

# Random Evolution Equations in Hydrology

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## ABSTRACT

A number of partial differential equations governing the dynamic behavior of different components of the hydrologic cycle in a region can be treated as abstract evolution equations in appropriate Hilbert spaces. When the source terms, the boundary conditions, the initial conditions, and/or the equation parameters are prescribed as stochastic processes, the resulting random evolution equations may be solved in terms of the input stochastic processes if the semigroup associated with the partial differential operator is known. This places in the hands of the modeler a very flexible tool to solve a large variety of problems. Several explicit examples of the methodology are illustrated. Finally, applications of the Itô's lemma in Hilbert spaces to the numerical solution of stochastic partial differential equations in hydrology are presented.

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## 1. INTRODUCTION

Many linear and semilinear partial differential equations governing the unsteady behavior of hydrologic systems may be treated as linear evolution equations in appropriate Hilbert spaces. The unsteady-groundwater-flow equation in a phreatic aquifer, the unsteady-groundwater-flow equation in a confined or leaky aquifer, the advective-dispersive equation describing solid transport in a system of aquifers, and the diffusion equation in lakes are some examples. Other equations, such as the pressure-head equation in unsaturated soil media, are highly nonlinear and require a special approximate

treatment before they can be represented as nonhomogeneous evolution equations.

The solution of these deterministic equations can be easily obtained if a semigroup associated with the particular spatial partial differential operator can be derived. An important advantage of this approach is that once the semigroup is known, other problems subject to more complex source functions can be solved in a systematic way. Furthermore, problems involving time-dependent boundary conditions or spatially distributed initial conditions may be solved. However, the most important advantage of an abstract representation is the possibility of solving the differential equation when the initial condition, the source terms, the boundary conditions, and/or the system parameters are prescribed as stochastic processes. This is especially important in that there is a high degree of uncertainty associated with the measurement and the study of the time and space variability of environmental information required by hydrologic models. The stochastic properties of the output process can then be calculated from the known stochastic properties of the input processes. Thus this approach gives the modeler a flexible tool for forecasting hydrologic variables and the possibility of studying increasingly complex problems.

This article presents the most important results on the analysis and solution of stochastic partial differential equations in hydrology. It is based on recent contributions to the theory of stochastic partial differential equations in Hilbert space. The most important contributions are related to the theory of stochastic partial differential equations and semigroups of operators ([5], [7], [9], and [30] among others). These works have defined a functional-analytic framework to study many stochastic equations in mathematical physics. In particular the theorems on the existence and uniqueness of solutions of the appropriate stochastic evolution equation have proved most useful in responding to fundamental questions about the abstract equations.

Applications of these concepts to the groundwater flow equation subject to stochastic initial conditions, stochastic boundary conditions, and stochastic recharge have generated important results [34–36]. The procedure involved the derivation of explicit forms of the strongly continuous semigroup associated with the corresponding partial differential operator and the use of the topological properties of the solution Sobolev space to construct a series of approximations towards a solution. In particular, the Wiener process was expanded as an infinite basis in a Hilbert space composed of independent unidimensional Wiener processes with incremental variance parameters. A process of this kind was found to successfully explain the stochastic nature of field measurements of groundwater levels at the Chalk River Nuclear Laboratories, Ontario, Canada [33]. A similar approach was recently applied to the solution of the advective-dispersive transport equation in porous media when

the pollution loads, the boundary conditions, or the dispersion coefficient were prescribed as stochastic processes [36]. The results were obtained by blending the concepts of random evolution equations and semigroups with some classical results on the elementary heat flow equation.

Another interesting approach combined an abstract formulation of the two-dimensional groundwater flow equation, subject to a stochastic free-surface boundary condition, with computer-oriented numerical techniques [37]. This work proposed a representation of the Itô's lemma in a Hilbert space to obtain a series of deterministic equations satisfied by the moments of the potential process. These equations were solved by the boundary-integral-equation method. Applications of this approach can be seen in [34].

Other approaches to the solution of stochastic partial differential equations in hydrology have involved the perturbation-expansion solution [10–14]. However, this approach is limited to differential equations subject to stochastic processes with “small” variance. Other authors have opted for a Monte Carlo simulation approach to the problem [39].

In Section 2, we briefly describe the relevant functional-analytic results without repeating the proofs of the theorems. In Section 3, several applications to the solution of stochastic partial differential equations in hydrology are illustrated. Finally, in Section 4 we illustrate the use of a formulation of the Itô's lemma in Hilbert spaces to derive moment equations for stochastic partial differential equations in higher-dimensional domains with irregular geometry. These moment equations can be solved by any analytical or numerical deterministic procedure, a feature which allows the use of the extensive literature on numerical solutions of deterministic partial differential equations.

## 2. THE MATHEMATICAL THEORY

The general three-dimensional stochastic partial differential equation in a hydrologic system may be treated as a stochastic evolution equation of the form

$$\begin{aligned} \frac{\partial u}{\partial t}(x, t, \omega) + A(x, t, \omega)u &= g(x, t, \omega), & (x, t, \omega) \in G \times [0, T] \times \Omega, \\ Q(x, t, \omega)u &= F(\omega), & (x, t, \omega) \in \partial G \times [0, T] \times \Omega, \\ u(x, 0, \omega) &= u_0(x, \omega), & (x, \omega) \in G \times \Omega, \end{aligned} \quad (2.1)$$

where  $u \in L^2(0, T; V)$  is the system output,  $g \in L_2(\Omega, B, P)$  is a second-order

random forcing function;  $G \subset \mathbb{R}^3$  is an open domain subset of the three-dimensional real space with boundary  $\partial G$ ;  $0 < T < \infty$ ,  $Q$  is a boundary operator;  $\Omega$  is the basic probability sample space of elements  $\omega$ ;  $L_2(\Omega, B, P) = L^2(\Omega)$  is the complete probability space of second-order random functions with probability measure  $P$  and Borel field (or class of  $\omega$  sets)  $B$ ;  $x$  represents three-dimensional spatial domain;  $A$  is an  $m$ th-order random partial differential operator in the space  $H^m(G)$ , given by

$$Au = \sum_{k,l \leq m} (-1)^k D^k (p_{kl}(x, t, \omega) D^l u), \quad (2.2)$$

where  $D$  is differentiation;  $p_{kl}(x, t, \omega)$  are randomly valued stochastic processes representing the system parameters, which are assumed bounded and mean-square continuous on  $[0, T]$ ;  $m$  is the order of the space; the space

$$L^2(0, T; V) = \left\{ f: [0, T] \rightarrow V: \int_0^T \|f\|_V^2 dt < \infty \right\}$$

for  $0 < T < \infty$ ;  $V = H^m$  is the Sobolev space of order  $m$  of  $L^2(\Omega)$ -valued functions;  $V \subset H \subset V'$ ;  $V$  is dense in  $H$ , where  $H = H^0$ ; the norm on  $V$  is denoted by  $\| \cdot \|_V$ ;  $V'$  is the dual of  $V$ ;  $g \in L_2(0, T; V')$ ; and  $u_0 \in H \times \Omega$  is the system initial condition. For a more complete description of the above definitions the reader is referred to the available functional-analytic literature [17, 19, 28, 38, 30, 5].

Theorems and proofs concerning the existence and uniqueness of the solution to a system (2.1) have been extensively treated in [30], [5], [7], and [9] among others.

Randomness may enter the system (2.1) in the following ways: (i) The random-initial-value problem, when  $u_0$  is random. (ii) The random-boundary-value problem, when  $F$  is random. (iii) The random-forcing problem, when  $g$  is random. (iv) The random-operator problem, when  $A$  or  $Q$  is random. (v) The random-geometry problem.

For cases (ii) and (iii) above, the operator  $A$  is deterministic, and in many practical applications it is a time-independent operator. Moreover, if we transform the functional space in (2.1) into an equivalent one in which the system has homogeneous boundary conditions, we have

$$\frac{\partial v}{\partial t}(x, t, \omega) + A(x)v = h(x, t, \omega), \quad (2.3)$$

$$v|_{\partial G} = 0,$$

$$v(x, 0) = v_0(x),$$

where  $v \in L^2(0, T; V)$  is the system output;  $V = H_0^m$  is a closed subspace of  $H^m$ ;  $H_0^m$  is the closure of  $C_0^\infty(G; L^2(\Omega))$  in  $H^m$ , that is,  $H_0^m$  is the  $m$ th-order Sobolev space of second-order random functions with compact support;  $h(x, t, \omega)$  includes the function  $g(x, t, \omega)$  and the appropriate function(s) resulting from the space transformation, including the boundary conditions; and  $v_0(x)$  includes  $u_0(x)$ .

The general solution of (2.3) is

$$v(t) = J_t v_0 + \int_0^t J_{t-s} h(x, s, \omega) ds, \tag{2.4}$$

where  $J_t \in l(H, H)$  is the strongly continuous semigroup associated with  $A$  [16, 23, 20, 2, 4, 8, 24, 6]. If a transformation of spaces is not done, it is clear that the semigroup operator must satisfy the prescribed boundary conditions. In general, if the operator  $A$  is time-independent and if the evolutionary operator  $J_t$  in the Hilbert space  $H$  satisfies

- (i)  $J_{t+s} = J_t J_s \geq 0$ ,
- (ii)  $J_0 = I$ , where  $I$  is the identity operator, and
- (iii)  $\|J_t v - v\|_H \rightarrow 0$  as  $t \rightarrow 0$  for all  $v \in H$ ,

where  $\| \cdot \|_H$  denotes the norm, then  $J_t$  is said to be a strongly continuous semigroup. Properties (i) and (ii) above give the semigroup structure, whereas property (iii) is topological and defines the “strong continuity.”

Theoretically, we may be interested in finding the joint distribution function of all orders that characterize the process  $v$ . This task is frequently too complicated and in many situations represents more than is needed. We often can consider simpler and necessarily less complete characterizations in the form of expectations, dispersions, covariances, joint moments, etc., which are called statistical measures. This view is supported by the fact that it is usually not feasible to collect enough field information to evaluate the joint probability density function of the input processes and the parameters. Therefore, from the practical point of view, it is possible to calculate only the first few low-order moments of the solution process  $v$ . The first two moments give considerable information of the joint probability density function of  $v$ .

The mean value of  $v$  is given by

$$E\{v(x, t)\} = J_t v_0 + \int_0^t J_{t-s} E\{h(x, s, \omega)\} ds, \tag{2.5}$$

where  $E\{ \}$  denotes the expectation operator.

Now the second moment of  $v$  in Equation (2.4) is given by

$$E\{v(t_1)v(t_2)\} = E\left\{\left(J_{t_1}v_0 + \int_0^{t_1} J_{t_1-s}h(s) ds\right)\left(J_{t_2}v_0 + \int_0^{t_2} J_{t_2-\xi}h(\xi) d\xi\right)\right\}, \quad (2.6)$$

where  $\omega$  has been omitted for convenience. Without loss of generality we assume that  $h$  is a zero-mean stochastic process. This is the residual stochastic process after the best possible deterministic analysis and trend have been subtracted. Thus the second term in Equation (2.5) is equal to zero, and Equation (2.6) becomes

$$E\{v(t_1)v(t_2)\} = J_{t_1+t_2}v_0 + \int_0^{t_1} \int_0^{t_2} J_{t_1+t_2-s-\xi} E\{h(s)h(\xi)\} ds d\xi. \quad (2.7)$$

Note that the calculation of Equation (2.7) requires knowledge of the correlation of  $h$  and that it is towards this end that the field measurements should be oriented. Higher-order moments may be easily calculated in a similar way.

If the operator  $A$  in Equation (2.1) is stochastic because one or more of the parameters are defined as a stochastic process, then Equation (2.3) should be written as

$$\begin{aligned} \frac{\partial v}{\partial t}(x, t, \omega) + A(x, t, \omega)v &= h(x, t, \omega), & v \in H_0^m(G), \\ v|_{\partial G} &= 0, \\ v(x, 0) &= v_0(x). \end{aligned} \quad (2.8)$$

We follow in this section a procedure similar to the one used in [1] to decompose an ordinary stochastic differential operator into an infinite series in order to approximate the corresponding stochastic Green's function. We shall present a semigroup formulation for a general stochastic partial differential operator, such as the one introduced in [32] for the groundwater flow equation subject to stochastic transmissivity.

Let us write the operator  $A$  as

$$A(x, t, \omega) = A(x) + R(t, \omega), \quad (2.9)$$

where  $A(x)$  is the deterministic, time-independent component and  $R(t, \omega)$  the random part—the part of the partial differential operator containing the parameters that are random in time. It is clear that this time-stochastic representation is equally valid for spatial stochasticity, and we prefer for now to solve the equation with time-stochastic parameters.

Using Equation (2.9), Equation (2.8) can be written as

$$\frac{\partial v}{\partial t} + Av = h - Rv,$$

$$v|_{\partial G} = 0,$$

$$v(x, 0) = v_0, \quad (2.10)$$

whose solution is given by Equation (2.4) as

$$v = J_t v_0 + \int_0^t J_{t-s} h(s) ds - \int_0^t J_{t-s} Rv(s) ds \quad (2.11)$$

It is not possible to solve Equation (2.11) explicitly because  $v$  is on the right-hand side. At this point we follow the method described by Adomian [1], in which an ordinary stochastic differential operator is decomposed in an infinite series approaching the operator Green's function in the limit. We assume that the method is equally valid for stochastic partial differential operators as suggested by Adomian, and that the decomposition series will approach, in the limit, the semigroup of the operator. Thus we decompose  $v$  on the right-hand side as an infinite series  $v = \sum_{i=1}^{\infty} v_i$ . Equation (2.11) becomes

$$v = J_t v_0 + \int_0^t J_{t-s} h(s) ds - \int_0^t J_{t-s} R(v_1 + v_2 + v_3 + \cdots) ds, \quad (2.12)$$

where the semigroup  $J$  is now deterministic. Identifying  $v_1$  as the preceding

term  $\int_0^t J_{t-s} h(s) ds$ , we can determine each  $v_i$  in terms of the preceding  $v_{i-1}$ . Thus

$$v = J_t v_0 + \int_0^t J_{t-s} h(s) ds + \int_0^t \int_0^s J_{t-s} R J_{s-\tau} h(\tau) d\tau ds \\ - \int_0^t \int_0^s \int_0^\tau J_{t-s} R J_{s-\tau} R J_{\tau-\xi} h(\xi) d\xi d\tau ds - \dots, \quad (2.13)$$

where the last term in the series contains  $v$ . The basic idea here is that a random semigroup operator, which may be difficult to derive in particular cases, can be determined in an easily computable series by decomposition of the differential operator  $A(x, t, \omega)$  into a deterministic operator  $A(x)$  whose semigroup is known or found with little effort, and a random operator  $R(t, \omega)$  whose contribution to the modification of the total semigroup  $J_{t, \omega}$  can be found in series form. The convergence question will not be treated here, since it has been discussed in detail by Adomian [1]. However, it is possible to see intuitively that each term in the series of integrals in (2.13) contains an increasing number of decay (usually exponential) functions of decreasing magnitude, which assure the convergence to a desired level of accuracy within a few steps. Recently [32] the authors conducted a sensitivity analysis of the effect of truncating the series at the fourth term in (2.13) on the accuracy with respect to the exact solution, and it was found that using  $v_1$  only would give quite acceptable accuracy for most hydrologic cases of interest.

The mean value of  $v$  is obtained by truncating Equation (2.13) and taking expectations:

$$E\{v\} = J_t v_0 + \int_0^t J_{t-s} E\{h(s)\} ds - \int_0^t \int_0^s J_{t-s} E\{R J_{s-\tau}\} E\{h(\tau)\} d\tau ds, \quad (2.14)$$

where statistical separability occurs between the semigroup and the forcing term. Physically this stems from the independent behavior of the input function and the system parameters.

Similarly, the correlation function is given by

$$\begin{aligned}
 E\{v(t_1)v(t_2)\} = E\left\{ \left( J_{t_1}v_0 + \int_0^{t_1} J_{t_1-s}h(s) ds - \int_0^{t_1} \int_0^s J_{t_1-s}RJ_{s-\tau}h(\tau) d\tau ds \right) \right. \\
 \times \left( J_{t_2}u_0 + \int_0^{t_2} J_{t_2-\rho}h(\rho) d\rho \right. \\
 \left. \left. - \int_0^{t_2} \int_0^\beta J_{t_2-\beta}RJ_{\beta-\gamma}h(\gamma) d\gamma d\beta \right) \right\}. \quad (2.15)
 \end{aligned}$$

Assuming  $h$  a zero-mean temporal stochastic process and solving,

$$\begin{aligned}
 E\{v(t_1)v(t_2)\} = J_{t_1+t_2}v_0^2 + \int_0^{t_1} \int_0^{t_2} J_{t_1+t_2-s-\rho} E\{h(s)h(\rho)\} ds d\rho \\
 - 2 \int_0^{t_1} \int_0^{t_2} \int_0^\beta J_{t_1+t_2-s-\beta} E\{RJ_{\beta-\gamma}\} E\{h(s)h(\gamma)\} d\gamma d\beta ds \\
 + \int_0^{t_1} \int_0^{t_2} \int_0^s \int_0^\beta J_{t_1+t_2-s-\beta} E\{RJ_{s-\tau}RJ_{\beta-\gamma}\} \\
 \cdot E\{h(\tau)h(\gamma)\} d\tau ds d\gamma d\beta. \quad (2.16)
 \end{aligned}$$

These are the results obtained by considering one term  $v_1$  in the series of Equation (2.12). Obviously calculations can be extended up to any desired degree of accuracy by considering more terms. However, it is known that the series converges rapidly and that in some circumstances by considering one term we have sufficient accuracy, as demonstrated by the sensitivity analysis on the semigroup for the stochastic Boussinesq equation in groundwater flow presented in [32].

In the following section several illustrative applications of the above method to the solution of stochastic hydrologic problems will be presented. The applications are chosen to satisfy the current modeling needs in hydrology. Use will be made of a well-known stochastic process in the applications, namely the white Gaussian process. This is done for simplicity and because the properties of this process closely resemble many physically realizable processes after the deterministic trend has been moved. It is clear, however, that any process in  $L_2(\Omega)$  could be used, the properties of which should be derived from sample field measurements followed by an estimation algorithm (i.e., [15]).

### 3. APPLICATIONS TO STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN HYDROLOGY

#### 3.1. *Modeling the Stochastic Evolution of Groundwater Pollution*

To begin the illustration on the applications of the theory presented in Section 2, consider the stochastically forced longitudinal dispersion in a semiinfinite aquifer. Assuming that the porous medium is homogeneous and isotropic, that no mass transfer occurs between the solid and liquid phases, and that the average groundwater velocity is constant throughout the length of the flow field, the differential equation is obtained after applying the divergence theorem to an integral-equation statement of mass conservation in a control aquifer volume, and combining this with the equation of Fick's first law [18]:

$$\frac{\partial C}{\partial t} - D \frac{\partial^2 C}{\partial x^2} + u \frac{\partial C}{\partial x} = \frac{d\beta}{dt}, \quad (3.1)$$

subject to

$$C(0, t) = 0, \quad C(\infty, t) = 0, \quad C(x, 0) = 0,$$

where  $C(x, t, \omega)$  is the stochastic process representing the concentration of a principal contaminant in the fluid (mg/l),  $D$  is the aquifer dispersion coefficient ( $\text{m}^2/\text{day}$ ),  $u$  is the average pore velocity, that is, the flux velocity divided by the average porosity of the medium ( $\text{m}/\text{day}$ ),  $x$  is the coordinate parallel to the flow,  $t$  is the time coordinate, and  $d\beta/dt = w$  is a white Gaussian noise process with the properties

$$E\{w(t)\} = 0, \quad E\{w(t_1)w(t_2)\} = q\delta(t_2 - t_1), \quad (3.2)$$

where  $q$  is the variance parameter ( $\text{mg}/\text{l day}$ ). The stochastically forced equation may be used to model the effect of unknown environmental quantities affecting the concentration distribution, errors generated in the development of the model, errors in the estimation of parameter values, and/or uncertain chemical reactions between the porous matrix and the fluid. In this section we study the kind of problems when a time-stochastic distributed source dominates the uncertainty of the system.

The operator  $A$  such that

$$AC = \left( -D \frac{\partial^2}{\partial x^2} + u \frac{\partial}{\partial x} \right) C$$

generates a strongly continuous semigroup  $J_t$  given by

$$J_t f(x) = \frac{1}{(4\pi Dt)^{1/2}} \int_0^\infty \left\{ \exp \left[ -\frac{(x-ut-s)^2}{4Dt} \right] - \exp \left[ -\frac{(x-ut+s)^2}{4Dt} \right] \right\} f(s) ds. \quad (3.3)$$

This is easy to see by analogy of Equation (3.3) with the classical heat flow (diffusion) equation. Thus, according to Equations (2.4) and (3.3), the solution to (3.1) is simply

$$C(x, t, \omega) = \int_0^t \frac{w(\tau, \omega)}{(4\pi D(t-\tau))^{1/2}} \int_0^\infty \left\{ \exp \left[ -\frac{[x-u(t-\tau)-s]^2}{4D(t-\tau)} \right] - \exp \left[ -\frac{[x-u(t-\tau)+s]^2}{4D(t-\tau)} \right] \right\} ds d\tau. \quad (3.4)$$

Simplifying the inner integral, we obtain

$$C(x, t, \omega) = \int_0^t \operatorname{erf} \left[ \frac{x-u(t-\tau)}{4D(t-\tau)} \right] w(\tau, \omega) d\tau. \quad (3.5)$$

By generating sample functions of  $w$  we may obtain sample functions of  $C$ . Sample functions are useful for testing models and for observing the qualitative behavior of the system due to different types of excitations. Now taking expectations on both sides of (3.5) and using (3.2), we find the mean concentration to be

$$E\{C(x, t)\} = 0. \quad (3.6)$$

Now following Equations (2.7), and (3.2), the second moment of  $C$  is

$$E\{C(x, t_1)C(x, t_2)\} = q \int_0^{t_1} \int_0^{t_2} \operatorname{erf} \left[ \frac{x - u(t_1 - \tau)}{[4D(t_1 - \tau)]^{1/2}} \right] \operatorname{erf} \left[ \frac{x - u(t_2 - \xi)}{[4D(t_2 - \xi)]^{1/2}} \right] \times \delta(t_1 - t_2) d\xi d\tau. \quad (3.7)$$

If we let  $t_1 = t_2 = t$ , Equation (3.7) becomes the variance of  $C$ :

$$\sigma_C^2 = q \int_0^t \operatorname{erf}^2 \left[ \frac{x - u(t - \tau)}{[4D(t - \tau)]^{1/2}} \right] d\tau. \quad (3.8)$$

Higher-order moments may be similarly computed. These moments provide considerable information on the stochastic properties of the concentration process.

As an example of the application of the above solution, numerical values of the mean concentration, a sample function, and the standard deviation of the concentration with time were computed. An average pore velocity  $u = 0.2$  m/day, a dispersion coefficient  $D = 0.1$  m<sup>2</sup>/day, a concentration at the origin  $C_0 = 10.0$  mg/l, and a variance parameter  $q = 0.01$  were assumed. The value of  $q$  is entirely arbitrary here. It is clear that the actual value should be determined from field measurements and an estimation algorithm. The integrals were numerically evaluated. Figure 1 is a digital plotter output

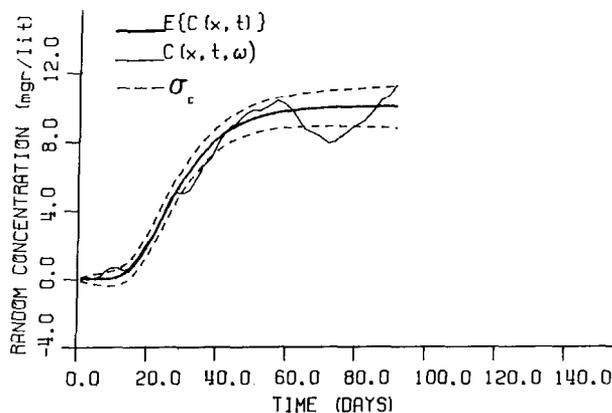


FIG. 1. Stochastic concentration with time at  $x = 6.0$  m (case 1).

of the program for a point in space  $x = 6.0$  m from the origin. The solid line represents the evolution with time of the mean concentration, in this case an assumed deterministic concentration due to a constant plane source at the origin. The continuous sinuous line represents the sample concentration, and the dashed lines represent the mean concentration plus and minus one standard deviation respectively. This information is useful in the forecasting of the stochastic properties of the concentration properties with time. Applications of this methodology to groundwater-transport problems in which sources are distributed randomly both in time and in space can be found in [31].

### 3.2. Random Unsteady Groundwater Flow in a Sloping Phreatic Aquifer

The above methodology may be applied to the case when the operator  $A$  in Equation (2.3) has a discrete spectrum. This is the case in many applications of the groundwater flow equations in finite domains where the boundary conditions are known and significantly control the potential distribution inside the aquifer. Consider for example the one-dimensional groundwater flow equation in a mildly sloped phreatic aquifer with a homogeneous media [33]:

$$\frac{\partial h}{\partial t} - \frac{T}{S} \frac{\partial^2 h}{\partial x^2} - \frac{\alpha K}{S} \frac{\partial h}{\partial x} = \frac{d\beta}{dt}, \quad (3.9)$$

$$h(0, t) = 0, \quad h(L, t) = 0, \quad h(x, 0) = h_0(x),$$

where  $h(x, t)$  is the elevation of the water table with respect to the sloping bedrock (m),  $t$  is the time coordinate (days),  $S$  is the aquifer specific yield,  $K$  is the aquifer hydraulic conductivity ( $\text{m day}^{-1}$ ),  $I$  is the input function representing deep percolation to the aquifer and assumed uniformly distributed along the horizontal distance  $x$  ( $\text{m day}^{-1}$ ), the white-noise process  $d\beta/dt$  represents the model uncertainty in the water-table fluctuation, and  $\alpha$  is the average bedrock slope with respect to the horizontal axis ( $\text{m} \cdot \text{m}^{-1}$ ). Note that Dupuit assumptions have been used in the derivation of the differential equation for a small slope  $\alpha$ . Also note that the boundary conditions are homogeneous and that a corresponding steady-state problem satisfies the actual field boundary conditions.

In this case the partial differential operator  $A$  is given by

$$Ah = \left( -\frac{T}{S} \frac{\partial^2}{\partial x^2} - \frac{\alpha K}{S} \frac{\partial}{\partial x} \right) h.$$

Although the operators in Equations (3.1) and (3.9) are similar, they generate different semigroups because of the boundary conditions imposed on the differential equations are different: In (3.1) the semiinfinite domain would produce a continuous spectrum in the semigroup operator, while in (3.9) the finite domain with known boundary conditions would produce a discrete spectrum in the semigroup operator. Therefore, this operator generates a strongly continuous semigroup given by [33]

$$J_t z = e^{-\alpha Kx/2T} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \sin \frac{n\pi x}{L} \cdot \frac{2}{L} \int_0^L z e^{\alpha Kx/2T} \sin \frac{n\pi x}{L} dx, \quad (3.10)$$

where the eigenvalues  $\lambda_n$  are given by

$$\lambda_n^2 = \frac{n^2 \pi^2 T}{L^2 S} + \frac{\alpha^2 K^2}{4TS}. \quad (3.11)$$

Thus, following Equation (2.4), the solution to Equation (3.9) is

$$\begin{aligned} h(x, t) = & + e^{-\alpha Kx/2T} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \sin \frac{n\pi x}{L} \cdot \frac{2}{L} \int_0^L h_0(x) e^{\alpha Kx/2T} \sin \frac{n\pi x}{L} dx \\ & + e^{-\alpha Kx/2T} \int_0^t \sum_{n=1}^{\infty} e^{-\lambda_n^2(t-s)} \sin \frac{n\pi x}{L} d\beta(s) \\ & \cdot \frac{2}{L} \int_0^L e^{\alpha Kx/2T} \sin \frac{n\pi x}{L} dx, \end{aligned} \quad (3.12)$$

where  $d\beta(s)$  is a Brownian-motion increment with the properties

$$E\{d\beta(t)\} = 0, \quad E\{d\beta(t)^2\} = q dt. \quad (3.13)$$

A naive attempt at modeling a distributed Brownian-motion increment is suggested by the fact that if the process is a Hilbert-space-valued Brownian-motion process, then it should be possible to expand it as a sequence

$$d\beta(t) = \sum_{n=1}^{\infty} db_n e_n, \quad (3.14)$$

where  $e_n$  is an orthonormal basis function and  $db_n$  is a unidimensional scalar

Brownian motion process with an incremental variance parameter given by [7]

$$E\{\|d\beta\|^2\} = q dt \sum_{n=1}^{\infty} \frac{1}{\lambda_n}, \quad (3.15)$$

where  $\|\cdot\|$  is the space norm and  $\lambda_n$  the eigenvalues. A classical interpretation of the series of (3.14) would imply that

$$db_n = \frac{2}{L} \int_0^L d\beta(t) e^{\alpha Kx/2T} \sin \frac{n\pi x}{L} dx. \quad (3.16)$$

Using (3.14) and (3.16), Equation (3.12) may be simplified as

$$\begin{aligned} h(x, t) = & e^{-\alpha Kx/2T} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \sin \frac{n\pi x}{L} \cdot \frac{2}{L} \int_0^L h_0(x) e^{\alpha Kx/2T} \sin \frac{n\pi x}{L} dx \\ & + e^{-\alpha Kx/2T} \int_0^t \sum_{n=1}^{\infty} e^{-\lambda_n^2(t-s)} db_n(s) \sin \frac{n\pi x}{L}. \end{aligned} \quad (3.17)$$

This equation may be used to generate sample functions of the process  $h$ . Now, by taking expectations on both sides of (3.12) and using (3.13), we obtain an expression for the mean of the potential:

$$E\{h(x, t)\} = e^{-\alpha Kx/2T} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \sin \frac{n\pi x}{L} \cdot \frac{2}{L} \int_0^L h_0(x) e^{\alpha Kx/2T} \sin \frac{n\pi x}{L} dx. \quad (3.18)$$

The second moment of  $h$  is given by Equation (2.6). We are particularly interested in the variance of  $h$  at any time  $t$ . This can be calculated in a similar manner to the calculation of the concentration variance (Section 3.1):

$$\begin{aligned} \sigma_h^2 = & \frac{4q}{L^2} e^{-\alpha Kx/T} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(m\pi x/L) \sin(n\pi x/L)}{\lambda_m^2 + \lambda_n^2} \\ & \times \frac{1 - e^{\alpha KL/2T} \cos m\pi}{1 + (\alpha^2 K^2 L^2 / 4T^2 m^2 \pi^2)} \frac{1 - e^{\alpha KL/2T} \cos n\pi}{1 + (\alpha^2 K^2 L^2 / 4T^2 n^2 \pi^2)} (1 - e^{-t(\lambda_m^2 + \lambda_n^2)}). \end{aligned} \quad (3.19)$$

Applications of this procedure to the Twin Lake Aquifer at the Chalk River Nuclear Laboratories, Ontario, Canada are found in [33].

### 3.3. Modeling Groundwater Flow Subject to Random Transmissivity

As an example of the application of the semigroup method to the solution of partial differential equations with stochastic parameters, consider the one-dimensional groundwater flow equation with Dupuit assumptions in a horizontal aquifer with a fully penetrating stream as a left boundary, and a topographic divide, or no flow, as a right boundary. This situation arises in a regional alluvial aquifer deposited over a consolidated impervious formation. The governing equation is given by [32]

$$\frac{\partial h}{\partial t} - \frac{\overline{T(t)} + T'(t, \omega)}{S} \frac{\partial^2 h(x, t, \omega)}{\partial x^2} = 0, \quad (3.20)$$

$$h(0, t) = 0, \quad \frac{\partial h}{\partial x}(L, t) = 0, \quad h(x, 0) = h_0(x),$$

where  $h(x, t, \omega)$  is the hydraulic head (m) or groundwater depth over the bedrock formation;  $x$  is the horizontal distance (m) from the left boundary;  $t$  is the time coordinate in months (mo);  $S$  is the aquifer storage coefficient or specific yield;  $L$  is the total length of the aquifer (m); and  $T$  is the aquifer transmissivity ( $\text{m}^2/\text{mo}$ ), which is represented as a stochastic process of the form  $T(t, \omega) = \overline{T(t)} + T'(t, \omega)$ , where  $\overline{T(t)}$  is the deterministic seasonal component and  $T'(t, \omega)$  is Gaussian white noise in time with

$$E\{T'(t)\} = 0, \quad E\{\overline{T'(t_1)}T'(t_2)\} = q\delta(t_1 - t_2), \quad t_1 \leq t_2. \quad (3.21)$$

This last assumption is not necessary, but it is useful for illustration. The correlation function of  $T'$  should be deduced from field measurements. Obviously (3.20) is part of a more specific model in which the exact left and right boundary conditions are included. For illustrative purposes we consider here the problem of homogeneous boundary conditions only.

Let us write Equation (3.20) in the form given by Equation (2.10):

$$\frac{\partial h}{\partial t} - \frac{\overline{T(t)}}{S} \frac{\partial^2 h}{\partial x^2} = \frac{T'}{S} \frac{\partial^2 h}{\partial x^2}, \quad (3.22)$$

$$h(0, t) = 0, \quad \frac{\partial h}{\partial x}(L, t) = 0, \quad h(x, 0) = h_0,$$

where the deterministic components are on the left-hand side and the random operator acts as a forcing term. We now approximate the monthly mean transmissivity  $\overline{T(t)}$  as a monthly step function so that the partial differential operator is time-independent at each time step (each month) and we can invoke a semigroup solution (2.11), for (3.22):

$$h = J_t h_0 + \int_0^t J_{t-s} R h(s) ds, \quad t_{i-1} < t < t_{i+1}, \quad (3.23)$$

where  $R$  is the random partial differential operator given by

$$R h = \left( \frac{T'(t, \omega)}{S} \frac{\partial^2}{\partial x^2} \right) h,$$

and  $J_t$  is the semigroup given by the deterministic operator

$$J_t h = \sum_{n=1}^{\infty} b_n(h) \Phi_n(x) M_n(t), \quad (3.24)$$

where

$$b_n(h) = \frac{2}{L} \int_0^L h \Phi_n(x) dx, \quad (3.25)$$

$$\Phi_n(x) = \sin \lambda_n x, \quad (3.26)$$

$$\lambda_n = \frac{2n-1}{2L} \pi, \quad n = 1, 2, 3, \dots, \quad (3.27)$$

$$M_n(t) = \exp\left(-\frac{\lambda_n^2 \overline{T(t)} t}{S}\right). \quad (3.28)$$

Now expand  $h$  in the right-hand side of Equation (3.23) as a series  $\sum h_i$  of the form given by Equation (2.12):

$$h = J_t h_0 + \int_0^t J_{t-s} R [h_1(s) + h_2(s) + h_3(s) + \dots] ds. \quad (3.29)$$

Identifying  $h_1$  as the preceding term or the deterministic solution  $J_t h_0$ , we

can determine each  $h_i$  in terms of the preceding term  $h_{i-1}$ . Equation (3.29) will become an infinite series in which the last term contains  $h$  [see (2.13)]:

$$h = J_t h_0 + \int_0^t J_{t-s} R(J_s h_0) ds + \int_0^t \int_0^s J_{t-s} R J_{s-\xi} R J_\xi h_0 d\xi ds + \cdots. \quad (3.30)$$

Sensitivity tests [32] have demonstrated that Equation (3.30) converges rapidly and that on considering only a few terms in the series the error is very small. Because the calculation of each subsequent term in the series is significantly more complex than the previous one, and significantly smaller in size, we shall limit our calculations to the first two terms on the right-hand side of (3.30).

Sample functions can be computed from Equation (3.30) by generating sample functions of  $T'$  (i.e., approximating it as a random walk) and approximating the time integral:

$$\begin{aligned} h(x, t, \omega) = & J_t h_0 - \frac{1}{S} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_m^2 \sin \lambda_n x \\ & \times \left[ \sum_{k=0}^t \exp\left(-\frac{\lambda_n^2 \overline{T(t)} (t-k)}{S}\right) \right. \\ & \left. \times \exp\left(-\frac{\lambda_m^2 \overline{T(t)} k}{S}\right) T'(\omega, k) \right] c_{nm} b_m(u_0), \\ & t_{i-1} < t < t_{i+1}, \quad (3.31) \end{aligned}$$

where  $t$  in this case will be equal to 1 because of the stepwise solution,

$$c_{nm} = \frac{2}{L} \left[ \frac{\sin(\lambda_n - \lambda_m)L}{2(\lambda_n - \lambda_m)} - \frac{\sin(\lambda_n + \lambda_m)L}{2(\lambda_n + \lambda_m)} - \frac{1}{2(\lambda_n - \lambda_m)} + \frac{1}{2(\lambda_n + \lambda_m)} \right], \quad n \neq m, \quad (3.32)$$

$$c_{nm} = 1 - \frac{\sin 2\lambda_n L}{2L\lambda_n}, \quad n = m. \quad (3.33)$$

The expected value of  $h$  from Equation (3.30) is given by Equation (2.14), and in this case it coincides with the deterministic solution:

$$E\{h(x, t)\} = J_t h_0. \quad (3.34)$$

Finally, the variance of  $h$  is given by

$$\sigma_h^2 = \frac{q}{S^2} \int_0^t J_{2(t-s)} [J_{2s} \lambda^4 (h_0^2)] ds. \tag{3.35}$$

Expanding,

$$\begin{aligned} \sigma_h^2 = & \frac{q}{\overline{T(t)} S} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{\lambda_m^2 \lambda_q^2 \sin \lambda_n x \sin \lambda_p x}{(\lambda_n^2 - \lambda_m^2 + \lambda_p^2 - \lambda_q^2)} c_{nm} c_{pq} b_m(h_0) b_q(h_0) \\ & \times \exp\left(-\frac{\overline{T(t)} t (\lambda_n^2 + \lambda_p^2)}{S}\right) \left[ \exp\left(\frac{\overline{T(t)} t (\lambda_n^2 - \lambda_m^2 + \lambda_p^2 - \lambda_q^2)}{S}\right) - 1 \right]. \end{aligned} \tag{3.36}$$

Numerical calculations of Equation (3.36) demonstrated that the series converges slowly because of the four summations [32]. It is interesting to note that the statistical measures, as computed by the present method, are almost identical in magnitude to the exact ones. The sample functions were very close when the exact and the approximate method were fed by the same noise realization. This is an important result, since the approximation method will allow one to accurately solve much more complex equations by calculating only the first random term in the series.

One inconvenience of the approximation method is the amount of computational time required. The approximation method spent about three times more as that required by the exact semigroup method to solve the same problem on a microcomputer.

#### 4. APPROXIMATE SOLUTIONS OF RANDOM PARTIAL DIFFERENTIAL EQUATIONS

Sections 2 and 3 were concerned with the solutions of partial differential equations in a one-dimensional domain. These solutions used the concept of semigroups associated with partial differential operators in Sobolev spaces. While the semigroup is relatively simple to derive for second-order equations in one-dimensional domains, the derivation for a two-dimensional domain becomes significantly more difficult, and yet the geometrical domains are limited to square or rectangular shapes because of the integral terms. Thus a numerical solution of a stochastic partial differential equation is a desirable

alternative. From the abstract theoretical point of view, some important contributions have been made [3, 40]. It has been noted that the discretization must be done not only in the spatial coordinates, but also in the probabilistic variables. The practical implementation of these schemes may still be a formidable task.

From the applied point of view, most of the work has been directed to the development of numerical techniques for random ordinary differential equations ([22, 26, 27, 25], and [43] as an application in hydrology). One of the first applications of numerical methods to the solution of a one-dimensional stochastic partial differential equations in hydrology was presented in [34]. In this work the Galerkin approximation was approached in the usual sense, except that the coefficients in the linear combination were random functions in conjunction with a Taylor series expansion to obtain the first two moments. Another approach [41] presented a method for the numerical solution of the one-dimensional advective-dispersive equation with random coefficients. This work used the method of decomposition of a stochastic partial differential operator [1] and functional-analytic methods to prove that the decomposition sequence converges towards the exact solution.

In the present section, we shall describe a different approach [37], which uses Itô's lemma as a suitable link between the abstract theoretical analysis of stochastic partial differential equations and the extensive literature on numerical methods for deterministic partial differential equations.

#### 4.1. The Itô's Lemma in Hilbert Spaces

Consider a separable Hilbert space  $H$ . Let  $a(t)$  be a bounded stochastic process with values in  $H$ . We define a continuous stochastic process in  $H$  by setting

$$z(t) = z(0) + \int_0^t a(s) ds + \int_0^t \Gamma(s) d\beta(s) ds,$$

where  $z(0)$  is a random variable in  $H$ ,  $d\beta(s)$  is Brownian-motion increment, and  $\Gamma(s)$  is a second-order stochastic process in  $H$ . Now, let  $\Phi(z, t)$  be a functional on  $H \times [0, T]$  which is twice continuously differentiable in  $H$  and once continuously differentiable in  $t$ . Additionally,  $\partial\Phi/\partial z$  and  $\partial^2\Phi/\partial z^2$  are assumed to be bounded on bounded sets of  $H$ . Hence, the Itô's lemma in Hilbert spaces may be written as (see [5], and [30] for proof)

$$\begin{aligned} \Phi(z(t), t) &= \Phi(z(0), 0) + \int_0^t \left( \frac{\partial\Phi}{\partial z}, a \right) ds + \int_0^t \left( \frac{\partial\Phi}{\partial z}, \Gamma d\beta(s) \right) \\ &\quad + \frac{1}{2} \int_0^t \text{tr} \left( \Gamma^* \frac{\partial^2\Phi}{\partial z^2} \Gamma q \right) ds + \int_0^t \frac{\partial\Phi}{\partial t} ds, \end{aligned} \quad (4.1)$$

where  $(\cdot, \cdot)$  implies the inner product determined by the norm of the Hilbert space in consideration. This equation can be derived from the usual Itô's lemma developed, for example, in [21].

Interpreting Equation (2.1) in the Itô sense, applying the Itô's lemma with  $z(t) = u(t)$ , and differentiating, we obtain

$$\frac{d\Phi(u)}{dt} = - \left( \frac{\partial\Phi}{\partial u}, Au \right) + \left( \frac{\partial\Phi}{\partial u}, \Gamma \frac{d\beta}{dt} \right) + \left( \frac{\partial\Phi}{\partial u}, g \right) + \frac{1}{2} \text{tr} \left( \Gamma^* \frac{\partial^2\Phi}{\partial u^2} \Gamma q \right) + \frac{\partial\Phi}{\partial t}, \quad (4.2)$$

where  $q$  is the white-noise variance parameter as in Equation (3.2). On taking expectation this yields

$$\begin{aligned} \frac{dE\{\Phi(u)\}}{dt} + E \left\{ \left( \frac{\partial\Phi}{\partial u}, Au \right) \right\} \\ = E \left\{ \left( \frac{\partial\Phi}{\partial u}, g \right) \right\} + \frac{1}{2} \text{tr} E \left\{ \Gamma^* \frac{\partial^2\Phi}{\partial u^2} \Gamma q \right\} + E \left\{ \frac{\partial\Phi}{\partial t} \right\}. \end{aligned} \quad (4.3)$$

Taking  $\Phi(u) = (h, u)$ ,  $h \in V^*$ , where  $h$  forms a basis in  $V^*$ , and  $V \subset H_0^1(G)$ , Equation (4.3) yields an equation for the mean of the solution  $M_1 = E\{u(t)\}$ :

$$\frac{dM_1}{dt} + AM_1 = g, \quad (4.4)$$

where  $M_1$  is a mean-value operator in the Hilbert space  $H^1$ . Next set  $\Phi(u) = (h_1, u)(h_2, u)$ , so that  $E\{\Phi(u)\} = (M_2 h_1, h_2)$  for  $h_1, h_2 \in V^*$ . Then (4.3) gives an equation for the correlation operator  $M_2$  or the second moment of  $u(t)$ :

$$\frac{dM_2}{dt} + (A \oplus A) M_2 = (g \otimes g) M_1 + (\Gamma^* \otimes \Gamma q) \quad (4.5)$$

where  $\oplus$  and  $\otimes$  denote the direct sum and tensor product of two operators on appropriate tensor product spaces. Basically  $A \oplus A$  implies the summation of the operator  $A$  on two orthogonal directions to form the complete space.

Similarly, higher-order moments may be obtained by defining  $M_n(h) = E\{\prod_{j=1}^n (h_j, u)\}$  and using Equation (4.3). These moment equations hold in the weak sense. We should remark that these are operator equations. A very detailed derivation of the moment equations can be found in the recent work

by Unny [42]. The interesting feature is that (4.4) and (4.5) are deterministic and may be solved by any analytical or approximate method available in the literature.

#### 4.2. Modeling 2-D Groundwater Flow with a Stochastic Phreatic Surface

As an application of the Itô's lemma to the numerical solution of moment equations in higher-dimensional domains, let us consider the two-dimensional groundwater flow equation subject to a linearized stochastic free-surface boundary condition [34]:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad (4.6)$$

$$u(0, z, t) = 0, \quad u(L, z, t) = 0, \quad \frac{\partial u}{\partial n}(x, \xi, t) = 0,$$

$$u(x, z, t) = \eta(x, z, t), \quad \frac{\partial u}{\partial t} = - \left( \frac{K}{n_e} \right) \frac{\partial u}{\partial z} + \frac{d\beta}{dt} \quad \text{on } z = \eta,$$

$$u(x, z, 0) = u_0(x, z),$$

where  $u$  is the hydraulic potential (m),  $n_e$  is the aquifer effective porosity,  $\eta$  is the potential at the free surface (m),  $z$  is the vertical coordinate (m),  $u_0$  is the initial potential in the aquifer (m),  $K$  is the aquifer hydraulic conductivity ( $\text{m}^2/\text{day}$ ),  $du/dn$  is the normal derivative of the potential at the bedrock level  $z = \xi$ ,  $L$  is the maximum horizontal dimension of the aquifer (m),  $x$  is the horizontal coordinate (m),  $t$  is the time coordinate (days), and  $d\beta/dt = w$  is a white-noise process in time with the properties specified by Equation (3.2).

Thus the mathematical problem is the solution of the Laplace equation subject to an unsteady, randomly perturbed, free-surface boundary condition. This random perturbation includes the several environmental fluctuations, such as barometric pressure, erratic precipitation, and recharge, which ultimately affect the position of the phreatic surface and thus the redistribution of the potential inside the aquifer.

Applying the Itô's lemma (4.3) to Equation (4.6), we obtain the first moment equation for  $u(x, z, t)$  given by  $M_1 = E\{u\}$  as

$$\frac{\partial^2 M_1}{\partial x^2} + \frac{\partial^2 M_1}{\partial z^2} = 0, \quad (4.7)$$

$$M_1(0, z, t) = 0, \quad M_1(L, z, t) = 0, \quad \frac{\partial M_1}{\partial n}(x, \xi, t) = 0,$$

$$M_1 = E\{\eta(x, z, t)\}, \quad \frac{\partial M_1}{\partial t} = -\left(\frac{K}{n_e}\right) \frac{\partial M_1}{\partial z} \quad \text{on } z = E\{\eta\},$$

$$M_1(x, z, 0) = u_0(x, z).$$

Finally, applying Equation (4.3) to Equation (4.6) [and using Equation (4.5)], the second moment equation for  $u$  is given by  $M_2 = E\{u^2\}$  as

$$\frac{\partial^2 M_2}{\partial x^2} + \frac{\partial^2 M_2}{\partial z^2} = 0, \quad (4.8)$$

$$M_2(0, z, t) = 0, \quad M_2(L, z, t) = 0, \quad \frac{\partial M_2}{\partial n}(x, \xi, t) = 0,$$

$$M_2 = E\{\eta^2(x, z, t)\}, \quad \frac{\partial M_2}{\partial t} = -2\left(\frac{K}{n_e}\right) \frac{\partial M_2}{\partial z} + q \quad \text{on } z = E\{\eta\},$$

$$M_2(x, z, 0) = u_0^2(x, z),$$

where  $q$  is the variance parameter of the white-noise process. Thus the moment equations may be solved by any numerical procedure. This makes the procedure very useful, since there is an extensive body of literature and software available on the numerical solution of deterministic partial differential equations.

An application of the above procedure to a field situation [34] involved the development of the moment equation for the Twin Lake aquifer at the Chalk River Nuclear Laboratories, Ontario, Canada, and the subsequent solution by the use of the boundary integral equation method, which was the most efficient for the homogeneous aquifer being treated.

## 5. CONCLUSIONS

A large number of partial differential equations in hydrology may be treated as abstract evolution equations in appropriate Sobolev spaces. The solution of these equations may be obtained if the semigroup associated with the particular partial differential operator is known. Once this fundamental solution is derived, differential equations subject to more complex random source terms, random boundary conditions, random initial conditions, and/or random coefficients may be solved. Potential applications of this approach include the solution of groundwater flow and groundwater transport equations subject to stochastic transmissivity, stochastic dispersion coefficient, stochastic recharge, stochastic pollution loads, stochastic reactive functions, and randomly fluctuating boundary conditions.

The Ito's lemma in Hilbert spaces is an efficient link between abstract stochastic partial differential equations and the computer-oriented numerical solutions of the corresponding moment equations. This technique allows the approximate solution of stochastic partial differential equations in two-dimensional and three-dimensional domains with irregular geometry.

## REFERENCES

- 1 G. Adomian, *Stochastic Systems*, Academic, New York, 1983.
- 2 L. Arnold, P. Kotelenetz, and R. F. Curtain, Linear stochastic evolution equation models for chemical reactions, in *Stochastic Nonlinear Systems in Physics, Chemistry and Biology* (L. Arnold and R. Lefever, Eds.), Springer-Verlag, 1981.
- 3 G. A. Becus, Variational formulation of some problems for the random heat equation, in *Applied Stochastic Processes* (G. Adomian, Ed.), Academic, New York, 1980.
- 4 A. Belleni-Morante, *Applied Semigroups and Evolution Equations*, Clarendon, Oxford, 1979.
- 5 A. Bensoussan, Control of stochastic partial differential equations, in *Distributed Parameter Systems* (W. H. Ray and D. G. Lainiotis, Eds.), Marcel Dekker, 1977.
- 6 P. L. Butzer and H. Berens, *Semigroups of Operators and Approximations*, Springer, 1967.
- 7 P. L. Chow, Approximate solution of random evolution equations, in *Approximate Solutions of Random Equations* (A. T. Bharucha-Reid, Ed.), North Holland, 1979.
- 8 R. F. Curtain and A. J. Pritchard, Infinite dimensional linear systems theory, in *Lecture Notes in Control and Inform. Sci.* 8 (A. V. Balakrishnan and M. Thoma, Eds.), Springer-Verlag, 1978.
- 9 R. F. Curtain and P. L. Falb, Stochastic differential equations in Hilbert space, *J. Differential Equations* 10:412-430 (1971).
- 10 G. Dagan, Solute transport in heterogeneous porous formations, *J. Fluid Mech.* 145:151-177 (1984).
- 11 G. Dagan, Stochastic modeling of groundwater flow by unconditional and

- conditional probabilities. 2. Solute transport, *Water Resources Res.* 18(4):835–848 (1982).
- 12 G. Dagan and E. Bresler, Solute dispersion in unsaturated heterogeneous soil at field scale. 1. Theory, *Soil Sci. Soc. Amer. J.* 43:461–467 (1979).
  - 13 L. W. Gelhar and C. L. Axness, Three-dimensional stochastic analysis of macrodispersion in aquifers, *Water Resources Res.* 19(1):161–180 (1983).
  - 14 L. W. Gelhar, A. L. Gutjahr, and R. J. Naff, Stochastic analysis of macrodispersion in a stratified aquifer, *Water Resources Res.* 15:1387–1397 (1979).
  - 15 V. P. Godambe and M. E. Thompson, Robust estimation through estimating equations, *Biometrika* 71(1):115–125 (1984).
  - 16 J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford U.P., New York, 1985.
  - 17 D. H. Griffel, *Applied Functional Analysis*, Ellis Horwood, 1981.
  - 18 B. Hunt, *Mathematical Analysis of Groundwater Resources*, Butterworths, London, 1983.
  - 19 V. Hutson and J. S. Pym, *Functional Analysis in Applied Mathematics*, Academic, 1980.
  - 20 A. Ichikawa, A semigroup model for parabolic equations with boundary and pointwise noise, in *Stochastic Space-Time Models and Limit Theorems* (L. Arnold and P. Kotelenetz, Eds.), Reidel, Dordrecht, 1985.
  - 21 A. H. Jazwinski, *Stochastic Processes and Filtering Theory*, Academic, New York, 1970.
  - 22 W. E. Kohler and W. E. Boyce, A numerical analysis of some first order initial value problems, *SIAM J. Appl. Math.* 27:167–179 (1974).
  - 23 P. Kotelenetz, On the semigroup approach to stochastic evolution equations, in *Stochastic Space-Time Models and Limit Theorems* (L. Arnold and P. Kotelenetz, Eds.), Reidel, Dordrecht, 1985.
  - 24 G. Ladas and V. Lakshmikantham, *Differential Equations in Abstract Spaces*, Academic, 1972.
  - 25 M. D. Lax, Approximate solutions of random differential and integral equations, in *Applied Stochastic Processes* (G. Adomian, Ed.), Academic, New York, 1980.
  - 26 G. N. Mil'shtien, G. N. Approximate integration of stochastic differential equations, *Theory Probab. Appl.* 19:557–562 (1974).
  - 27 N. J. Rao, J. D. Borwanker, and D. Ramkrishna, Numerical solutions of Ito integral equations, *SIAM J. Control* 12:124–139 (1974).
  - 28 J. T. Oden, *Applied Functional Analysis*, Prentice-Hall, 1977.
  - 29 B. Sagar, Galerkin finite element procedure for analyzing flow through random media, *Water Resources Res.* 14(6):1035–1044 (1978).
  - 30 Y. Sawaragi, T. Soeda, and S. Omatu, Modeling, estimation, and their applications for distributed parameter systems, in *Lecture Notes in Control and Inform. Sci.* 2 (A. V. Balakrishnan and M. Thoma, Eds.), Springer-Verlag, 1978.
  - 31 S. E. Serrano, General solution of the random advective-dispersive transport equation in porous media. Part I: Stochasticity in the sources and the boundaries, *J. Stoch. Hydrol. Hydraul.* 2, No. 2 (1988).
  - 32 S. E. Serrano and T. E. Unny, Semigroup solutions to stochastic unsteady groundwater flow subject to random parameters, *J. Stoch. Hydrol. Hydraul.* 1(4):281–296 (1987).

- 33 S. E. Serrano and T. E. Unny, Predicting groundwater flow in a phreatic aquifer, *J. Hydrol.* 95:241–268 (1987).
- 34 S. E. Serrano and T. E. Unny, Boundary element solution of the two-dimensional groundwater flow equation with stochastic free-surface boundary condition, *Numer. Methods Partial Differential Equations* 2:237–258 (1986).
- 35 S. E. Serrano, T. E. Unny, and W. C. Lennox, Analysis of stochastic groundwater flow problems. Part I: deterministic partial differential equations in groundwater flow. A functional-analytic approach, *J. Hydrol.* 82(3–4):247–263 (1985).
- 36 S. E. Serrano, T. E. Unny, and W. C. Lennox, Analysis of stochastic groundwater flow problems. Part II: Stochastic partial differential equations in groundwater flow. A functional-analytic approach, *J. Hydrol.* 82(3–4):265–284 (1985).
- 37 S. E. Serrano, T. E. Unny, and W. C. Lennox, Analysis of stochastic groundwater flow problems. Part III: Approximate solution of stochastic partial differential equations, *J. Hydrol.* 82(3–4):285–306 (1985).
- 38 R. E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Pitman, 1977.
- 39 E. A. Sudiky and J. A. Cherry, Field observations of tracer dispersion under natural flow conditions in an unconfined sandy aquifer, *Water Pollut. Res. Canada* 14:1–17 (1979).
- 40 T. C. Sun, A finite element method for random differential equations, in *Approximate Solution of Random Equations* (A. T. Bharucha-Reid, Ed.), Elsevier, North Holland, 1979.
- 41 D. H. Tang and G. F. Pinder, Analysis of mass transport with uncertain physical parameters, *Water Resources Res.* 15(5):1147–1155 (1979).
- 42 T. E. Unny, Stochastic partial differential equations in groundwater hydrology, presented at Symposium on Stochastic Structural Dynamics, Univ. of Illinois, Urbana-Champaign, 1988.
- 43 T. E. Unny, Numerical integration of stochastic differential equations in catchment modeling, *Water Resources Res.* 20(3):360–368 (1984).