Modeling infiltration in hysteretic soils

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Existing vertical infiltration models either assume a constant infiltration rate at the soil surface and a unique set of soil-water functional relationships (hysteresis is neglected), or use empirical expressions which disregard physical laws. In this article a method for analyzing vertical infiltration in hysteretic soils subject to the random variations of point rainfall in watersheds and a procedure for solving the resulting stochastic partial differential equation is presented. The effect of the time random variability in point rainfall produces a water content process in the upper soil layer which can be modeled as a shot noise process. The hysteretic loops resulting from the natural wetting and drying cycles generate a correlated random soil-water diffusivity process, *D*. The modeling problem reduces to the solution of the infiltration equation subject to a shot noise boundary condition and a colored noise soil-water diffusivity. A new semi-group solution of this evolution equation is obtained and expressions for sample functions, the mean and the variance of the water content in space and time are derived. A computational procedure for each of the components of the stochastic solution is presented, and suggestions for the reduction of the predicted variance are given. It is hoped that the present methodology will provide the modeler with a better tool to model infiltration in natural soils subject to the uncertainties associated with the rainfall regime and the hysteresis in the soil, and to encourage the use of physically based models for infiltration rather than empirical equations.

Key Words: Infiltration, hysteresis, stochastic partial differential equations.

INTRODUCTION

Modeling infiltration in soils has been approached in the past in two main ways. In the first approach, hydrologists have recognized the difficulties associated with the solution of physically-based unsaturated flow equations and have opted for a large variety of empirical expressions with parameters to calibrate in optimization procedures[41]. This approach, which has produced acceptable results for surface hydrologic computations, has not generated much understanding on the phenomenon of infiltration and distribution of water in unsaturated soils. In the second approach, soil physicists have attempted to produce solutions to physically-based equations describing horizontal or vertical infiltration in soils. Several quasi-analytical solutions of the non-linear unsaturated flow equation have been reported in the literature[23-26]. Recently, exact non-linear solutions for constant flux infiltration using Lie-Backlund transformations was reported[3,29]. This approach has given much understanding of the phenomenon of infiltration in soils and in general the solutions have been in good agreement with experimental data. Other solutions in this category use a numerical algorithm to implement in a computer[6,16,19,21]. These models have given valuable computational information to use in watershed simulation models.

Most analytical solutions restrict their validity to specific laboratory conditions in which the infiltration rate at the source boundary is constant in time, and the soil is continuously wetted. In this situation the natural effects of hysteresis in the soil-water functional relationships are minimized. Therefore the application of these solutions to field conditions in watersheds where the rainfall regime is a highly erratic process composed of periods of rainfall followed by periods of drought is limited. Rainfall periods will produce a partial wetting cycle in the soil profile and drought periods will produce a partial drying cycle in the soil profile. This situation will produce a set of soil-water functional relationships in which hysteresis is very important. The solution of the unsaturated groundwater flow equation subject to hysteretic functional relationships is very difficult to obtain.

Most numerical solutions can handle time variability in the source boundary for specific rainfall patterns known *a priori*. Nevertheless, hysteresis in the soil-water functional relationships is seldom incorporated and a unique, usually empirical, set of relationships is used. The use of empirical, single-valued, functional relationships obviates the difficulties associated with the unpredictability of hysteresis, although the predicted water content values do not represent the natural variability associated with the hysteretic soil.

Other approaches have attempted to use stochastic concepts in an effort to describe the uncertainty of the infiltration phenomenon in a statistical manner. One of these studies[5] defined the saturated hydraulic conductivity as a log-normally distributed random variable and, by assuming analytical, single-valued, expressions for

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the soil-water functional relationships a solution of a simplified 'piston flow' model of the infiltration equation was obtained. A constant flux infiltration during rainfall was assumed and spatial variability of infiltration was studied. Another interesting study[18] derived two ordinary differential equations describing the surplus and the deficit conditions in the unsaturated zone. These equations are forced by the random infiltration process, modeled as a Poisson process, and the evapotranspiration, modeled as a Brownian motion process. In an innovative approach, the study in mention solved the resulting random differential equations and derived expressions for the first two moments. The above study motivates two questions. The first question, one may ask, is what would be the random form of the soil-water diffusivity, D, which results from a Poissonian type of infiltration? This is a question which cannot be answered without considering hysteresis in the soil-water functional relationships. The next question is: is it possible to obtain a solution of the physically-based non-linear unsaturated flow equation when the top boundary is a Poisson process and D is a stochastic process? This is a question which involves the solution of a random partial differential equation, which would describe the time and space variability of the water content. The present article attempts to answer these questions.

This article presents a new methodology for the analysis and solution of the unsaturated groundwater flow equation subject to the uncertainty inherent to hysteresis. It is a theoretical analysis on the general form of the point precipitation process forcing infiltration in natural watersheds and the subsequent hysteretic loops in the soil-water functional relationships. The solution of the infiltration equation subject to the resultant random processes is the main objective of the study with the hope to develop a more realistic statistical representation of the infiltration phenomenon. In the first section, a simplified Poisson process is assumed to represent the time variability of point rainfall in a hypothetical watershed. The water content variation over time at the top layer due to the Poissonian rainfall pattern is found to follow a shot noise process, in agreement with existing literature on the topic. In the second section a simulation experiment to synthesize realizations of the D process in a hypothetical soil is performed. In this experiment, sample values of the shot noise process describing water content in the root zone are used to reproduce the hysteretic loops in the water content versus pressure-head relationship and in the hydraulic conductivity versus pressure head relationship, by emulating experimental scanning curves published in the literature. Using these simulated functional relationships, a sample function of D is obtained. While this approach only gave qualitative information on the form of D, it was observed that the hysteretic wetting and drying cycles produced a highly erratic D process, which would be better described as a random process of an exponential type of correlation. It was then assumed that D can be represented by a colored noise. In the third section a new semigroup solution of the infiltration equation subject to a shot noise boundary condition and a colored noise D was obtained. Finally, in section four a computational procedure of each of the components of the stochastic solution, and of the first two moments is presented, along with observations about the role of each of the components and the procedure for reduction of the model variance in practical field situations.

It is hoped that the results will encourage hydrologists and water resources scientists to use physically based equations subject to the natural environmental fluctuations encountered in the field, rather than empirical expressions. Since the theory and solutions of stochastic partial differential equations are now available, the modeler can now use a predictive tool which may give a better insight on the physical processes.

THE WATER CONTENT AT THE ROOT ZONE: DEFINITION OF THE TOP BOUNDARY CONDITION

We begin our analysis by considering a typical homogeneous soil profile in a natural watershed having a gentle slope and a deep mean water table elevation for now. The regional hydrology is such that in the area considered, the vertical infiltration is the main source of aquifer recharge. This would be the situation in an agricultural watershed whose regional groundwater flow occurs through an alluvial aquifer. The problem we consider is the statement and the solution of the boundary value problem modeling the vertical infiltration at a point in the recharge zone, and in particular the prediction of the evolution of the volumetric soil-water content.

The first difficulty the hydrologist faces is the fact that the upper boundary is the ground surface which is subject to the complex time variation of the rainfall occurrence in the area, alternated by dry periods which allow the redistribution of moisture in the soil. This erratic nature of the rainfall time distribution would pose serious difficulties in the exact solution of the differential equation governing infiltration, except in limited cases when assumptions of constant infiltration rate at the top boundary are made. The problem is further aggravated by recent hydrologic evidence of the existence of a macropore flow zone in the top layer of most natural soils[38] usually in the first 10 to 20 cm of the soil profile corresponding to the agricultural A horizon. This macropore zone is created by the penetration of roots from plants, animal burrows and natural soil weathering.

The exact hydraulic interaction between the rainfall and the soil-water content in the root zone is still unknown, although some mathematical models have been proposed in the literature. One of these models[40] approximates point rainfall depth as a Poisson process in time. This is indeed a simplification since it is well known that point rainfall follows a much more complex stochastic process (see Ref. 11). Behind this assumption is the recognition of the root zone as a filtering entity of the rainfall intensity signal. For example, the complex hourly rainfall intensity curves would generate a daily infiltrated rainfall time series which can be easily observed as a Poisson process such that[22]

$$p(N(t)=n)=\frac{(\lambda t)^n}{n!}\,e^{-\lambda t}, \tag{1}$$

where $p(N(t) = n)$ is the probability of n storms in the interval $(0, t]$; t is time (days); $n = 0, 1, 2, \ldots$; and λ is the mean storm arrival rate in $(0, t]$. This process has the properties

$$E\{N(t)\} = \lambda t, \; E\{N^2(t)\} = \lambda^2 t^2 + \lambda t, \tag{2}$$

where $E\{\}$ represents the expectation operator. The time $T = t_a$ in days between storms can be modeled as an exponential distribution of the form

$$p_T(t_a) = \lambda e^{-\lambda t}, \tag{3}$$

with the properties

$$E\{T\} = \frac{1}{\lambda}, \qquad E\{T^2\} = \frac{2}{\lambda^2}. \tag{4}$$

Tsakiris[40] assumed that the moisture depletion during rainless periods in the root zone is a function of the potential evapotranspiration and the field capacity and adopted the Thornthwaite and Mather equation for this purpose:

$$\theta_0(t) = \theta' \, e^{-(PET/FC)t}, \tag{5}$$

where $\theta_0(t)$ is the water content in the root zone (mm); θ' is the initial water content (mm); PET is the potential evapotranspiration (mm); and FC is the soil field capacity (mm).

Because of its simplicity, in the present study we will adopt the Tsakiris *et al.*[40] model to represent the water content in the root zone in order to obtain a description of the conditions at the top boundary, which we will call 'boundary layer', in our boundary value problem. This transition layer will resemble a similar concept in fluid dynamics except that in this case the representative scale (the depth) of the boundary layer is significantly greater than that used in fluid dynamics problems because of the presence of the porous media. Therefore, acceptable typical dimensions for the soil boundary layer will have to be of the order of the scale of existing measurement devices for soil moisture (see Cushman[4]). For the present study we assumed a boundary layer depth of 10 cm, a dimension appropriate for core sampling which also corresponds to the depth of the A horizon in many natural soils.

Recognizing that this is only an approximative abstraction of the complex conditions in the top boundary, this model will serve our objective to develop a methodology to solve the infiltration equation when the top boundary is a time stochastic process. It is clear that the hydrologist will have to identify the particular time stochastic process representing soil moisture in the root zone by measuring the water content at the root zone over time. Then s/he can use a methodology such as the one presented in this article to predict the statistical nature of the evolution of the water content at different depths.

Summarizing, we assume that the input infiltration to the boundary layer is a Poisson sequence of pulses of the form[22]

$$Z(t) = \sum_{i=0}^{N(t)} X_i \, \delta(t - t_i), \tag{6}$$

where $Z(t)$ is the moisture depth input (mm) in $(0, t]$; $N(t)$ is the number of pulses in $(0, t]$, which is Poisson distributed as in equation (1); X_i is the infiltration pulse magnitude at t_i, which is modeled in this case as an exponential distribution of the form of equation (2) with parameter γ; t_i are the random points in time with the intervals $(t_i - t_{i-1})$ modeled as an exponential distribution; and $\delta(\,)$ is the Dirac's delta function.

Following equation (5) we regard the unit impulse response, $h(t)$, of the boundary layer as

$$h(t) = e^{-\alpha t}, \tag{7}$$

where α is a parameter to determine. Then the output water content of the boundary layer system is found as

$$\theta_0(t) = \int_0^t Z(\tau) h(t - \tau) \, d\tau = \int_0^t \sum_{i=0}^{N(t)} X_i \, \delta(\tau - t_i) e^{-\alpha(t-\tau)} \, d\tau. \tag{8}$$

Thus the water content at the boundary layer will be

$$\theta_0(t) = \sum_{i=0}^{N(t)} X_i \, e^{-\alpha(t - t_i)}, \tag{9}$$

where the random variables $N(t)$, X_i and t_i are assumed to be independent. The mean of $Z(t)$ is

$$E\{Z(t)\} = \lambda E\{X\}, \tag{10}$$

and the mean of $\theta_0(t)$ is given by

$$E\{\theta_0(t)\} = \int_0^t E\left\{ \sum_{i=0}^{N(t)} X_i \, \delta(t - t_i) \right\} e^{-\alpha \tau} \, d\tau. \tag{11}$$

Solving,

$$E\{\theta_0(t)\} = \frac{\lambda}{\alpha\gamma} (1 - e^{-\alpha t}). \tag{12}$$

Now the correlation of $Z(t)$ is found as

$$E\{Z(t_1)Z(t_2)\} = [\lambda^2 + \lambda \, \delta(t_1 - t_2)] E\{X^2\}. \tag{13}$$

Thus the correlation function of $\theta_0(t)$ is given by

$$E\{\theta_0(t_1)\theta_0(t_2)\} = \int_0^{t_1} \int_0^{t_2} \times [\lambda^2 + \lambda\delta(\tau - \rho) E\{X^2\}] e^{-\alpha(t_1 - \tau)} e^{-\alpha(t_2 - \rho)} \, d\rho \, d\tau. \tag{14}$$

Solving, we obtain,

$$E\{\theta_0(t_1)\theta_0(t_2)\} = \frac{2\lambda}{\alpha\gamma^2} \left[\frac{\lambda}{\alpha} - \frac{\lambda}{\alpha} e^{-\alpha t_2} - \frac{\lambda}{\alpha} e^{-\alpha t_1} + \left(\frac{\lambda}{\alpha} - \frac{1}{2} \right) e^{-\alpha(t_1 + t_2)} + \frac{1}{2} e^{-\alpha(t_1 - t_2)} \right]. \tag{15}$$

The process $\theta_0(t)$ now defines the time distribution of the water content in the boundary layer. As a simulation experiment, two months of daily water content data in the root zone were generated using arbitrary values for the parameters which reflected typical conditions of a homogeneous sandy soil in the torrid zone. It was decided for simplicity to model the number of storms $N(t)$ as an exponential distribution with parameter $\zeta = 0.2$ day^{-1}; the interarrival times t_i as an exponential distribution with parameter $\lambda = 0.2$ day^{-1}; the infiltration depths X_i as an exponential distribution with parameter $\gamma = 0.3$ mm^{-1}; the removal rate of moisture from the root zone was assumed to be $\alpha = 0.1$ day^{-1}; the depth of the root zone $d = 100.0$ mm; a residual minimum water content was assumed as $r_m = 10\%$; and a maximum saturated water content (or porosity) of $\theta_{max} = 30\%$. These values do not intend to specify actual field conditions in a particular soil, but rather some

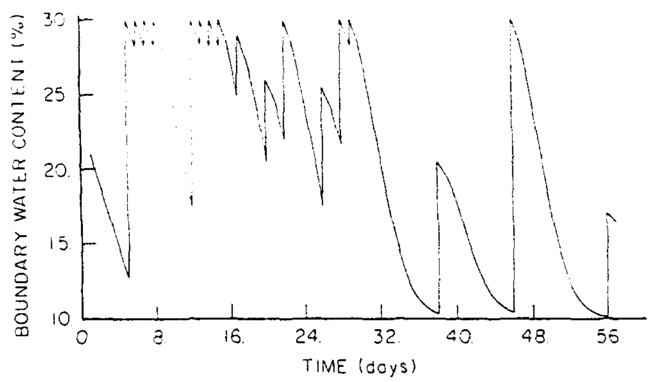

Fig. 1. Water content at the upper boundary versus time

hypothetical conditions in a soil in order to investigate the effect of time and space distribution of soil moisture due to general Poissonian conditions of the water content at the top boundary. These Poissonian conditions at the top boundary will reflect the effect of the complex time variability of point rainfall in real situations.

Using appropriate algorithms for the generation of exponential realizations[8] and equation (9), Fig. 1 was obtained. It describes the time variability of the water content with respect to time at the root zone. Note that wetting takes place almost instantaneously, whereas drying is a slow decay process. Such a wide variability in the water content should be expected in the root zone of natural soils.

THE RANDOM NATURE OF THE SOIL-WATER DIFFUSIVITY: A SIMULATION EXPERIMENT

In the vertical infiltration equation the soil-water diffusivity, $D(\theta)$, is defined as

$$D(\theta) = K(\theta)\frac{d\psi}{d\theta}, \tag{16}$$

where θ is the soil-water content; $K(\theta)$ is the soil-water hydraulic conductivity (m.day^{-1}); and ψ is the soil-water pressure head (m).

Several laboratory techniques have been proposed to determine the functional relationships of θ versus ψ and of K versus ψ in soil samples[12,13,15,17,27,39,42,43]. These functional relationships are used to numerically solve equation (16) to obtain the relationship of D versus θ, which are subsequently used to solve the vertical infiltration equation. Most of the existing solutions of the vertical infiltration equation assume a fixed set of soil functional relationships and a unique relationship of D versus θ. However, it is well known that the soil functional relationships are not unique and are subject to hysteretic effects (see Hillel[9] for discussion). Thus the shape of the soil functional relationships will be dependent on the way the soil was wetted or dryed in the experiment. For example, a continuous and gradual wetting of the soil will generate the 'main wetting curves' in the soil functional relationships, and a continuous and gradual drying of the soil will generate the 'main drying curves' in the soil functional relationships. When neither continuous wetting or continuous drying occurs, that is when the soil is subjected to cycles of partial wetting followed by partial drying, secondary or 'scanning curves' between the main wetting curve and the main drying curve are generated. The location of the scanning curves will depend on the value of the relationships at the instant in time when a cycle changes from wetting to drying or from drying to wetting, and from soil physical properties not yet well understood.

The unpredictability of the soil functional relationships subject to the cyclic conditions will produce an uncertain relationship between D and θ, which will in turn generate an uncertain solution of the vertical infiltration equation. The degree of uncertainty will depend on the degree of hysteresis in the soil in question. The hysteretic phenomenon, which has puzzled scientists for a long time, is the reason why solutions of the vertical infiltration equation have only been obtained under idealistic conditions of constant infiltration. These conditions are only realizable in laboratory settings which are rarely attained in natural soils of hydrologic watersheds, where rainfall periods are followed by dry periods. Hysteresis is also the reason why hydrologists mainly rely on empirical equations, rather than physically-based equations, for the calculation of infiltration and for the watershed simulation models requiring infiltration estimates.

In order to investigate the effect of hysteresis in the soil functional relationships on the form of the soil-water diffusivity a simulation experiment was performed. The simulation was based on the results of Liakopoulus[14], where detailed experimental information on the soil functional relationships of a fine sand was presented. We used the Liakopoulus data to represent the physical bounds of the main wetting curve, the main drying curve and the general direction of the scanning curves of a hypothetical sandy soil. The hypothetical soil was assumed to represent a typical soil profile in a watershed whose point rainfall regime follows the Poisson process described in the previous section.

We showed in the previous section that the water content in the root zone of a soil forced by a Poissonian rainfall follows a shot noise process. A sample function of the shot noise process was used to generate (Fig. 1) realizations of daily water content at the root zone. Using this information on the time variablity of the water content, realizations of the daily pressure head in the root zone were generated. The pressure head was computed by emulating the main wetting curve, the main drying curve and the secondary scanning curves according to the Liakopoulus data. Figure 2 shows a digital plotter output of 57 days of simulation. Note the main wetting and drying curves and the cyclic loops (scanning curves) as a result of partial wetting and drying of the rainfall inputs. We remark that the objective of the experiment was not to calculate exact absolute values of the pressure head, but rather to reproduce the phenomenon of hysteresis in the soil functional relationships and to observe the sort of stochastic process representing pressure head in a soil subject to the complex rainfall characteristics affecting real watersheds.

A similar procedure was used to generate 57 realizations of the hydraulic conductivity. Figure 3 shows the results of one of such simulations. Subsequently, the two sets of synthetic data (Figs 2 and 3) were used in conjunction with equation (16) to approximate realizations of the daily soil-water diffusivity. It is known that

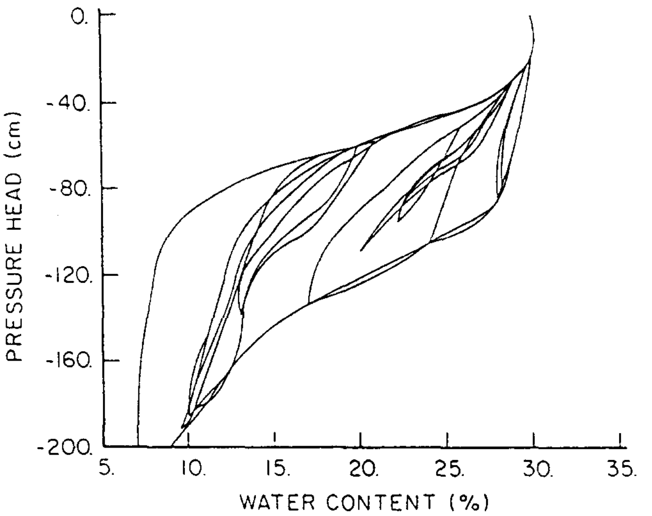

Fig. 2. Water content versus pressure head relationship

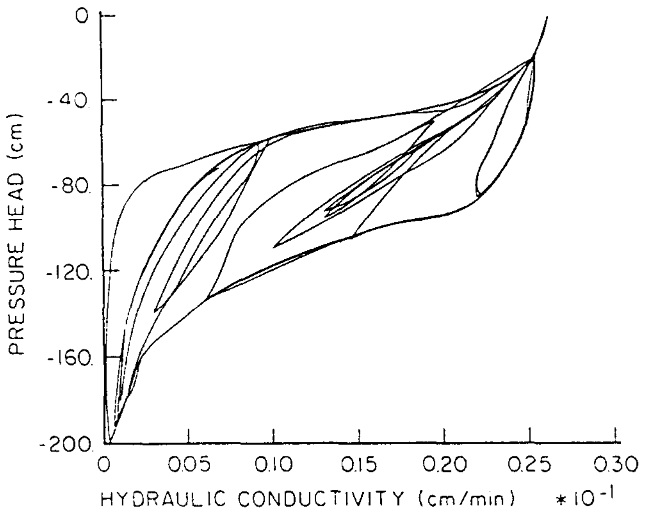

Fig. 3. Hydraulic conductivity versus pressure head relationship

the numerical approximation of the derivative in equation (16) is not very accurate, but again the objective is to observe the nature of the time stochastic process representing daily soil-water diffusivity.

Figure 4 shows the 57 realizations of such process. Note the high variability in D and that, at least in the root zone, D 'jumps' to high values in relatively short periods of time, whereas D recedes slowly in low values. The reason behind this resides in the fact that wetting and drying may be significantly attenuated as depth increases. There is also an indication that the extreme high values in D are probably unattainable in real field conditions. In the simulation, these extreme high values are obtained as the soil-water content approaches saturation. However, it is known that, except under prolonged ponding in the ground surface, and because of capillary fringe effects in fine soils with a shallow water table[7], a condition not considered here, saturation rarely occurs. Under conditions of intense rainfall, the value of the water content would be somewhat below saturation because of the presence of entrapped air. This would suggest that the variance in the D process is probably lower than the one exhibited by our simulation.

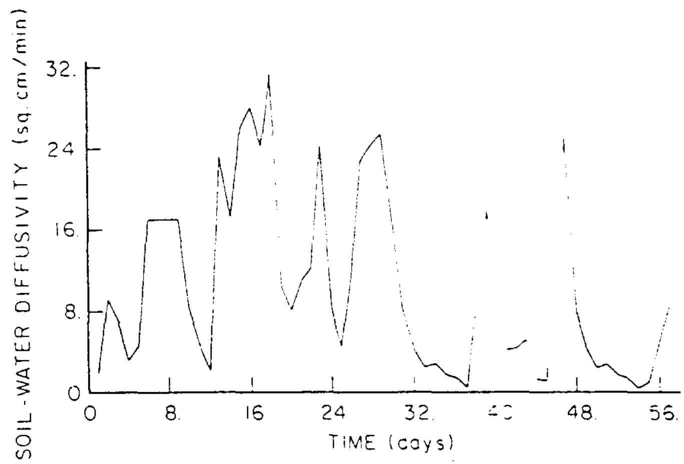

Fig. 4. Soil-water diffusivity time series for the root zone

It is also noted that during dry periods some persistence exists in the daily time series of D. To obtain a quantitative evaluation of this aspect, the serial correlation coefficient of the generated D series was computed. It was found that the correlation coefficient follows an exponential decay of the form

$$r_l = e^{-\rho l}, \tag{17}$$

where r_l is the lag l serial correlation coefficient; and ρ a recession parameter found to be equal to 0.5 on this example. No further analysis of the D series was done since it was considered that the data only represents a hypothetical condition rather than an actual field situation. Further research is needed to determine the correlation structure of the time variability in the soil-water diffusivity, its marginal probability density function, and its degree of stationarity. The research should involve some repetitive type of test in a soil profile in which the soil is excited by a rainfall process of known stochastic properties. The soil functional relationships and the soil-water diffusivity are determined on a continuous basis.

The preliminary results of this simulation indicate that the random time variability in the rainfall occurrence generates hysteretic loops and an important degree of uncertainty in the time variability of the soil-water diffusivity which should be accounted for in the solution of the vertical infiltration equation. This uncertainty in D will generate an important uncertainty in the water content evolution in the soil profile. Since the main aim of the present study was the development of a methodology to solve the vertical infiltration equation which accounted for the time variability in the rainfall input and the variability in the soil-water diffusivity resulting from the inherent hysteretic process, several assumptions were made on the stochastic properties of the D process. Knowing that this is only an illustration of the methodology in the next section, future research will have to be done to determine the exact stochastic properties of D in actual soil profiles.

It was assumed that the D process can be represented as

$$D = \bar{D} + D'(t, \omega), \tag{18}$$

where $\bar{D}$ represents a deterministic component; and

$D'(t,\omega)$ represents a random component in the probabilistic variable ω. Following the previous observation on the boundedness of D we adopted a significantly lower value for the mean $\bar{D}$, that is $\bar{D} = 0.0162\ \mathrm{m^2\,day^{-1}}$. The effect of seasonality was neglected in this example, knowing that its inclusion will not significantly complicate the procedure. The random component was assumed to follow a colored Gaussian noise. The latter follows in view of the evidence of persistence in our previous simulation of D:

$$E\{D'(t)\} = 0, \qquad E\{D'(t_1)D'(t_2)\} = qe^{-\rho(t_1 - t_2)}, \tag{19}$$

where q is the variance parameter equal to $1.56\ \mathrm{m^2 \cdot day^{-1}}$.

SOLUTION OF THE VERTICAL INFILTRATION EQUATION

Following the definition of the top boundary condition and the form of the soil-water diffusivity in the last two sections, we attempt, in this section to solve the vertical infiltration equation. The partial differential equation governing the one-dimensional vertical infiltration in a homogeneous soil is given by[2]

$$\frac{\partial\theta}{\partial t} - \frac{\partial}{\partial z}\left[D\frac{\partial\theta}{\partial z}\right] + \frac{\mathrm{d}K}{\mathrm{d}\theta}\frac{\partial\theta}{\partial z} = 0, \tag{20}$$

$$\theta(0,t,\omega) = \theta_0(t,\omega), \qquad \theta(\infty,t) = 0, \qquad \theta(z,0) = \theta_i(z),$$

where θ is the volumetric water content; t is the time coordinate (days); z is the vertical spatial coordinate, positive downward (m); D is the soil-water diffusivity ($\mathrm{m^2 \cdot day^{-1}}$), which we propose to be a stochastic process of the form of equation (18) when the top boundary is subject to the rainfall regime in a watershed; $K(\theta)$ is the unsaturated hydraulic conductivity of the soil ($\mathrm{m^2 \cdot days^{-1}}$); $\theta_0(t,\omega)$ is the stochastic process representing the time variability in the water content at the top boundary, the 'boundary layer', which we assume to follow a shot noise process of the form of equation (9) when this layer is subject to the time variabiltiy typical of natural point rainfall patterns; and $\theta_i(z)$ is the known initial water content along the soil profile.

The assumption on the form of D implies that we are neglecting its spatial variability. This means that in this study the time random variations in D occur in the entire soil profile, a condition feasible in small reaches of soil where a bulk D value can be measured. The effect of the spatial variability in D on the evolution of θ is currently being investigated using the method we describe in this section and the results will soon be published.

The term $\mathrm{d}K/\mathrm{d}\theta = u$ is assumed constant in the present study and equal to $0.0216\ \mathrm{m \cdot day^{-1}}$ which is an average value taken from the Liakopoulus data. Clearly this term is not constant and varies directly with θ. Thus, strictly speaking, u would be another time stochastic process presumably correlated with the D process. In the absence of experimental data on the correlation structure of this term, we leave the investigation of this aspect to a future study.

Another simplification of the formulation is the assumption of independence between the boundary process, θ_0 and the D process. Intuitively one may think that high rainfall intensity is associated with high values in D and low rainfall intensity is associated with low values in D. Therefore, some correlation could be expected between the above two processes. However, because of the attenuation effect of hydrodynamic dispersion in unsaturated soils, this possible degree of correlation may be significantly reduced as depth increases. The issue is further complicated because the nature and the behaviour of D is not well understood yet. Once again we are in unknown territory and the absence of experimental information refrains us from speculating further. Thus in the following treatment we assume that D is a system parameter which is physically independent of the input functions. This statistical independence between system parameters and environmental inputs has been reported in many other physical systems (see Ref. 1).

The objective in this section is therefore to solve the partial differential equation (20) subject to a colored noise soil-water diffusivity and a shot noise boundary condition. The following solution of the stochastic partial differential equation (20) is based on previous work by the author[30–37], where functional analytic concepts were applied to obtain new solutions to similar stochastic partial differential equations in subsurface hydrology. For this reason, many details of the mathematical derivation will not be repeated here and the reader is referred to those works.

Let us replace equation (18) in equation (20) and put the random component on the right hand side to obtain

$$\frac{\partial\theta}{\partial t} - \bar{D}\frac{\partial^2\theta}{\partial z^2} + u\frac{\partial\theta}{\partial z} = D'\frac{\partial^2\theta}{\partial z^2}, \tag{21}$$

subject to the same boundary and initial conditions of equation (20). The solution to this transport evolution equation is[30,36]

$$\theta(z,t) = \Phi(z,t) + J_t\theta_i(z) + \int_0^t J_{t-\tau}D'(\tau)\frac{\partial^2\theta}{\partial z^2}\,\mathrm{d}\tau, \tag{22}$$

where $\Phi(z,t)$ is the solution due to the stochastic boundary condition given by[30]

$$\Phi(z,t) = \frac{z}{(4\pi\bar{D})^{1/2}}\int_0^t \exp\left[-\frac{(z-u(t-\tau))^2}{4\bar{D}(t-\tau)}\right]\frac{\theta_0(\tau)}{(t-\tau)^{3/2}}\,\mathrm{d}\tau. \tag{23}$$

$J_t\theta_i(z)$ in equation (22) is the solution due to the deterministic initial condition, where J_t is the strongly continuous semigroup[36] associated with the evolutional operator in equation (20). The semigroup operator in this case is given by[30]

$$J_t\theta_i(z) = \frac{1}{(4\pi\bar{D}t)^{1/2}}\int_0^\infty\left\{\exp\left[-\frac{(z-ut-s)^2}{4\bar{D}t}\right] - \exp\left[-\frac{(z-ut+s)^2}{4\bar{D}t}\right]\right\}\theta_i(s)\,\mathrm{d}s. \tag{24}$$

We now define θ in the right hand side of equation (22) as the series $\theta = \theta_1 + \theta_2 + \theta_3 + \cdots$. Equation (22) becomes

$$\theta(z,t) = J_t\theta_i(z) + \Phi(z,t) + \int_0^t J_{t-\tau}D'(\tau)\frac{\partial^2}{\partial z^2}(\theta_1 + \theta_2 + \ldots)\,\mathrm{d}\tau. \tag{25}$$

Now set θ_1 equal to the previous part of the solution, $\Phi(z, t)$, and truncate at the first term to obtain

$$\theta(z, t) = J_t\theta_i(z) + \Phi(z, t) + \int_0^t J_{t-\tau}D'(\tau) \frac{\partial^2}{\partial z^2} \Phi(z, \tau)\, d\tau. \tag{26}$$

(For a justification of the above approximation procedure and a discussion on the convergence see Serrano[31]). We are truncating at the first term in the expansion for simplicity and because we assume we are dealing with relatively small variances in D. Obviously, more terms will have to be included in the case of arbitrary large variances. Note that the solution we are presenting is not a perturbation solution, and therefore it is not limited to small variances in the stochastic functions, which is the most important limitation in the existing perturbation solutions.

From equation (26) we may obtain sample functions of the water content at different depths and at different times. These sample functions help us observe the evolution of the water content distribution under different conditions. We are also very interested in obtaining statistical measures of the water content, which characterize the stochastic properties of the water content process. In particular it is very desirable to derive expressions to calculate the mean and the variance of the water content as a function of the same measures of D and θ_0. In most engineering applications the modeler only has the first two moments of the input processes available and rarely information of their joint probability density function. Therefore the first two moments of the solution are the only feasible statistical measures. In order to obtain such measures, we first examine each of the terms in equation (26).

The mean of the boundary solution is given by

$$E\{\Phi(z, t)\} = \frac{z}{(4\pi\bar{D})^{1/2}} \int_0^t \times \exp\left[-\frac{(z - u(t-\tau))^2}{4\bar{D}(t-\tau)}\right] \frac{E\{\theta_0(\tau)\}}{(t-\tau)^{3/2}}\, d\tau \tag{27}$$

Using equation (12) this becomes

$$E\{\Phi(z, t)\} = \frac{\lambda z}{\alpha\gamma(4\pi\bar{D})^{1/2}} \int_0^t \times \exp - \left[\frac{(z - u(t-\tau))^2}{4\bar{D}(t-\tau)}\right] \frac{(1 - e^{-\alpha\tau})}{(t-\tau)^{3/2}}\, d\tau. \tag{28}$$

The correlation function of the boundary solution is, from equation (23),

$$E\{\Phi(t_1)\Phi(t_2)\} = \frac{z^2}{4\pi\bar{D}} \int_0^{t_1} \int_0^{t_2} \exp\left[-\frac{(z - u(t_1-\tau))^2}{4\bar{D}(t_1-\tau)}\right] \exp\left[-\frac{(z - u(t_2-\xi))^2}{4\bar{D}(t_2-\xi)}\right] \cdot \frac{E\{\theta_0(\tau)\theta_0(\xi)\}}{(t-\tau)^{3/2}(t-\xi)^{3/2}}\, d\xi\, d\tau. \tag{29}$$

Using equation (15) this becomes

$$E\{\Phi(t_1)\Phi(t_2)\} = \frac{2\lambda z^2}{4\alpha\gamma^2\pi\bar{D}} \int_0^{t_1} \int_0^{t_2} \times \exp\left[-\frac{(z - u(t_1-\tau))^2}{4D(t_1-\tau)}\right] \times \exp\left[-\frac{(z - u(t_2-\xi))^2}{4\bar{D}(t_2-\xi)}\right] \cdot \left[\frac{\lambda}{\alpha} - \frac{\lambda}{\alpha} e^{-\alpha\xi} - \frac{\lambda}{\alpha} e^{-\alpha\tau} + \left(\frac{\lambda}{\alpha} - \frac{1}{2}\right) e^{-\alpha(\tau+\xi)} + \frac{1}{2} e^{-\alpha(\tau-\xi)}\right] \frac{d\xi\, d\tau}{(t-\tau)^{3/2}(t-\xi)^{3/2}}. \tag{30}$$

Using equations (28) and (30) the variance of Φ will then be

$$\mathrm{Var}\{\Phi(z, t)\} = E\{\Phi^2(z, t)\} - E^2\{\Phi(z, t)\}. \tag{31}$$

Now the mean of the third term, which we shall call $I(t)$, in the solution of equation (26) is simply

$$E\{I(t)\} = 0, \tag{32}$$

in which equation (19) has been used. Using equation (19) again, we find an expression for the correlation function of $I(t)$:

$$E\{I(t_1)I(t_2)\} = \int_0^{t_1} \int_0^{t_2} J_{t_1-\tau}J_{t_2-\xi}E\left\{D'(\tau)D'(\xi) \frac{\partial^2}{\partial z^2} [\Phi(\tau)] \frac{\partial^2}{\partial z^2} [\Phi(\xi)]\right\} d\xi\, d\tau. \tag{33}$$

Using equations (32), (33) and (19), setting $t_1 = t_2 = t$, and the previously mentioned assumption of independence between Φ and D', we obtain the variance of $I(t)$:

$$\mathrm{Var}\{I(t)\} = q \int_0^t \int_0^t J_{t-\tau}J_{t-\xi}e^{-\rho(\tau-\xi)} \frac{\partial^2}{\partial z^2} \times [E\{\Phi(\tau)\Phi(\xi)\}]\, d\xi\, d\tau, \tag{34}$$

where the correlation of Φ is given by equation (30). We are now in a position to calculate the first two moments of the water content. In equation (26) the mean water content is given by

$$E\{\theta(z, t)\} = E\{\Phi(z, t)\} + J_t\theta_i(z), \tag{35}$$

where equation (32) has been used, and the mean of Φ is given by equation (28). Finally, using equations (31) and (34), it is easy to show that the variance of the water content is given by

$$Var\{\theta(z, t)\} = E\{\theta^2(z, t)\} - E^2\{\theta(z, t)\} = \mathrm{Var}\{\Phi(z, t)\} + \mathrm{Var}\{I(z, t)\}. \tag{36}$$

The above derived expressions for sample functions, the mean and the variance were used in the numerical computations of the next section. Computation of each of the partial terms involved in the expressions was done in order to observe the contribution of each of the stochastic components to the combined random behaviour of the water content. The objective of the exercise was to evaluate the relative importance of each of the terms in order to identify the individual components crucial in the reduction of the variance of the water content forecast.

COMPUTATIONAL RESULTS AND ANALYSIS

In this section we explore the computational features of each of the terms in the stochastic solution of the vertical infiltration equation, its mean and its variance as developed in the previous section. We begin by considering the deterministic component in the solution of equation (26), $J_t\theta_i(z)$, which is defined by equation (24). Assuming that the typical scale of the instrument measuring field water content is 0.1 m, then the bulk water content will be constant at 0.1 m intervals. Thus the deterministic component will be reduced to

$$J_t\theta_i(Z) = \sum_{j=1}^{n} \theta_j M_j, \tag{37}$$

where the depth Z takes the discrete values 0.1, 0.2, 0.3, ...; θ_j is the measured initial water content at the above discrete intervals; $j = 1, \ ,2, \ldots, N$; and the function M_j is the spatial integral of equation (24) solved at 0.1 m intervals in which θ_j is constant. For example, when $j = 1$,

$$M_1 = \frac{1}{(\pi b)^{1/2}} \int_0^{0.1} \left\{ \exp\left[-\frac{(a-s)^2}{b} \right] - \exp\left[-\frac{(a+s)^2}{b} \right] \right\} ds, \tag{38}$$

where $a = Z - ut$; and $b = 4\bar{D}t$. This equation can be written as

$$M_1 = \frac{1}{\pi^{1/2}} \left\{ \int_{-a/b^{1/2}}^{(0.1-a)/b^{1/2}} e^{-\xi^2} d\xi - \int_{a/b^{1/2}}^{(0.1+a)/b^{1/2}} e^{-\xi^2} d\xi \right\} \tag{39}$$

which in turn can be written as

$$M_1 = \frac{1}{2} \left\{ erfc\left(-\frac{a}{b^{1/2}} \right) - erfc\left(\frac{0.1-a}{b^{1/2}} \right) - erfc\left(\frac{a}{b^{1/2}} \right) + erfc\left(\frac{0.1+a}{b^{1/2}} \right) \right\}, \tag{40}$$

where $erfc(\,)$ denotes the 'error function complement'. In general, for any j

$$M_j = \frac{1}{2} \left\{ erfc\left(\frac{j - 0.1 - a}{b^{1/2}} \right) - erfc\left(\frac{j-a}{b^{1/2}} \right) - erfc\left(\frac{j-0.1+a}{b^{1/2}} \right) + erfc\left(\frac{j+a}{b^{1/2}} \right) \right\}. \tag{41}$$

Interestingly, the convergence of equation (37) to a desired accuracy was achieved after two or three steps. The accuracy of the above scheme was tested by setting $\theta_j = 1.0$, for all j, and noting that $J_t\theta_i$ was equal to one. It was also found that a stepwise computation, at 1 day intervals, was the most accurate.

The deterministic solution results from a linearization of the infiltration equation. It is known that this solution by itself is a poor model for vertical infiltration. In the present study the deterministic solution was compared with the observed data presented by Liakopoulus[14] by computing equation (37) with the same initial condition presented in the above study. It was found that the deterministic solution substantially underestimates the values of the water content. This may indicate that the simulated values of D and u are probably lower in magnitude than the ones adopted. However this modification of the parameters would not produce a good fit with the observed data because the deterministic solution does not include the physical dependence between the parameters and θ. This suggests that a better deterministic solution should account for the high values in D and u in zones of high θ and the low values in the parameters in zones of low θ. However, a deterministic solution which includes this parameter dependence with the water content is very difficult to derive. No further analysis or 'calibration' of this component was attempted, since the motivation of the present work is the understanding that the deterministic, linearized, solution is not an appropriate tool to represent real field conditions. In the present study this component is only a part of the total stochastic solution.

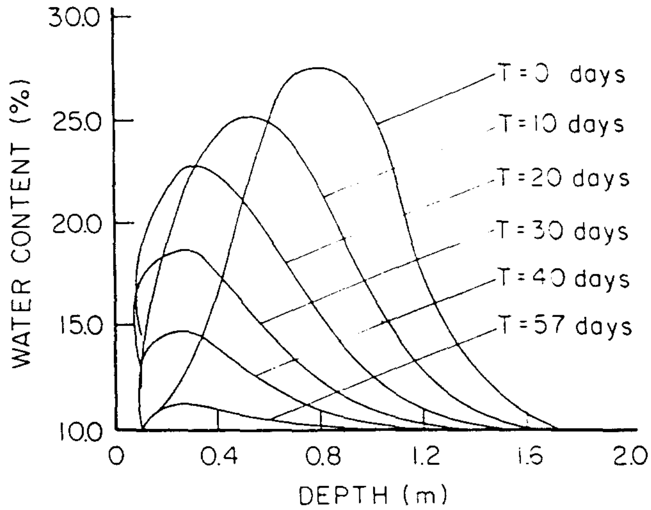

Fig. 5. Deterministic component of water content versus depth

For the rest of this study an arbitrary initial condition was assumed and its evolution over time was computed after equation (36). Figure 5 illustrates the behaviour of the deterministic component.

Now the second term in equation (26) is the partial solution due to the stochastic boundary condition, $\Phi(z, t)$, which is given by equation (23). Here the sample boundary water content values were discretized at 1 day intervals. Thus

$$\Phi(z, t, \omega) = \frac{Z}{(4\pi\bar{D})^{1/2}} \sum_{j=1}^{j=t} \int_{j-1}^{j} \times \exp\left[-\frac{(Z - u(t-\tau))^2}{4\bar{D}(t-\tau)} \right] \frac{\theta_j' e^{-\alpha\tau}}{(t-\tau)^{3/2}} d\tau, \tag{42}$$

where θ_j' is the initial value of the boundary water content, θ_0, at the left boundary of the time interval (that is at the time $\tau = j - 1$); and $\tau \in (j-1, j]$. Each stepwise integral was solved by a 24 point Gauss-Legendre quadrature with good results. Again, accuracy was tested first by setting θ_j' and $e^{-\alpha\tau}$ equal to one and assuring that Φ was equal to one.

Figure 6 shows the boundary component of the stochastic solution with respect to time at three different depths. It was found that the closer the observation point is to the upper boundary, the higher the variability in the water content due to the random nature of rainfall. This would explain the difficulty in forecasting the water content near the ground surface. As depth increases, the boundary component of the water content takes more time to react and its variation over time is smoother. Thus for depths beyond 0.8 m the water content will slowly increase to a steady mean value and its variability over time with respect to this mean value will be small. This feature is in agreement with field observations of the water content which seem to indicate that the water content in deep soils tends to maintain a steady constant value even though the rainfall on the surface is highly variable.

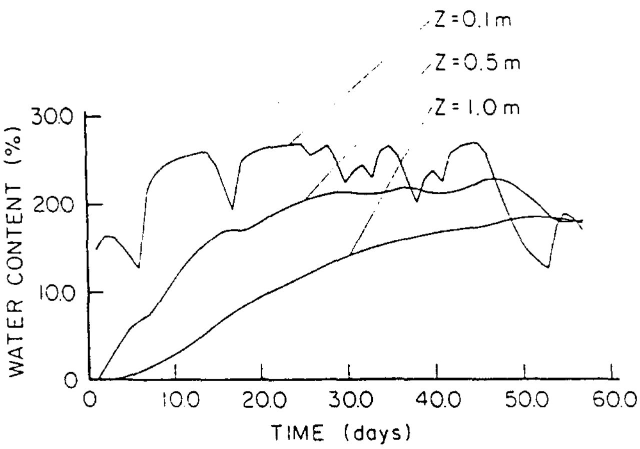

Fig. 6. Sample functions of the boundary component of water content versus time at different depths

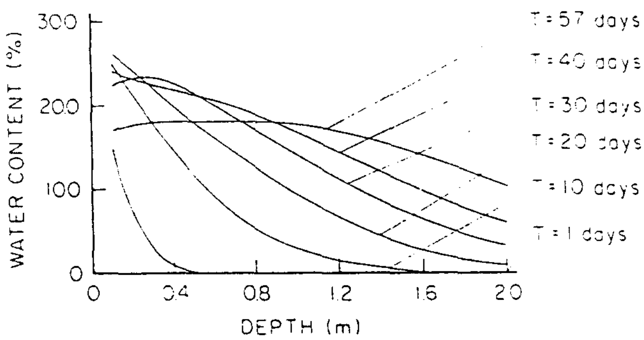

Fig. 7. Sample functions of the the boundary component of water content versus depth at different times

It is interesting to observe the spatial variability of the water content at different times. Figure 7 illustrates the water content versus depth at different times. Note that as time increases the water content profile tends to a smooth steady curve.

The calculation of the third term in equation (26) involves the generation of sample functions of the colored noise process representing the time variations in the soil-water diffusivity, D'. Generation of sample functions of a colored noise process with correlation function given by equation (19) can be easily achieved by making the transformation $q = \sigma^2\rho/2$ and noting that a zero-mean stationary Gaussian process with exponential correlation

$$E\{D'(t_1)D'(t_2)\} = \frac{\sigma^2\rho}{2}\, e^{-\rho(t_1 - t_2)} \tag{43}$$

can be generated by an ordinary differential equation forced by white Gaussian noise[10]:

$$\frac{dD'(t)}{dt} + \rho D'(t) = \sigma\rho\,\frac{d\beta(t)}{dt}, \tag{44}$$

subject to $D'(t=0) = D_0'$, where D_0' follows a normal distribution $N(0, \sigma^2\rho/2)$; and $d\beta(t)/dt = w(t)$ is white Gaussian noise with

$$E\{w(t)\} = 0, \qquad E\{w(t_1)w(t_2)\} = \delta(t_1 - t_2). \tag{45}$$

The solution of equation (44) is

$$D'(t) = D_0' e^{-\rho t} + \sigma\rho \int_0^t e^{-\rho(t-s)}\, w(s)\, ds, \tag{46}$$

which can be used to generate sample functions of D'.

This equation can be further reduced after recalling that the typical time scale in our simulations is one day and assuming that any time variation in an interval of less than a day is not recognized. This is equivalent to assuming that the w process is constant in intervals of one day. Thus equation (46) becomes

$$D'(t) = D_0' e^{-\rho t} + \sigma\rho \sum_{j=1}^{j=t} w(j) \int_{j-1}^{j} e^{-\rho(t-s)}\, ds, \tag{47}$$

where $w(j)$ is the value of the white Gaussian noise process at discrete times $j = 1, 2, \ldots, t$. Solving the integral this reduces to

$$D'(t) = D_0' e^{-\rho t} + \sigma e^{-\rho t}(1 - e^{-\rho}) \sum_{j=1}^{j=t} w(j) e^{-\rho j}. \tag{48}$$

This equation was used to generate realizations of the D' process which was needed in the calculation of the third term in equation (26). Since the colored noise variance parameter was previously chosen as $q = 1.56\ \mathrm{m}^2 \cdot \mathrm{day}^{-1}$, and $\rho = 0.5$, then $\sigma = 2.5$.

The third third term in equation (26) is due to the random component in the soil-water diffusivity, D', in the stochastic equation (20). This term was first approximated as

$$I(t) = \sum_{j=1}^{j=t} D_j' \int_{j-1}^{j} J_{t-\tau}\Delta^2(J_\tau \Phi_{j-1})\, d\tau,$$

where $\Delta^2(\ )$ is a suitable finite difference approximation of the spatial second derivative; D_j' is the value of D' at time j; and Φ_{j-1} is the value of the boundary solution along depth at time $j-1$. Since the differentiation operation may be an unstable procedure, some smoothing of the function $J_\tau\Phi_{j-1}$ may be useful. Each one of the t integrals was solved using a Gaussian quadrature. It was found that this procedure took too long in a microcomputer and a further simplification was formulated as follows:

$$I(t) = \sum_{j=t-1}^{j=t} J_{t-j} D_j' \Delta^2(J_j \Phi_{j-1}).$$

In this case $I(t)$ was approximated stepwise and computational time was drastically reduced.

Figure 8 shows a sample function of the I component of the water table generated from equation (50). Note

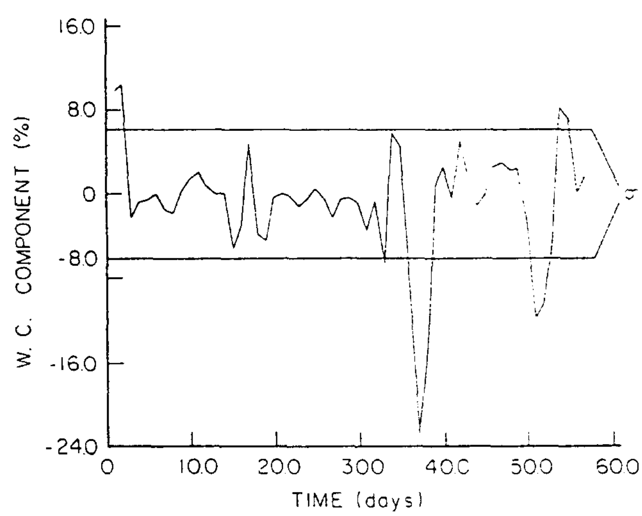

Fig. 8. Sample function of the component due to D' of water content versus time and standard deviation

that sometimes this component will lead the total value of the water content beyond the interval 0–30%. As with any statistical analysis, the physical bounds will limit the numerical value of the sample function.

Adding the three components of equation (26) (Figs 5, 6 and 8, corresponding to equations (37) (42) and (50) respectively), we obtain the total sample water content, an illustration of which is shown in Fig. 9. In this example the first 0.6 m of the soil profile is maintained at saturation. This only reflects the particular choice of parameters and the nature of the forcing rainfall used in the present example. It was earlier noted that saturation may rarely occur in the boundary layer and therefore, the boundary water content values used in this example are probably too high. In any case, the soil moisture profiles will be spatially smooth curves, varying erratically in time, which describe the random nature of the water content due to the random nature of the forcing rainfall and the random nature of the hysteresis process.

The next step is the calculation of the mean water content as given by equation (35). First, we compute the mean boundary component, $E\{\Phi(z,t)\}$, by using equation (28). This equation was approximated using a similar procedure to the one used to calculate the sample $\Phi(z,t)$ values, except that in this case the integral was much easier to solve. Figure 10 shows the mean boundary component with respect to time at three different depths in the soil. It is illustrative to compare these means with the sample boundary components at corresponding depths (Fig. 6). Using this information and

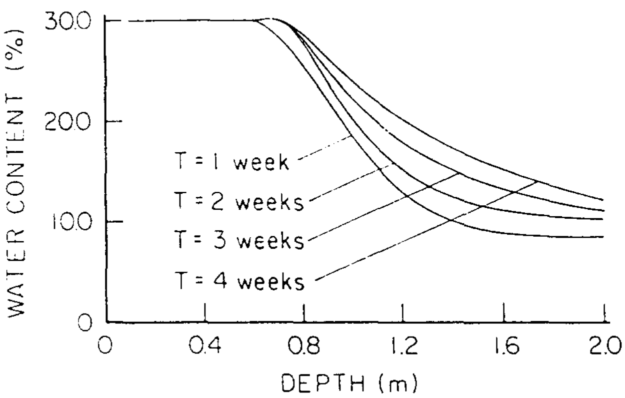

Fig. 9. Sample water content versus depth at different times

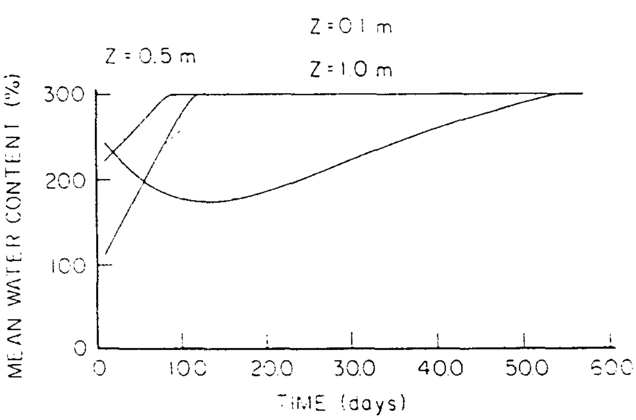

Fig. 10. Mean boundary component of the water content versus time at different depths

equation (35), the mean water content was computed and plotted with respect to time for the same typical depths (Fig. 11). Again, the shape of these curves reflects the particular choice in the parameters, the properties of the forcing rainfall, and the particular choice in the initial condition for the simulation.

The next step is the calculation of the variance of the water content as described by equation (36). First, we compute the variance of the boundary component $\mathrm{Var}\{\Phi\}$, which was done in a similar way to the calculation of the mean, $E\{\Phi\}$. Figure 12 describes the mean and one standard deviation of the boundary component with respect to time at $z = 1.0$ m. The large values in the variance reflect the high variability of the rainfall used in the simulation. It was also found that low values in the parameter α, that is an upper layer of soil which quickly looses water through evapotranspiration result in significantly lower values in the variance of the boundary component at depth.

The second term in equation (36) is the variance of I, $\mathrm{Var}\{I(t)\}$, which was approximated from equation (34) as follows:

$$\mathrm{Var}\{I(t)\} = q \sum_{k=t-1}^{k=t} \sum_{j=t-1}^{j=t} J_{t-k} J_{t-j} e^{-\rho(k-j)} \Delta^2 [E\{\Phi(k)\Phi(j)\}] \tag{51}$$

where the correlation of Φ is given by similar approximation of equation (30). It was found that the mean of I was about 7.2 and the standard deviation was 9.6. In Fig. 8 the mean plus and minus one standard deviation of I was plotted. Finally, equation (36) will give the variance of the water content.

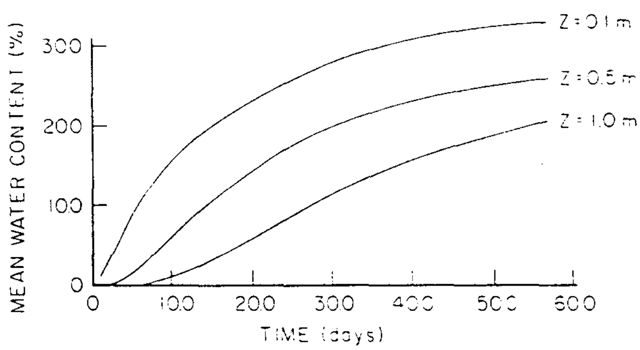

Fig. 11. Mean water content versus time at different depths

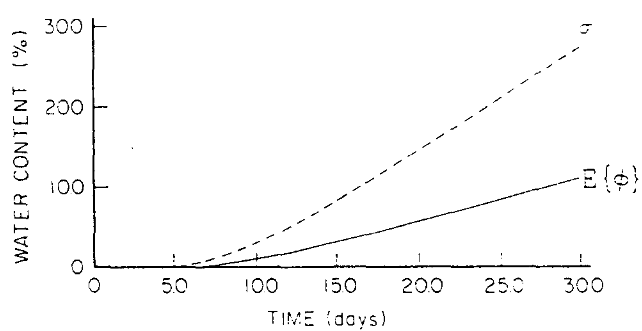

Fig. 12. Mean and standard deviation of the boundary component of the water content versus time at one meter depth

The results indicate that the variance of the predicted water content is largely dominated by the variance of the boundary component. The particular selection in the parameters of the boundary component produced large variance values which in some cases force the theoretical water content beyond the physically realizable values. This indicates that the variability of rainfall selected for the example calculations is probably too high. The results also suggest that the modeler should perform a series of measures of the upper-soil water content in order to determine the stochastic properties of the boundary component. This exercise will in turn tend to reduce the variance in the predicted water content at depth.

CONCLUSIONS

A method for analyzing vertical infiltration in hysteretic soils subject to the random variations of point rainfall in watersheds and a procedure for solving the resulting stochastic partial differential equation is presented. The effect of the time random variability in point rainfall produces a water content process in the upper soil layer which can be modeled as a shot noise process. Preliminary simulation experiments indicate that the hysteretic loops resulting from the random wetting and drying cycles generate a correlated random soil-water diffusivity process. The modeling problem reduces the solution of the infiltration equation subject to a shot noise boundary condition and a colored noise soil-water diffusivity. A new semigroup solution of this evolution equation was obtained and expressions for sample functions, the mean and the variance of the water content in space and time were derived. It is hoped that the present methodology will provide the modeler with a better tool to model infiltration in natural soils subject to the uncertainties associated with the rainfall regime and the hysteresis in the soil. This procedure will encourage the use of physically based models for infiltration rather than empirical equations.

The exact stochastic properties of the soil-water diffusivity process remain to be investigated. Further field or experimental research should be done by forcing a soil profile with a known random process representing point rainfall at a geographic location, observing the random shape of the soil-water functional relationships and deriving long-term sample functions of the soil-water diffusivity at different depths. This implies the conception of the soil-water functional relationships as being random in nature for field applications. The present research assumes one random in time soil-water functional reationships for the entire soil profile. The effect of spatial variability of these relationships, which in turn would produce a time and a space random process for the diffusivity, remains to be investigated. It is hoped, however, that by implementing the method presented in this article a solution to the resulting differntial equation can be easily obtained.

The computational example of each one of the components of the infiltration equation and the corresponding moments indicated that the variance of the predicted water content is controlled by the rainfall inputs. This suggests that accurate knowledge the stochastic process governing point rainfall may help in decreasing the variance of the water content forecasts at depth. Furthermore, by implementing a water content measurement program at the soil surface the variance of the predicted water content at depth may be substantially reduced.

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