in measurement, environmental fluctuations, unknown local effects and errors incurred as a consequence of adopting all the assumptions and calculations described so far in the paper.

One may hope that the probability distribution of the deviation  $d\beta(., t)$  is Gaussian, since a Gaussian process is easier to handle, only two parameters completely determine its probability distribution, and extensive theory is available for Gaussian processes. One may also expect a Gaussian process because the random variable  $d\beta(., t)$  is the resultant of several other components, each of them with a particular probability distribution. By the central limit theorem the resulting density function is Gaussian. It is difficult to prove the above in our case, since only a few sample points of measurement are available. However, a preliminary analysis of the random variables  $d\beta(25, t)$ ,  $d\beta(50, t)$  and  $d\beta(75, t)$ , resulting from the projected wells TL-28, TL-24 and TL 22-23, respectively, showed that each variable may be following a Gaussian law with similar parameters.



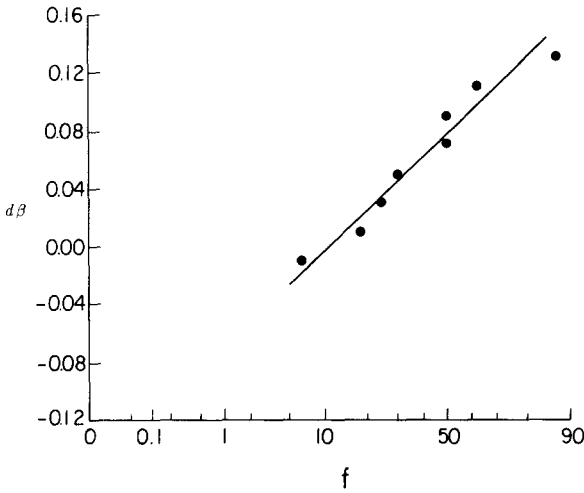


Fig. 4. TL-24 groundwater level fluctuation  $d\beta(50, t)$ .

Figures 4 and 5 show plots on normal paper of  $d\beta(50, t)$  (at TL-24) and  $d\beta(75, t)$  (at TL 22-23) respectively, against frequency. Although the number of points is not enough to state a definite conclusion, and no extreme values were used in the plots, there is evidence indicating that  $d\beta(\cdot, t)$  may follow a Gaussian law with a very similar variance. The differences in the mean values may be due to the distances between the wells and the simulation line X-X (see Fig. 2). Furthermore, it would be easy to show that the random variable  $\{d\beta(x, t) - E[d\beta(x, t)]\}$  for all  $x$ 's follows a normal distribution. Therefore the assumption that a single temporal stochastic process  $d\beta(t)$ , which perturbs the entire aquifer and is smooth in space is reasonable. In well TL-24, the closest to the simulation line, the mean value  $\mu_{d\beta}$  tends to zero and the standard deviation

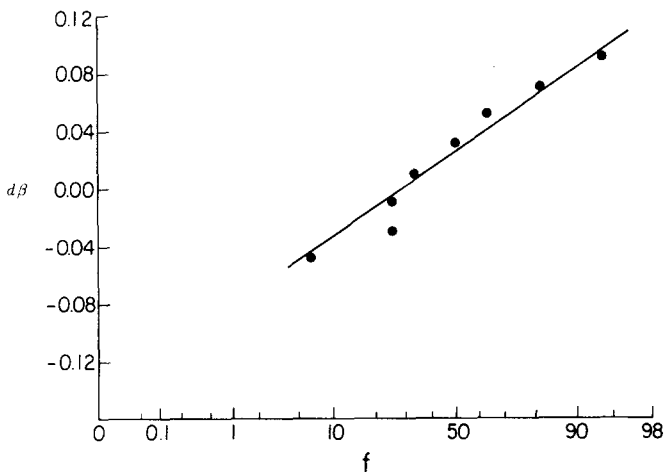


Fig. 5. TL 22-23 groundwater level fluctuation  $d\beta(75, t)$ .

represents purely random fluctuations with a standard deviation  $\sigma_{d\beta} = 0.06$  m.

As for the time correlation of  $d\beta(x, t)$ , no daily measurements are available to study the correlation structure of this stochastic process. Intuitively, we may suspect that daily groundwater level fluctuation is a strictly local in time process, that is the random disturbances act for very short periods of time, and that the correlation between successive fluctuations is negligible. Therefore a delta-correlated process is assumed. If more detailed measurements gave further evidence of a wide band process, or colored noise, the theory could still be applied, since it is known that colored noise can be generated by a linear stochastic differential equation forced by white noise (Jazwinski, 1970). Then, instead of eqn. (15) we could have a vector stochastic PDE.

It is now clear the reasons for assuming a spatially smooth white Gaussian noise process  $d\beta(t)/dt = w(t) = \zeta(t)/S$  to model the random component in eqn. (9) with properties given by eqns. (13) and (14).

In the solution of eqn. (A15) and in eqn. (A17) the Brownian motion increment  $d\beta(t)$  is assumed to have a constant variance parameter  $q = 0.0042 \text{ m}^2 \text{ d}^{-1}$  as suggested from the comparison between mean simulated and measured values. In order to obtain sample functions of Brownian motion increments, it is necessary to replace  $q$  by  $q/\Delta t$  (Jazwinski, 1970) as variance parameter, so that the variance of the output will remain constant as  $\Delta t \rightarrow 0$ .

Thus sample functions may be obtained by solving numerically the time integral of eqn. (15):

$$\begin{aligned}
 h(x, t, \omega) = & V(x) + e^{-\alpha Kx/2T} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \sin\left(\frac{n\pi x}{L}\right) \\
 & \times \frac{2}{L} \int_0^L [h_0(x) - V(x)] e^{\alpha Kx/2T} \sin\left(\frac{n\pi x}{L}\right) dx \\
 & + e^{-\alpha Kx/2T} \sum_{m=0}^t \sum_{n=1}^{\infty} e^{-\lambda_n^2 (t-m)} U_n(m) \sin\left(\frac{n\pi x}{L}\right) \quad (24)
 \end{aligned}$$

where  $\omega$  indicates that  $h(x, t, \omega)$  is a sample function from the stochastic process  $h(x, t)$  and  $\omega \in \Omega$ , the probability space;  $V(x)$  is the steady-state function given by eqn. (16);  $U_n(m)$  is a normally distributed random number with mean zero and incremental variance, eqn. (A19),  $\sigma_n^2 = (1/\lambda_n)q$ ; and the rest of the parameters as before.

The phreatic surface profile obtained by eqn. (24) is a smooth curve which fluctuates with time above and below the corresponding mean phreatic surface. In agreement with the assumptions, fluctuation is maximum in the middle of the aquifer and minimum nearby the boundaries. Sample functions are useful for testing our aquifer to various conditions, computing sample groundwater velocities and fluxes for design purposes.

Daily sample functions were obtained for the entire period of simulation. Figures 6 and 7 show the measured profiles and the mean simulated heads on October 27, 1982 and on March 29, 1983. Tables 1 and 2 show the corresponding mean simulated values and one sample realization for the above dates.

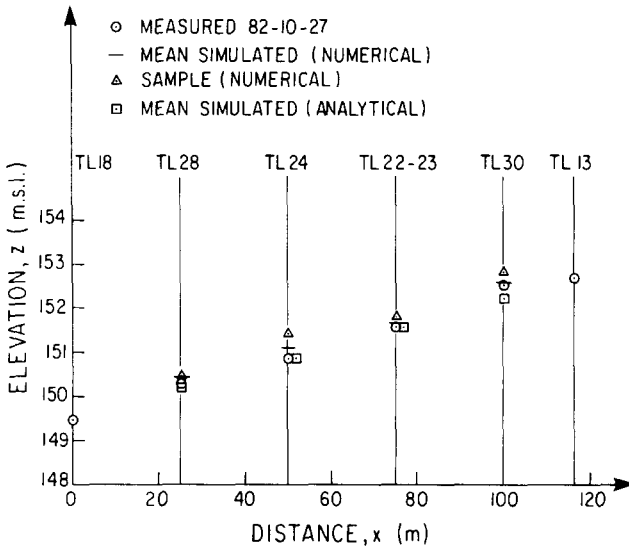


Fig. 6. Free surface potentials on October 27, 1982.

Correlations and covariances may now be computed. We are more interested at this point in calculating the standard deviation of the groundwater table at different locations across the aquifer. We used eqn. (19) for that purpose and it turned out that the convergence of the double summation sequence was slow. Truncated values for the points of interest on the above two days appear in Tables 1 and 2. Again, the standard deviation is maximum at the center of the aquifer and minimum at the extremes. This indicates that the model uncertainty is maximum at the center of the aquifer and minimum near the boundaries.

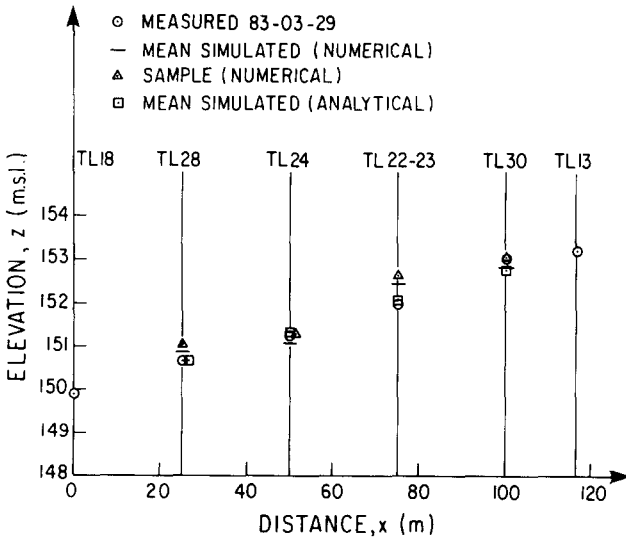


Fig. 7. Free surface potentials on March 29, 1983.

TABLE 1

Free surface heads  $h$  and potentials  $\phi$  on October 27, 1982 (m)

$x$	Method	$E[h(x, t)]$ or $E[\phi(x, t)]$	Sample $h(x, t, \omega)$ or $\phi(x, t, \omega)$	$\sigma_h$ or $\sigma_\phi$
25.0	Analytical	150.18	150.04	0.037
25.0	Numerical	150.26	150.46	0.168*
50.0	Analytical	150.86	150.64	0.050
50.0	Numerical	151.21	151.43	0.185*
75.0	Analytical	151.54	151.35	0.046
75.0	Numerical	151.56	151.79	0.185*
100.0	Analytical	152.22	152.13	0.027
100.0	Numerical	152.69	152.86	0.161*

\*Taken at half aquifer depth.

Boundary conditions: left = 149.501, right = 152.669. Deep percolation rate  $I = 0.0 \text{ mm d}^{-1}$ .

Calculation of the statistical measures of the groundwater potential using Model 2 were later done in order to compare the results with respect to the analytical solutions derived above. The details of these calculations are presented by Serrano and Unny (1986). The properties of the perturbing stochastic process for this model was derived in a manner similar to Model 1. In the time-numerical integration procedure for the free-surface boundary condition, sample values of the random process are calculated by subtracting the mean simulated potential at the free surface from the measured potential at the same place and time. The mean and standard deviation of these differences constitute the sample first two moments of the random process (Serrano and Unny, 1986). For Model 2, the parameter  $q$  was equal to 0.034. Figure 8 shows the boundary element grid used in the computational algorithm, the mean potential distribution and the standard deviation of the potential (shown at the

TABLE 2

Free surface heads  $h$  and potentials  $\phi$  on March 29, 1983 (m)

$x$	Method	$E[h(x, t)]$ or $E[\phi(x, t)]$	Sample $h(x, t, \omega)$ or $\phi(x, t, \omega)$	$\sigma_h$ or $\sigma_\phi$
25.0	Analytical	150.62	150.71	0.037
25.0	Numerical	150.86	151.05	0.168*
50.0	Analytical	151.33	151.47	0.050
50.0	Numerical	151.06	151.29	0.185*
75.0	Analytical	152.05	152.17	0.046
75.0	Numerical	152.45	152.65	0.185*
100.0	Analytical	152.76	152.82	0.027
100.0	Numerical	152.85	153.03	0.161*

\*Taken at half aquifer depth.

Boundary conditions: left = 149.908, right = 153.228. Deep percolation rate  $I = 0.0 \text{ mm d}^{-1}$ .

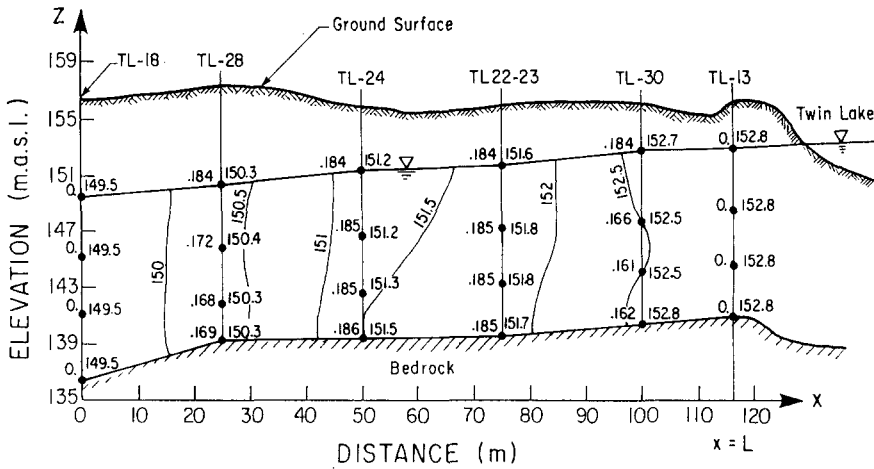


Fig. 8. Boundary element grid, mean potential distribution and standard deviation on October 27, 1982.

left of certain nodes) on October 27, 1982. Figure 9 shows the boundary element grid and a sample potential distribution on the same date. The corresponding figures are summarized in Table 1. Table 2 also summarizes the results for March 29, 1983. The results indicate that the highest standard deviation, and therefore the highest uncertainty, occurs near the center of the aquifer and that the minimum standard deviation occurs near the boundaries. Generally the standard deviations produced by the numerical model are significantly higher than the corresponding ones produced by the analytical model. This indicates that the analytical model is more reliable in this case. However this

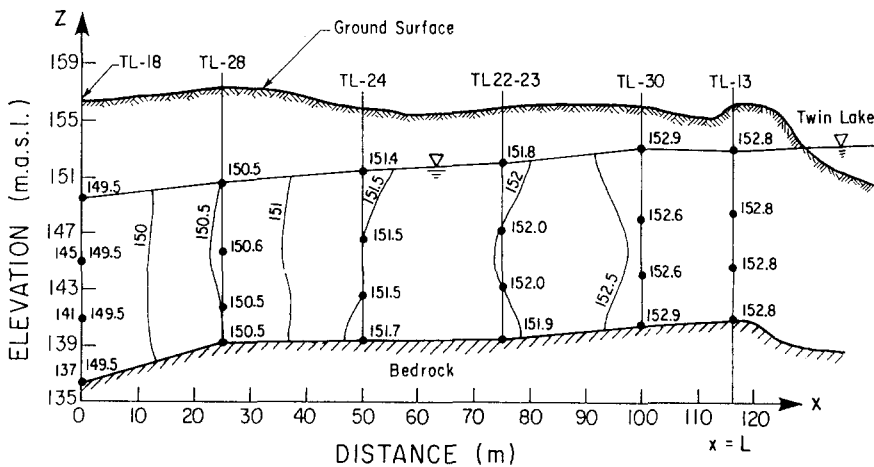


Fig. 9. Boundary element grid and sample potential distribution on October 27, 1982.

loss in accuracy is compensated by the ability of the numerical model to handle two-dimensional domains with complex geometrical shapes.

Following the discussion in the introduction, the groundwater model could be validated by noting that a reliable model for predictive purposes is one which generates groundwater heads whose statistical properties approach the corresponding statistical properties of measured heads. This is easily done by observing that mean simulated values correspond with the mean measured values in both models. By studying Tables 1 and 2, it is possible to see that for the analytical model the maximum standard deviation of 0.050 m converges to the standard deviation of the perturbing stochastic process, which was obtained by studying the frequency behaviour of the difference between the mean simulated and measured values, that is  $q^{1/2} = 0.065$ . For the numerical model, the maximum standard deviation of 0.185 m converges to the standard deviation of the perturbing stochastic process for this model, that is  $q^{1/2} = 0.184$ . Since the statistical measures calculated from the models are close to the corresponding ones of the perturbing stochastic processes derived from the measured heads, we conclude that the model produces forecast values with the same statistical properties of field values.

## 5. CONCLUSIONS AND RECOMMENDATIONS

Groundwater mathematical models are subject to several sources of uncertainty and errors. Field data for mathematical models are also subject to measurement errors and random environmental fluctuations. Groundwater forecasting models should be adapted to predict the behaviour of the groundwater potential in a statistical sense. These models will provide more information and a more realistic representation of the nature of groundwater flow, which specifically accounts for the errors in the formulation and solution of the model equations and the errors and environmental fluctuations associated with the measured data. Deterministic models can be considered as a very special, or ideal, case of the more general stochastic case.

From the application of the two models to the Twin Lake aquifer, it is concluded that stochastic analytical models provide relatively lower measures of uncertainty, and consequently higher degree of reliability of the forecast, than corresponding stochastic numerical models. This is compensated by the greater versatility of numerical models and their ability to handle two- or three-dimensional domains with complex geometrical boundaries. The two stochastic models, and specifically the analytical, will be used for the study of the groundwater flow regime in the Twin Lake aquifer and its random variability. These models will also serve as important structural components in the calculation of the statistical properties of groundwater transport from low level nuclear waste disposal facilities at the CRNL, and in the long term prediction of future extent of contamination.

Validation of groundwater mathematical models should be assessed based on the ability of the model to reproduce the statistical properties of the measured values, instead of on the ability to reproduce individual sample measurements.

## ACKNOWLEDGEMENTS

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## APPENDIX: ANALYTICAL SOLUTION OF THE STOCHASTIC BOUSSINESQ EQUATION

Eqn. (9) is a random PDE and as such cannot be analyzed and solved by using classical mathematics. As stated in section 1, analysis and solution of stochastic PDE was only possible as a result of recent developments in the functional analysis of abstract evolution equations with partial differential operators. For a theoretical description of these equations, the reader is referred to the works by Adomian (1964, 1970, 1971, 1976, 1983), Becus (1977, 1980), Bensoussan (1977), Bharucha-Reid (1964), Curtain (1978), Curtain and Falb (1971), Curtain and Pritchard (1977, 1978), Gopalsamy and Bharucha-Reid (1975) and Sawaragi et al. (1978). For a more readable summary of this theory and several examples in groundwater flow see Serrano and Unny (1987a), Serrano et al. (1985a, b, c, 1986). These authors have developed function spaces (Sobolev spaces) of random functions having much of the same properties found in equivalent Sobolev spaces of deterministic functions (compactness, the existence of an orthogonal basis and an  $\varepsilon$  net, infinite dimensionality, a norm based on inner products, boundedness, convergent sequences of approximations, etc.). Fundamental questions concerning the existence and uniqueness of solutions have been solved by the statement and proof of appropriate theorems. The determination of a particular Sobolev space where the solution belongs has permitted the description of some of the properties of the solution process like continuity, differentiability, etc. Finally, the use of the topological properties of these spaces has opened the way for the development of explicit solutions, which could be interpreted in the same way as classical solutions are with deterministic PDE.

Because in this paper we are more concerned with the practical application of these equations, a theoretical analysis of the problem is not needed. It is enough to say, for now, that in order to use the theory to find a solution, it is necessary to transform the function spaces. In the system of eqns. (9)–(12),  $h \in H^1(0, L)$ , where  $H^1(0, L)$  is the first-order Sobolev space. By defining  $u(x, t) = h(x, t) - V(x)$ , where  $V(x)$  is a smooth known function satisfying the boundary conditions, then the equivalent system of equations in  $u$ , where  $u \in H_0^1(0, L)$  and  $H_0^1(0, L)$  is the first-order Sobolev space with compact support, may be treated as an abstract evolution equation and solved directly. This process is called homogenization in classical mathematics. Obviously in the present case  $V(x)$  corresponds to the steady-state solution, but in the general case of time dependent boundary conditions  $V$  is a time dependent function. Hence the transformed boundary value problem in  $u$  is given by:

$$\frac{\partial u}{\partial t} - \frac{T}{S} \frac{\partial^2 u}{\partial x^2} - \frac{\alpha K}{S} \frac{\partial u}{\partial x} = \frac{d\beta}{dt} \quad (\text{A1})$$

$$u(0, t) = 0 \quad (\text{A2})$$

$$u(L, t) = 0 \quad (\text{A3})$$

$$u(x, 0) = h_0(x) - V(x) = u_0(x) \quad (\text{A4})$$

$V$  satisfies the deterministic boundary value problem:

$$\frac{T}{S} \frac{d^2 V}{dx^2} + \frac{\alpha K}{S} \frac{dV}{dx} + \frac{I}{S} = 0 \quad (\text{A5})$$

$$V(0) = h_1 \quad (\text{A6})$$

$$V(L) = h_2 \quad (\text{A7})$$

whose solution is easily found as (Spiegel, 1980):

$$V(\alpha) = h_1 + \frac{\left(h_2 - h_1 + \frac{IL}{\alpha K}\right)}{\left(e^{-\alpha KL/T} - 1\right)} \left(e^{-\alpha Kx/T} - 1\right) - \frac{I}{\alpha K} x \tag{A8}$$

This is eqn. (16) in section 3.

As for the system of eqns. (A1)–(A4), it can be treated as an abstract stochastic evolution equation of the form:

$$\frac{\partial u}{\partial t} + Au = w, \quad u \in H_0^1(0, L) \tag{A9}$$

$$u_{x(0, L)} = 0 \tag{A10}$$

$$u(0, x) = u_0(x) \tag{A11}$$

where  $A$  is a linear partial differential operator given by:

$$Au = \left(-\frac{T}{S} \frac{\partial^2}{\partial x^2} - \frac{\alpha K}{S} \frac{\partial}{\partial x}\right)u$$

The solution of (A9)–(A11) is given by:

$$u = J_t u_0 + \int_0^t J_{t-s} d\beta(s) \tag{A12}$$

where  $J_t$  is the strongly continuous semigroup associated with  $A$ , which has several interesting properties we could use (Curtain and Pritchard, 1978; Butzer and Berens, 1967). The semigroup operator is easily found by setting  $d\beta/dt = 0$  in eqn. (A1). The resulting simple deterministic PDE may be solved by well-known procedures (i.e., Powers, 1979; Spiegel, 1980; Farlow, 1982). The semigroup is thus given as:

$$J_t z = e^{-\alpha Kx/2T} \sum_{n=1}^{\infty} e^{-j_n^2 t} \sin\left(\frac{n\pi x}{L}\right) \frac{2}{L} \int_0^L z e^{\alpha Kx/2T} \sin\left(\frac{n\pi x}{L}\right) dx \tag{A13}$$

where the eigenvalues  $\lambda_n$  are given by:

$$\lambda_n^2 = \frac{n^2 \pi^2 T}{L^2 S} + \frac{\alpha^2 K^2}{4TS} \tag{A14}$$

Knowing the semigroup operator significantly increases the efficiency and accuracy of finding solutions for more complex deterministic PDE with different boundary and initial conditions, different forcing functions or PDE involving stochastic functions. Clearly the functional analytical approach provides not only notational economy, flexibility and efficiency, but also presents a rigorous approach to the problem of stochastic PDE.

Substituting eqn. (A13) in eqn. (A12) and expressing the solution in terms of  $h$ :

$$\begin{aligned} h(x, t) = & V(x) + e^{-\alpha Kx/2T} \sum_{n=1}^{\infty} e^{-j_n^2 t} \sin\left(\frac{n\pi x}{L}\right) \frac{2}{L} \int_0^L [h_0(x) - V(x)] e^{\alpha Kx/2T} \sin\left(\frac{n\pi x}{L}\right) dx \\ & + e^{-\alpha Kx/2T} \int_0^t \sum_{n=1}^{\infty} e^{-j_n^2(t-s)} \sin\left(\frac{n\pi x}{L}\right) d\beta(s) \frac{2}{L} \int_0^L e^{\alpha Kx/2T} \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned} \tag{A15}$$

where  $d\beta(s)$  is Brownian motion increment with the properties (Jazwinski, 1970):

$$E[d\beta(t)] = 0 \tag{A16}$$



$$E[\|d\beta(t)\|^2] = qdt \tag{A17}$$

A naive attempt at modeling a distributed Brownian motion increment is provided by the fact that if the process is a Hilbert space-valued Brownian motion process, then it should be possible to expand it as a sequence:

$$d\beta(t) = \sum_{n=1}^{\infty} db_n e_n \tag{A18}$$

where  $e_n$  is an orthonormal basis function and  $db_n$  is a unidimensional scalar Brownian motion process with an incremental variance parameter given by (Chow, 1979; Curtain and Pritchard, 1978):

$$E[\|d\beta(t)\|^2] = qdt \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \tag{A19}$$

where  $\|\cdot\|$  is the space norm and  $\lambda_n$  the eigenvalues. A classical interpretation of the series (A18) would imply that:

$$db_n = \frac{2}{L} \int_0^L d\beta(t) e^{\alpha Kx/2T} \sin\left(\frac{n\pi x}{L}\right) dx \tag{A20}$$

Using (A18) and (A20), eqn. (A15) may be simplified as:

$$\begin{aligned} h(x, t) = & V(x) + e^{-\alpha Kx/2T} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \sin\left(\frac{n\pi x}{L}\right) \frac{2}{L} \int_0^L [h_0(x) - V(x)] e^{\alpha Kx/2T} \sin\left(\frac{n\pi x}{L}\right) dx \\ & + e^{-\alpha Kx/2T} \int_0^t \sum_{n=1}^{\infty} e^{-\lambda_n^2(t-s)} db_n(s) \sin\left(\frac{n\pi x}{L}\right) \end{aligned} \tag{A21}$$

This is eqn. (15) in section 3.

Now, by taking expectations on both sides of eqn. (A15) and using (A16), we obtain an expression for the mean of the solution, which coincides with the deterministic solution:

$$E[h(x, t)] = V(x) + e^{-\alpha Kx/2T} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \sin\left(\frac{n\pi x}{L}\right) \frac{2}{L} \int_0^L [h_0(x) - V(x)] e^{\alpha Kx/2T} \sin\left(\frac{n\pi x}{L}\right) dx \tag{A22}$$

This is eqn. (18) in section 3. This could also be written as:

$$E[h(x, t)] = V(x) + J_t u_0 \tag{A23}$$

For notational economy it is easier to derive the correlation function in terms of the semigroup of the process  $u$ . From (A12):

$$E[u(t_1)u(t_2)] = E\left\{ \left[ J_{t_1} u_0 + \int_0^{t_1} J_{t_1-s_1} d\beta(s_1) \right] \left[ J_{t_2} u_0 + \int_0^{t_2} J_{t_2-s_2} d\beta(s_2) \right] \right\} \tag{A24}$$

After solving, using (A17) and some properties of random integrals (Jazwinski, 1970), we obtain:

$$E[u(t_1)u(t_2)] = J_{t_1+t_2} u_0^2 + q \int_0^{t_1} J_{t_1+t_2-2s} ds \tag{A25}$$

Similarly the covariance in (A12) gives:

$$E(\{u(t_1) - E[u(t_1)]\}\{u(t_2) - E[u(t_2)]\}) = \text{Cov}[u(t_1)u(t_2)] = q \int_0^{t_1} J_{t_1+t_2-2s} ds \quad (\text{A26})$$

We are particularly interested in the standard deviation of  $h$  at any time  $t$ . Then if in eqn. (A26) we make  $t_1 = t_2 = t$ , expand and solve the integrals, we obtain:

$$\sigma_h = E(\{h(x, t) - E[h(x, t)]\}^2) = \frac{4q}{L^2} e^{-\alpha Kx/T} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)}{\lambda_m^2 + \lambda_n^2} \times \frac{[1 - e^{\alpha KL/2T} \cos(m\pi)] [1 - e^{\alpha KL/2T} \cos(n\pi)]}{\left(1 + \frac{\alpha^2 K^2 L^2}{4T^2 m^2 \pi^2}\right) \left(1 + \frac{\alpha^2 K^2 L^2}{4T^2 n^2 \pi^2}\right)} (1 - e^{-\mu(\lambda_m^2 + \lambda_n^2)}) \quad (\text{A27})$$

This is eqn. (19) in section 3.

Having completed the application example, the authors would like to add a few conclusions concerning the methodology used. Functional analysis presents several advantages when solving a system equation for the first time. Considerable notational economy and efficiency is gained, and a rigorous treatment of stochastic PDE under an integrated theoretical framework with the equivalent deterministic PDE is possible. Functional analysis presents an encouraging alternative to solve increasingly complex stochastic hydrologic problems, where classical mathematics methods offer serious difficulties. In the example of this paper, functional analysis and semigroup theory facilitated a simple solution for the mean groundwater level, the correlation function, the covariance and the standard deviations with time and space. These expressions were obtained with a minimum of analytical steps and notation, thus reducing the risks for error.

The derivation of a semigroup for a particular partial differential operator is best accomplished by solving the corresponding homogeneous deterministic PDE with homogeneous boundary conditions. Once the semigroup is obtained, it may be used to solve more complex problems, such as the randomly-forced equation, with source terms and/or random, time-varying boundary conditions and/or random initial conditions. Hence considerable flexibility is obtained by using the semigroup. In the present example, a semigroup for a PDE in a sloping phreatic aquifer is used.

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