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Stochastic Differential Equation Models of Erratic Infiltration

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Laboratory and field infiltration data exhibit a degree of erratic variability usually associated with measurement errors and uncertainties in the phenomenon of unsaturated porous media flow. Traditionally, these uncertainties are ignored and averaged soil characteristic curves are used in the inverse and direct modeling problems. However it is desirable to develop models capable of reproducing the inherent variability of soil moisture in order to study the erratic physics of flow at the laboratory level and to reproduce infiltration data in natural watersheds. In the present article, two exploratory models are tested as to their ability to replicate the erratic variability of experimental horizontal infiltration data. The first is based on the full partial differential equation of infiltration, and the second on the Boltzmann-reduced differential equation. Both models are subject to a space or a time and space random soil-water diffusivity defined as uncertainty term. The solution of the equations is presented, statistical properties of the water content function are described and verification of the models is conducted. Both models satisfactorily reproduced the statistical properties of the experimental data. While the first model easily relates to real space and time variables, the second required less computer time. As an application of the methodology, a third model is introduced as a new approach to predict vertical infiltration in hysteretic soils in natural watersheds. For this purpose, the effect of time variability of point rainfall is represented as a shot noise process, the hysteretic loops resulting from the natural wetting and drying cycles generate a correlated random soil-water diffusivity process, and a solution of the vertical infiltration equation is presented along with statistical properties of the water content.

1. INTRODUCTION

The hydrology of unsaturated flow in natural soils is an important determinant of the overall fate of storm water in a watershed. The magnitude and evolution of infiltration will dictate the proportion of water moving as subsurface storm flow, the proportion recharged to the groundwater reservoir (the slow component), and the proportion available for overland runoff supply (the fast component). Thus the availability of models which adequately describe the qualitative and quantitative behaviour of unsaturated flow is a desirable feature for the estimation of parameters in other components of the hydrologic cycle.

Mathematical models used to predict the time and space evolution of infiltration are based on laboratory and field measurements of infiltration. Infiltration data in natural soils exhibit a degree of erratic variability usually associated with instrument and measurement errors, environmental fluctuations and parameter uncertainty associated with the measurement scale [see *Cushman*, 1987]. This poses a problem when an attempt is made to use the information to develop parameters or to verify models of unsaturated porous media flow. Traditionally, these uncertainties are ignored and averaged soil characteristic curves are used in the inverse and direct modeling problems. However, valuable information on the random nature of porous media flow is lost by this procedure and the resulting models will be able to simulate only idealistic experimental or field conditions. Therefore, it would be desirable to develop models capable of reproducing the inherent variability of soil moisture in order to study the physics of flow at the laboratory level and to reproduce realistic infiltration data in natural watersheds. These models would use all the information provided by the laboratory or field measurements and they would be able to generate values comparable with measured ones. Verification of models would easily and naturally be done by comparing the statistical properties of measured values with respect to statistical properties of simulated values, instead of comparing a single, highly uncertain, realization of measured values with a single, highly uncertain, realization of a model output [*Serrano and Unny*, 1987*b*].

Other researchers have approached the modeling of unsaturated flow in a statistical manner. *Dagan* [1983] defined the saturated hydraulic conductivity as a log-normally distributed random variable and by assuming analytical, single-valued, expressions for the soil-water functional relationships a solution to a simplified "piston flow" model of the infiltration equation was obtained. *Mtundu and Koch* [1987] derived two ordinary differential equations describing the surplus and deficit conditions in the unsaturated zone. These equations were forced by the random infiltration process and by the evapotranspiration process, and the resulting random differential equations were solved. *Tsakiris et al.* [1988] developed a shot noise model for the generation of soil moisture data in the root zone. *El-Kadi* [1987] used Monte-carlo simulation techniques to study the effect of variability of infiltration on soil parameters. Other researchers have used small perturbation techniques to solve the stochastic unsaturated flow equation [eg., *Yeh et al.*, 1985]. However, it has been known for a long time that perturbation techniques are limited to small variances in the uncertainty terms and that in some circumstances they may lead to important errors [see *Cushman*, 1987; *Adomian*, 1983].

In the present article the techniques of stochastic ordinary differential equations introduced by Mtundu and Koch will be extended to the cases when the stochastic partial differential equations for horizontal and vertical infiltration are considered. The techniques used in the solution of the equations can be related to simple linear and non-linear systems theory, which were derived using functional-analytic concepts in previous publications of the author.

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These techniques do not require the use of Montecarlo simulations, spectral decomposition or small perturbation assumptions, and they are not limited to small variances or one random process.

Two models are tested as to their ability to replicate the erratic variability of horizontal infiltration data collected in a series of laboratory experiments. In section 2, the experiments are described and the classical procedure of analysis of the information is summarized. In section 3, the development of the two models is presented. The first model is based on the full partial differential equation of infiltration, and the second model is based on the Boltzmann-reduced differential equation. Both models are subject to a space or a time and space random soil-water diffusivity defined as the uncertainty term. The solution of the equations is presented, and expressions for the statistical properties of the water content function are derived. In section 4, verification of the models is conducted.

As an application of the methodology, a third model is introduced in section 5 as a new approach to predict vertical infiltration in hysteretic soils in natural watersheds. For this purpose, the effect of time variability of point rainfall is represented as a shot noise process (using the Tsakiris et al model), the hysteretic loops resulting from the natural wetting and drying cycles generate a correlated random soil-water diffusivity process, and the modeling problem is reduced to the solution of the vertical infiltration equation subject to a shot noise top boundary condition and a colored noise soil-water diffusivity. The solution of this equation is presented and statistical properties of the water content are studied.

2. INFILTRATION EXPERIMENTS

The infiltration experiments were conducted by the author at the soil physics laboratory of the Department of Land Resource Science of the University of Guelph, Canada, while one of a team investigating various aspects of hydrodynamic dispersion during infiltration in soils. Several horizontal infiltration experiments were carried on a Guelph clay loam (17% sand, 48% silt, and 35% clay). The dominant clay mineral in the soil was illite. Data were obtained from experiments terminated at different elapsed times.

The objective of each experiment was to observe the evolution of the volumetric water content as it traveled through a horizontal column of soil sample due to the natural forces of suction. Thus it was necessary to obtain a relationship between the volumetric water content in finite sections of the soil sample with respect to the horizontal distance to an origin. The origin was the left boundary condition, where the water content was maintained constant throughout the duration of the experiment.

For each experiment, sectioned lucite columns were packed uniformly with soil having an initial water content of $\theta_n = 0.086$ cm^3/cm^3 [*Elrick et al.*, 1979]. A glass bead base, approximately 2 mm thick, with two ports in the entry plate, one for water and one to allow the escape of air, were helpful in establishing the constant boundary condition at $x = 0$. After about 30 s the initially small positive inflow head was lowered to approximately -1 mbar. The laboratory temperature for all the experiments varied between 21 and 23°C. The cumulative volume of infiltrated water, which was measured with a horizontal burette, as well as the distance

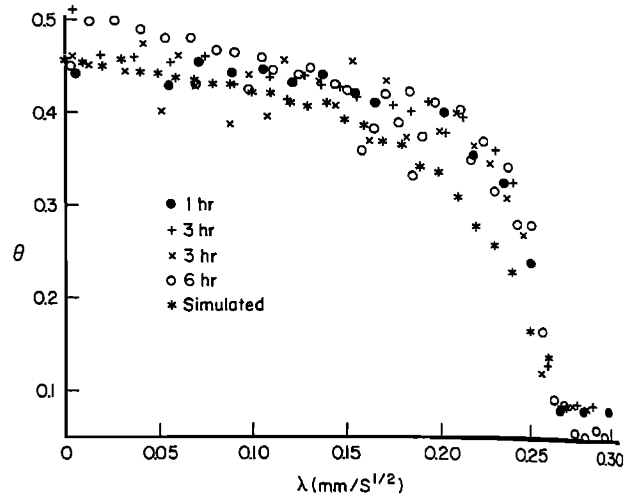

Fig. 1. Observed and simulated sample water content versus λ.

from the origin to the wetting front was recorded as a function of [time$^{1/2}$]. A linear relationship between the above two variables indicated the preservation of similarity with regard to water flow, in agreement with previous research [*Elrick et al.*, 1979]. At the end of each infiltration experiment, the time was recorded, the column was sliced into sections and the soil in each section, of known volume, was used to calculate the bulk density and the volumetric water content.

Figure 1 illustrates the observed water content, θ, as a function of the variable λ (where $\lambda = x/t^{1/2}$) for one experiment terminated after 1 h, two experiments of 3 h duration each and one experiment of 6 h. Note the erratic variability of the water content profiles. These uncertain values of the water content are always present regardless of how carefully the experiment is conducted. Initially the variability was ignored and the analysis proceeded formally by "filtering" and smoothing the curves of all the experiments in order to obtain and average curve of θ versus λ. Next the relationship of the soil-water diffusivity, D, to θ was derived by using the standard procedure of numerically integrating the equation [*Philip*, 1955]

$$D(\theta_x) = -\frac{1}{2}\frac{d\lambda}{d\theta}\bigg]_{\theta_x} \int_{\theta_n}^{\theta_x} \lambda \, d\theta$$

Finally, the direct problem on the prediction of the relationship of θ versus λ was conducted by using the derived $D(\theta)$ function along with the Philip solution [*Philip*, 1955] of the horizontal infiltration equation, equation (1) [*Elrick et al.*, 1979].

After the above classical deterministic analysis of horizontal infiltration was concluded, the author reconsidered the possibility of further studying the experimental water content profiles without the smoothing procedure. Recognizing that the erratic variability of the experimental values actually provide precious information on the inherent uncertainty associated with the phenomenon of unsaturated flow in porous media, it was decided to develop a methodology to produce new mathematical models which would use all the experimental information and replicate the variability in the water content values.

3. Modeling Spatial Infiltration Variability

We begin our analysis by considering the partial differential equation governing the one-dimensional horizontal infiltration in a homogeneous soil as given by [*Bear*, 1979]

$$\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial \theta}{\partial x}\right] = 0 \tag{1}$$

$$\theta(0, t) = \theta_0 \qquad \theta(\infty, t) = \theta_n \qquad \theta(x, 0) = \theta_{in}$$

where θ is the volumetric water content; t is the time coordinate (h); x is the horizontal spatial coordinate (mm); and D is the soil-water diffusivity (mm^2/h).

Assuming that the most important element of variability, due to the combined effect of uncertainties in the unsaturated transport phenomenon and measurement errors, is the uncertainty in the spatial variability of D, we should expect that the general form for this parameter would be

$$D(x, \omega) = \bar{D}(x) + D'(x, \omega)$$

where $\bar{D}(x)$ represents the expected value in the soil-water diffusivity, $D'(x, \omega)$ represents the random spatial variability of this parameter; and ω is the probability variable. The deterministic trend, $\bar{D}(x)$ is a difficult function to estimate. For the experiment in question, it should be a smooth function which slowly decays in magnitude as x increases, thus exhibiting high values with high values of θ and low values with low values of θ. For a soil in a natural watershed experiencing cycles of partial wetting followed by partial drying, $\bar{D}(x)$ should be a periodic function of x, presumably with a small amplitude. The first approximation to a solution which includes the erratic variability in D should be one in which $\bar{D}$ is assumed equal to a constant average of the value. Now the statistical properties of the random component, $D'(x, \omega)$, will depend on the uncertainty or the erratic nature of D. For the present experiment we have fixed conditions on the left boundary, and then it seems natural to assume that the uncertainty in D increases with distance. Thus we assume for the present study a soil-water diffusivity of the form

$$D(x, \omega) = \bar{D} + D'(x, \omega) \tag{2}$$

Since the variability in D increases erratically with distance, we choose a suitable random process such as a Brownian motion process given by [*Jazwinski*, 1970]

$$E\{D'(x)\} = 0 \qquad E\{D'(x_1)D'(x_2)\} = q \min(x_1, x_2) \tag{3}$$

where $E\{\ \}$ denotes the expectation operator; q is the variance parameter; and $\min(x_1, x_2)$ denotes the minimum magnitude of the distance x_1 or x_2. The parameter estimation procedure for stochastic processes involved in differential equations is still a matter of research (see *Godambe and Thompson* [1984]; *Unny* [1989] for a recent contribution). In this study we use a stochastic process, whose correlation structure is known, and whose qualitative behaviour replicates the form of D'.

Substituting equation (2) into equation (1), placing the terms containing random coefficients in the right hand side of the equation and rearranging, we obtain

$$\frac{\partial \theta}{\partial t} - \bar{D}\frac{\partial^2 \theta}{\partial x^2} = D'\frac{\partial^2 \theta}{\partial x^2} + w\frac{\partial \theta}{\partial x} \tag{4}$$

where $w = \partial D'/\partial x$ is the formal derivative of the Brownian motion process, that is a White Gaussian noise process given by (*Jazwinski*, 1970)

$$E\{w(x)\} = 0 \qquad E\{w(x_1)w(x_2)\} = q\delta(x_1 - x_2) \tag{5}$$

where $\delta(\ )$ is the Dirac's delta function. In the above equations ω has been dropped for convenience, but it is clear that because the differential equation contains random functions, the dependent variable, θ, is also a random process. Equation (4) can be treated as a stochastic evolution equation whose solution is given by [*Serrano et al*, 1985*a*, *b c*; *Serrano and Unny*, 1987*a*]

$$\theta(x, t) = J_t\theta_{in} + \Phi(x, t) + \int_0^t J_{t-\tau}R\theta(\tau)\, d\tau \tag{6}$$

where J_t is the strongly continuous semigroup associated with the diffusion operator $\bar{D}(\partial^2/\partial x^2)$ and it is given by [*Serrano*, 1988*b*]

$$J_t\theta(x, t) = \frac{1}{(4\pi\bar{D}t)^{1/2}}\int_0^\infty \left\{\exp\left[-\frac{(x-s)^2}{4\bar{D}t}\right] - \exp\left[-\frac{(x+s)^2}{4\bar{D}t}\right]\right\}\theta(s, t)\, ds \tag{7}$$

$\Phi(x, t)$ in equation (6) is the particular solution due to the source boundary condition given by [*Serrano*, 1988*a*]

$$\Phi(x, t) = \frac{x}{(4\pi\bar{D})^{1/2}}\int_0^t \exp\left[-\frac{x^2}{4\bar{D}(t-\tau)}\right]\frac{\theta_0 - \theta_n}{(t-\tau)^{3/2}}\, d\tau \tag{8}$$

Since the source boundary condition is constant over time, this equation becomes

$$\Phi(x, t) = (\theta_0 - \theta_n)\operatorname{erfc}\left[\frac{x}{(4\bar{D}t)^{1/2}}\right] \tag{9}$$

where erfc [ ] denotes the "error function complement". Finally the operator R in equation (6) represents

$$R\theta(x, t) = \left[D'(x, \omega)\frac{\partial^2}{\partial x^2} + w(x, \omega)\frac{\partial}{\partial x}\right]\theta(x, t) \tag{10}$$

The third term in the right hand side of equation (6) contains θ. Thus we will approximate this integral by expanding θ as an infinite series of partial solutions $\theta = \theta_1 + \theta_2 + \cdots$. For a description of this approximation procedure of non-linear equations and a discussion of the convergence speed, the reader is referred to *Serrano and Unny* [1987*a*], and *Serrano* [1988*b*]. Since the initial condition is a constant equal to the right boundary condition, $\theta_{in} = \theta_n$, equation (6) reduces to

$$\theta(x, t) = \theta_n \operatorname{erf}\left[\frac{x}{(4\bar{D}t)^{1/2}}\right] + \Phi(x, t) + \sum_{i=1}^{\infty}\int_0^t J_{t-\tau}R\theta_i(\tau)\, d\tau \tag{11}$$

For dissipative systems such as the one in question, the convergence speed of this approximation series is extremely fast and ordinarily only a few terms in the series are needed. Furthermore, since this is not a perturbation approximation, arbitrarily large variances in the stochastic terms can be included. We initiate the approximation by setting $\theta_1 = \Phi$, which is the previous approximation and compute recursively subsequent terms in the series by setting

$$\theta_i = \int_0^t J_{t-\tau} R\theta_{i-1}\, d\tau$$

Equation (11) can be used to generate sample functions of the water content, and to replicate experimental infiltration data, by generating sample functions of D' and w and solve for θ numerically. We are also interested in computing the mean and the variance of the water content as a function of x and t. After taking expectations on both sides of equation (11) one obtains

$$E\{\theta(x, t)\} = \theta_n \operatorname{erf}\left[\frac{x}{(4\bar{D}t)^{1/2}}\right] + \Phi(x, t) \qquad (12)$$

Similarly from equation (11) it is possible to derive an expression for the second moment of θ and after some algebraic manipulation one obtains

$$\sigma_\theta^2 = q \int_0^t \int_0^t \int_0^\infty \int_0^\infty f(x, s, t-\tau) f(x, u, t-\xi) \cdot E\{R(s)R(u)\}\Phi(s, \tau)\Phi(u, \xi)\,du\,ds\,d\xi\,d\tau \qquad (13)$$

where the series approximation has been truncated at the first term due to the small magnitude of the higher order terms;

$$f(x, s, t-\tau) = \frac{1}{(4\pi\bar{D}(t-\tau))^{1/2}} \cdot \left\{\exp\left[-\frac{(x-s)^2}{4\bar{D}(t-\tau)}\right] - \exp\left[-\frac{(x+s)^2}{4\bar{D}(t-\tau)}\right]\right\} \qquad (14)$$

$$E\{R(s)R(u)\} = [\min(s, u)\nabla_s^2\nabla_u^2 + U(s-u)\nabla_s^2\nabla_u + U(u-s)\nabla_s\nabla_u^2 + \delta(s-u)\nabla_s\nabla_u] \qquad (15)$$

$\nabla_s^2 = \partial^2/\partial s$; $U(\ \ )$ denotes the unit step function, and the rest of the terms as before.

As an alternative to model equation (4) and its solution, it was decided to study the Boltzmann-reduced equation of equation (1) because of its importance as a classical solution of the corresponding deterministic problem. Recall that by defining a new independent variable $\lambda = x/t^{1/2}$, equation (1) reduces to the non-linear ordinary differential equation given by [*Hillel*, 1980]

$$\frac{d}{d\lambda}\left(D\frac{d\theta}{d\lambda}\right) + \frac{\lambda}{2}\frac{d\theta}{d\lambda} = 0 \qquad (16)$$

subject to

$$\theta(0) = \theta_0,\ \theta(\infty) = \theta_n,\ \frac{d\theta}{d\lambda}(\infty) = 0$$

For consistency we define the uncertainty term D as a stochastic process of (λ, ω) following the same rules of equations (2), (3) and (5). Thus equation (16) becomes

$$\frac{d^2\theta}{d\lambda^2} + \alpha\lambda\frac{d\theta}{d\lambda} = 2\alpha R(\lambda, \omega)\theta \qquad (17)$$

where the operator

$$R(\lambda, \omega)\theta = \left[-D'(\lambda, \omega)\frac{d^2}{d\lambda^2} - w(\lambda, \omega)\frac{d}{d\lambda}\right]\theta \qquad (18)$$

and $\alpha = 1/2\bar{D}$.

The general solution of equation (17) is given by

$$\theta(\lambda) = \Psi(\lambda) + \int_0^\infty G(\lambda, \xi)R(\xi)\theta(\xi)\,d\xi \qquad (19)$$

where the function Ψ is the solution satisfying the boundary conditions, and the kernel G is the impulse response function. With the aim of deriving a practical solution to model the water content variability in the soil domain, it was decided to simplify the right boundary condition of equation (16) and to represent the unsaturated flow problem as a two-point boundary value problem. Thus a simple expression for the Ψ function is

$$\Psi(\lambda) = \theta_0 - A\lambda + B\lambda^3 - C\lambda^5 \qquad 0 \le \lambda \le a \qquad (20)$$

where a is the limiting right value of λ as given by the laboratory experiment, and A, B and C are constants. Equation (20) is actually an approximation of the series solution

$$\Psi(\lambda) = \theta_0 - E\left(\lambda - \frac{\alpha\lambda^3}{3!} + \frac{3\alpha^2\lambda^5}{5!} - \cdots\right)$$

where E is a constant. In view of the above, the impulse response function in equation (19) is given by (see *Adomian* [1983] for a description of the procedure)

$$G(\lambda, \xi) = U(\lambda-\xi)\left\{\frac{Z(\xi)Z(\lambda)}{Z'(\xi)Z(a)} - \frac{Z(\xi)}{Z'(\xi)}\right\} + U(\xi-\lambda)\left\{\frac{[Z(\xi)-Z(a)]Z(\lambda)}{Z'(\xi)Z(a)}\right\} \qquad (21)$$

where $Z(\xi) = [\Psi(\xi) - \theta_0]/E$, and $Z'(\xi)$ is the first derivative of Z evaluated at ξ.

The integral term in equation (19) contains θ. Thus we approximate this integral successively in the same way as that of equation (6). That is we define $\theta = \Sigma_{i=1}^\infty \theta_i$ and equation (19) reduces to

$$\theta(\lambda) = \Psi(\lambda) + \sum_{i=1}^\infty \int_0^a G(\lambda, \xi)R(\xi)\theta_i(\xi)\,d\xi \qquad (22)$$

where the first approximation $\theta_1 = \Psi$, and subsequent approximations are recursively evaluated as

$$\theta_i(\lambda, \xi) = \int_0^a G(\lambda, \xi)R(\xi)\theta_{i-1}(\xi)\,d\xi$$

From equation (22) the mean of the water content is

$$E\{\theta(\lambda)\} = \Psi(\lambda) \qquad (23)$$

and the variance of the water content can be deduced after some algebraic manipulation to be

$$\sigma_\theta^2 = \int_0^a \int_0^a G(\lambda, \xi)G(\lambda, \rho)E\{R(\xi)R(\rho)\}\Psi(\xi)\Psi(\rho)\, d\rho\, d\xi \tag{24}$$

where $E\{R(\xi)R(\rho)\}$ is given by equation (15).

4. Verification of Models

A somewhat qualitative verification of model equation (11) is provided through a visual comparison between observed water content versus distance profiles with respect to simulated sample functions of the water content versus distance at the same times of breakthrough. This was easily accomplished by adopting an average diffusivity $\bar{D}$ = 100 cm^2/h observed in the experiment, generating a Brownian motion sequence and a White Gaussian noise sequence for an interval Δx = 1 cm [*Jazwinski*, 1970] and solving equation (11) numerically for the same experimental times t = 1, 3 and 6 h respectively.

Unfortunately, the comparison of observed and simulated values of water content at corresponding times was not satisfactory. The reason for this discrepancy is the fact that at the heart of equation (11) is the function Φ given by equation (9), which is a solution of the linearized deterministic unsaturated flow equation. A substantial improvement was obtained after the function Φ was taken as a solution to the non-linear deterministic equation. Thus setting $\Phi = \Psi(x, t)$, where Ψ is given by equation (20) for a fixed breakthrough time, produced reasonably good qualitative agreement between observed water content profiles and simulated sample functions. However the computation of sample functions took considerable more computer time than the sample functions of the second model.

A more objective verification of model equation (11) is provided after a comparison of the mean and variance of observed water content profiles with respect to the corresponding simulated mean and variance values of the water content [*Serrano and Unny*, 1987*b*]. The mean observed values were deduced from the experiments and the mean simulated values were computed after equation (12) for the same experimental times above. The observed means were in good agreement with the simulated means, but computer times required for the calculation of simulated means required significantly more time than the calculation of the simulated means of the second model.

Now the observed variance was computed from the experimental values at the different breakthrough times, after assuming that the observed variances were generated by a stationary random process. This would produce a constant value of the variance with respect to distance for every breakthrough time. We realize it is not possible to assess this assumption, but this is the best one can do with the available information, and at least the experimental variance values will help establish some bounds for the simulated variances. The observed variance at t = 1 h was equal to 1.00 × 10^{-4}. The first experiment at t = 3 h produced a variance in the water content equal to 2.55 × 10^{-4}, and the second experiment at t = 3 h produced a variance equal to 6.81 × 10^{-4}. The observed variance at t = 6 h produced a variance equal to 6.43 × 10^{-4}. The simulated variances were computed using a numerical approximation of equation (13), which exhibited constant magnitudes with respect to distance for every specific time. For an average value of q = 0.0035 in equation (13), the simulated values of the variance at times t = 1, 3 and 6 h were respectively 6.12 × 10^{-4}, 5.07 × 10^{-4} and 3.46 × 10^{-4}. Thus the simulated variances fall within the same range of corresponding experimental variances. It was then concluded that equations (11), (12), and (13) are a satisfactory model for the sample functions, the mean of the water content evolution, and for the replication of the variability of soil moisture around the mean. The advantages of this model are the direct dependency of θ on real time and space, but an important disadvantage is the higher cost in terms of computer time with respect to the second model.

Following a similar procedure, the qualitative verification of the second model we presented in section 3 is accomplished by comparing observed experimental profiles of θ versus λ with corresponding sample functions generated from equation (22). The following parameter values based on the experiments were used: $\bar{D}$ = 2.77 mm^2/s, α = 0.1801 s/mm^2 (assumed from recommendation by Tsakiris et al, 1988), α = 2.7 mm/s$^{1/2}$, θ_0 = 0.458, θ_n = 0.086, q = 0.0002, $A = E$ = 3.217 × 10^{-3}, B = 9.651 × 10^4, C = 2.6057 × 10^{12}, and the computation interval $\Delta\lambda$ = 0.1 mm/s$^{1/2}$. It was noted that only two or three terms in the summation series in equation (22) were necessary since the convergence speed was very fast and the computer time required was substantially less than the required for the previous model. Figure 1 illustrates the qualitative verification of this second model by depicting the evolution of θ with respect to λ for all of the experiments and a plot of one simulated sample function. Other simulated sample functions not illustrated were found to be similar and to evolve in a similar manner with respect to experimental values.

A more objective verification of this second model was accomplished after comparing the experimental mean values and variances of the water content with respect to the corresponding simulated mean values and variances. The simulated mean of the water content as a function of λ was easily computed from equation (23) and the simulated variance as a function of λ was computed after an approximation of equation (24). Figure 2 shows a plot of the observed mean, the simulated mean, the observed mean plus and minus one observed standard deviation, and the simulated mean plus and minus one simulated standard deviation, all with respect to λ. Generally the observed and simulated means are close one another, and the only discrepancy worth noting is the somewhat lower slope of the simulated mean at the wetting front. Also the simulated mean does not level to zero slope at $\lambda = a$, following the simplification of the right boundary condition described in section 3, a small price to pay in the name of simplicity. Now the observed standard deviation is a constant equal to 0.0214, whereas the simulated standard deviation starts with zero at λ = 0 and monotonically increases with λ up to a value of about 0.09 at $\lambda = a$. Both simulated and observed are within the same order of magnitude. Thus it was concluded that the second model based on Boltzmann-reduced equation represents a good tool for the prediction of the variability of soil moisture evolution, and one that requires substantially less computer time than the first model.

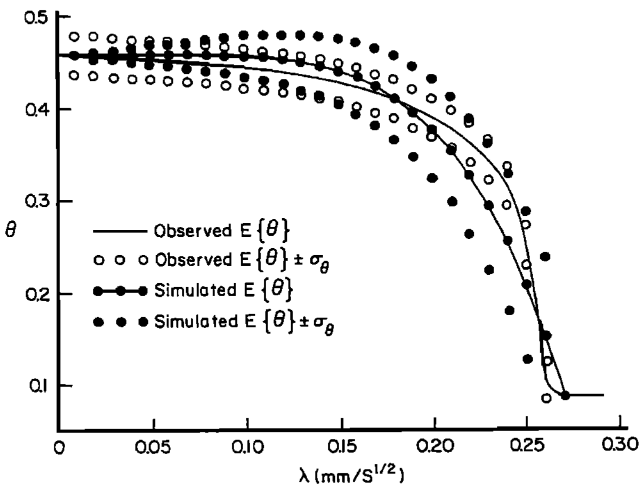

Fig. 2. Observed and simulated moments of water content versus λ.

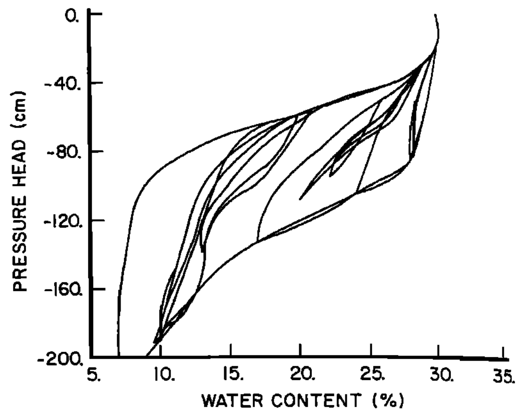

Fig. 4. Sample water content versus pressure head relationship.

5. Applications to Modeling Infiltration in Hysteretic Soils

The present methodology offers promising possibilities to study complex infiltration evolution in watersheds. Consider the problem of modeling vertical infiltration in a homogeneous soil of a watershed exhibiting a moderate slope. Existing infiltration models either assume a constant infiltration rate at the soil surface and a unique set of soil-water functional relationships (hysteresis is neglected), or use empirical expressions which disregard physical laws. There are a number of deterministic models which consider hysteresis [i.e., *Paolovassilis*, 1962; *Kool and Parker*, 1988; *Scott et al.*, 1983]. The problem consists in keeping track of the history of wetting and drying in the simulation model. A generalization of this process should characterize the wetting-drying cycles in a statistical manner reflecting the variability of point rainfall. In the present section we would like to develop a model to represent vertical infiltration when the infiltration rate at the ground surface responds to the natural random variability of point rainfall.

Tsakiris et al. [1988] introduced a shot noise process to describe the variability of water content in the root zone when the time evolution of point rainfall was modeled as a Poisson process:

$$\theta_0(t, \omega) = \sum_{i=0}^{N(t)} X_i e^{-\alpha(t - t_i(\omega))} \tag{25}$$

where $\theta_0(t, \omega)$ is the water content depth in the root zone (in mm or dimensionless if divided by the bulk soil depth); X_i is the depth magnitude of infiltrated water in the top soil per rainfall event (mm), in this case modeled as an exponential random variable; $N(t)$ is the number of rainfall events in the interval $(0, t]$, modeled as a Poisson random variable; $t_i(\omega)$ represents random points in time; and α is a soil parameter representing the rate of moisture depletion during rainless periods, function of the ratio of potential evapotranspiration to the field capacity.

A simulated sample function of the process given by equation (25) for a parameter $\alpha = 0.1$ day^{-1} is shown in Figure 3. This illustrates the rapid wetting of the root zone during a rainfall event and a slow decline in water content during rainless periods. Let us investigate the effect of repeated sample functions of the shot noise water content on the form of the soil-water diffusivity, D. For this purpose, a computer experiment was designed based on the laboratory experimental soil characteristic curves presented by *Liakopoulus* (1965) for a very fine sand. The generated sample water content from the shot noise process was used in order to emulate secondary scanning curves in the water content versus pressure head relationship and in the hydraulic conductivity versus pressure head relationship. Figures 4 and 5 illustrate a digital plotter output of this computer experiment. The main wetting and the main drying curves can be

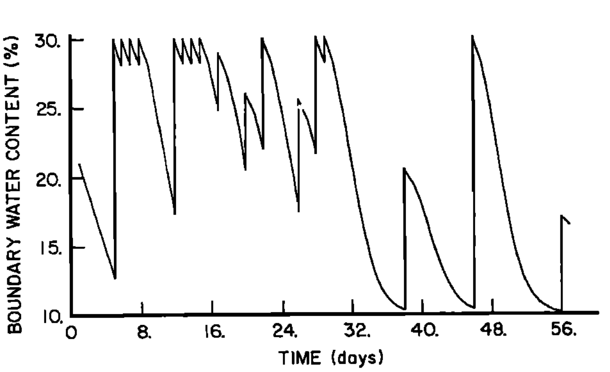

Fig. 3. Water content at the upper boundary versus time.

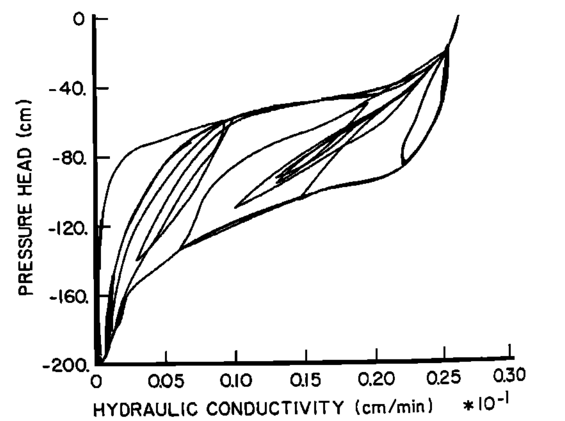

Fig. 5. Sample hydraulic conductivity versus pressure head relationship.

observed along with several secondary scanning curves resulting from the partial wetting and partial drying loops very similar to the ones which occur in a real watershed. These hysteretic phenomenon, which has puzzled scientists for a long time, has posed tremendous obstacles to the solution of the vertical infiltration equation.

Nevertheless, we proceed formally, and derive a sample function of the soil water diffusivity by using the sample soil-water characteristic curves (Figures 4 and 5) and a numerical approximation of the physical relationship $D(\theta) = K(\theta)(d\Psi/d\theta)$, where K is the hydraulic conductivity, and Ψ is the pressure head. The time series obtained for D showed that this parameter could be described as a colored noise process of the form

$$D = \bar{D} + D'(t, \omega) \qquad E\{D'(t)\} = 0$$
$$E\{D'(t_1)D'(t_2)\} = qe^{-p(t_1 - t_2)} \tag{26}$$

It should be noted that in this model the fluctuations in D are driven by fluctuations in θ, whereas in the stochastic models of section 3 fluctuations in D occur even when θ is fixed. Given the erratic variability of point rainfall, fluctuations driven by changes in the boundary condition are much more important than other fluctuations, which are comparatively insignificant. It was then concluded that D could be functionally related to a random function in time when the point rainfall at the surface is a Poisson process and hysteresis is considered. The spatial redistribution of D' remains to be investigated, but in the absence of more experimental information we refrain from speculating about it.

We now study the solution of the vertical infiltration equation subject to a colored noise soil-water diffusivity and a shot noise top boundary condition. The governing differential equation is [*Bear*, 1979]

$$\frac{\partial\theta}{\partial t} - \frac{\partial}{\partial z}\left[D(t, \omega)\frac{\partial\theta}{\partial z}\right] + \frac{dK}{d\theta}\frac{\partial\theta}{\partial z} = 0 \tag{27}$$

subject to

$$\theta(0, t, \omega) = \theta_0(t, \omega) \qquad \theta(\infty, t) = 0 \qquad \theta(z, 0) = \theta_{in}(z)$$

where we now stress the functional dependency of D on t and ω, as demonstrated by the computer experiment; K is the soil-water hydraulic conductivity; and z is the vertical spatial coordinate, positive downwards. By taking $D(\theta)$ as a random function of time, as demonstrated by the simulations, the non-linearity in the unsaturated flow equation is removed and the resulting linear stochastic partial differential equation can be easily solved. Following the same methodology described in section 3 for the horizontal infiltration equation, equation (27) could be written as a stochastically forced equation of the form (see equation (4))

$$\frac{\partial\theta}{\partial t} - \bar{D}\frac{\partial^2\theta}{\partial z^2} + u\frac{\partial\theta}{\partial z} = D'\frac{\partial^2\theta}{\partial z^2} \tag{28}$$

where $u = dK/d\theta$. In this case u is taken as constant at this stage, since relative fluctuations in D are larger than those in K, because of the extra factor $d\Psi/d\theta$ taken along an erratic curve $\Psi(\theta(t))$. The solution of this transport evolution equation is (see equation (6))

$$\theta(z, t, \omega) = J_t\theta_{in}(z) + \Phi(z, t, \omega) + \int_0^t J_{t-\tau}D'(\tau, \omega)\frac{\partial^2\theta}{\partial z^2}\,d\tau \tag{29}$$

where $\Phi(z, t, \omega)$ is the solution due to the shot noise top boundary condition, and it is given by (see equation (8))

$$\Phi(z, t, \omega) = \frac{z}{(4\pi\bar{D})^{1/2}} \cdot \int_0^t \exp\left[-\frac{[z - u(t-\tau)]^2}{4\bar{D}(t-\tau)}\right]\frac{\theta_0(\tau, \omega)}{(t-\tau)^{3/2}}\,d\tau \tag{30}$$

Once again, J_t in equation (29) is the impulse response function given by (see equation (7), and *Serrano*, 1988*a* for details)

$$J_t\theta_{in}(z) = \frac{1}{(4\pi\bar{D}t)^{1/2}}\int_0^\infty\left\{\exp\left[-\frac{(z-ut-s)^2}{4\bar{D}t}\right] - \exp\left[-\frac{(z-ut+s)^2}{4\bar{D}t}\right]\right\}\theta_{in}(s)\,ds \tag{31}$$

As before, the term containing θ in the right hand side of equation (29) is expanded as an infinite series (see equation (11)), and equation (29) becomes

$$\theta(z, t, \omega) = J_t\theta_{in}(z) + \Phi(z, t, \omega) + \sum_{i=1}^{\infty}\int_0^t J_{t-\tau}D'(\tau, \omega)\frac{\partial^2}{\partial z^2}\theta_i\,d\tau \tag{32}$$

where the first term in the approximation can be taken as $\theta_1 = \Phi$ and subsequent terms can evaluated recursively as

$$\theta_i = \int_0^t D_{t-\tau}D'(t, \omega)\frac{\partial^2\theta_{i-1}}{\partial z^2}\,d\tau$$

Equation (32) can be used to generate sample functions of the water content and thus to observe the qualitative behaviour of the water content evolution at different depths, but most importantly, expressions for the mean and the variance of the water content can also be derived from equation (32). For this purpose, a numerical approximation of equation (32) was attempted and the time integral was recursively computed at intervals of 1 day, and 0.1 m. The parameter $\bar{D}$ was chosen as equal to 0.0162 m^2/day, the parameter q as equal to 1.56 m^2/day, and the parameter u was assumed constant for this illustration and equal to 0.0216 m/day. The details on the approximation are lengthy and are not included.

Figure 6 shows the results of the simulations to compute sample water content functions at typical depths for an initially dry soil and when the random part of D is not included. This figure illustrates the effect of a highly variable Poisson rainfall on the water content evolution at depth. Note that the variability is attenuated as depth increases. This would indicate that a more appropriate, and more general, representation for the variance of D' should include a smooth attenuation with depth. Figure 7 shows the same effect of the random rainfall on the water content profile with respect to depth at different times. Figure 8 shows the effect

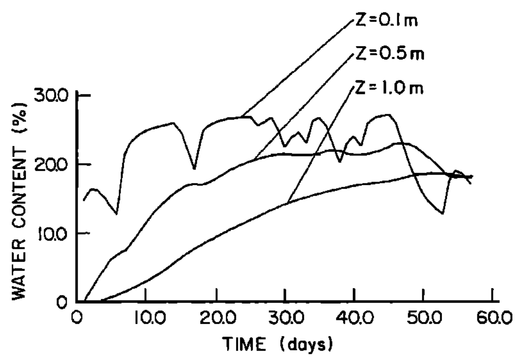

Fig. 6. Sample water content due to random boundary condition alone versus time at different depths.

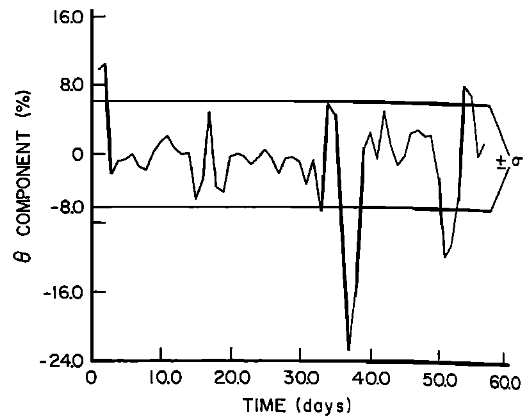

Fig. 8. Sample function of the component due to D' of water content versus time and standard deviation at the boundary.

of the random component of the diffusivity D' as a function of time and its standard deviation.

From equation (32) the mean and the variance functions were derived. Figure 9 shows the mean of the random boundary component with respect to time at different depths. Figure 10 shows the mean and the mean plus one standard deviation of the boundary component at one meter of depth. Note that the high uncertainty generated by the random rainfall would dictate the field measurement of soil moisture at intervals spaced no more than 10 days in order to maintain the standard deviation of water content forecasts within meaningful accuracy.

Thus the methodology introduced can be used to develop models for soils moisture forecast and for the study of interesting and complex physics of the phenomenon of unsaturated porous media flow in real watersheds.

6. Summary and Conclusions

The movement of water in the unsaturated zone of natural watersheds ultimately controls the proportion of storm water which follows the different paths of the hydrologic cycle. Therefore it is important to have good models describing the qualitative and quantitative behaviour of soil moisture. These mathematical models are based on laboratory and field data of infiltration, which is usually subject to uncertain variability. Although traditional models ignore this erratic variability, it is desirable to develop models which actually reproduce this variability in order to study the soil moisture evolution, at the laboratory or field scale, in a general statistical fashion.

Two models were tested as to their ability to replicate the erratic variability of experimental horizontal infiltration data. The first model is based on the full partial differential equation of infiltration, and the second model is based on the Boltzmann-reduced differential equation. Both models were subject to a space or a time and space random soil-water diffusivity defined as uncertainty term. The solution of the equations was presented, statistical properties of the water content function were described and verification of the models was conducted. It was found that both models closely reproduced the statistical properties of the experimental data. The first model is easier to reproduce since it relates to real time and space as independent variable. However the second model required substantially less computer time to evaluate the sample functions and the moments to the water content.

As an application of the methodology, a third model was introduced as a new approach to predict vertical infiltration in hysteretic soils in real watersheds. For this purpose, the effect of time variability of point rainfall was conceived as a shot noise process, the hysteretic loops resulting from the natural wetting and drying cycles generated a correlated random soil-water diffusivity process, and a solution of the vertical infiltration equation was presented. Statistical properties of the water content were studied.

It is hoped that the present methodology will encourage further research on the use of stochastic partial differential equation models, rather than empirical equations, to simulate and study infiltration in natural watersheds. Applications in higher dimensional domains and the combination of numerical methods with these type of stochastic models are of much interest [*Serrano and Unny*, 1986].

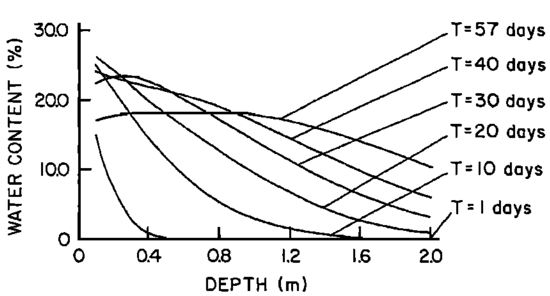

Fig. 7. Sample water content due to random boundary condition alone versus depth at different times.

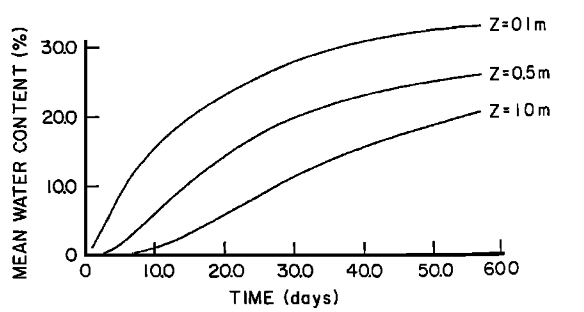

Fig. 9. Mean water content due to random boundary conditions alone versus time at different depths.

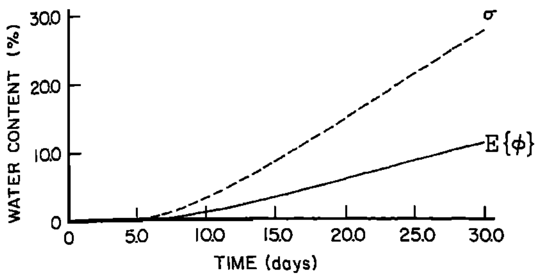

Fig. 10. Mean and standard deviation of the water content due to random boundary conditions versus time at 1-m depth.

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